ON THE k-COCHAINS OF A SPECTRUM

BY
DANIEL M. KAN[1]

1. Introduction

Just as a group may be described by generators and relations, so the homotopy type of a connected CW (or semisimplicial) complex X may be described ([4], [5]) by the homotopy groups $\pi_i(X)$ and a sequence of cocycles $k^3, k^4, \cdots, k^{i+2}, \cdots$. Here each k^{i+2} is a cochain on a space whose homotopy type depends on the groups $\pi_1(X), \cdots, \pi_i(X)$ and the cocycles $k^3, \cdots, k^{i+1}$, while $\pi_{i+1}(X)$ is the coefficient group. The k^{i+2} are usually called "k-invariants", although "k-cocycles" might be a more appropriate name.

It is our purpose to give a similar result for homotopy types of (semisimplicial) spectra. It will be shown that, using a suitably generalized notion of cochain, the homotopy type of a spectrum Y may be described by the homotopy groups $\pi_i Y$ $(-\infty < i < \infty)$ and a collection of cochains k_i^j (one for every pair of integers (i, j) with $i < j$), where each k_i^j is a cochain on a spectrum which depends only on $\pi_i Y$, while the coefficients depend only on $\pi_j Y$.

The paper is written semisimplicially, and we shall freely use the results of [2].

§§2 and 3 deal with the analogues for spectra of the notions path space, fibre map, and loop space, while §4 is concerned with the inverse of "taking loops".

In §5 cochains are generalized. This generalization is mainly based on the facts that (i) a cochain complex may be considered as a chain complex (if the degree of a q-cochain is taken to be $-q$), and (ii) a chain complex determines [2, §5] an essentially unique abelian group spectrum containing it. The cocycles of a cochain complex then correspond exactly with the simplices (of the corresponding abelian group spectrum) of which all faces are $= *$, and the cochains with those of which all faces, except possibly the 0-face, are $= *$. This suggests considering the other simplices of the abelian group spectrum as well.

Decomposing a spectrum into abelian group spectra one gets a string of diamond-shaped diagrams which we call a "decomposition". These decompositions are considered in §6. With each decomposition one may associate a collection of (generalized) cochains, and it is shown in §7 that under suitable circumstances a decomposition is completely determined by these cochains. In §8 we then use this to show that the homotopy type of a spectrum Y may be

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described by the homotopy groups $\pi_i Y$ and a suitable collection of cochains k_i^j.

2. Path spectra

2.1 Definition. Let $\mathcal{S}p_E$ denote the category of set spectra which satisfy the extension condition [2, §7]. For $Y \in \mathcal{S}p_E$ its path spectrum ΛY will be the spectrum which has a simplex Λy for every simplex $y \in Y$; the face and degeneracy operators and the gradation are given by

$$d_i \Lambda y = \Lambda d_{i+1} y \qquad \text{for all } i,$$

$$s_i \Lambda y = \Lambda s_{i+1} y \qquad \text{for all } i,$$

$$\text{degree } \Lambda y = \text{degree } y - 1.$$

Clearly $\Lambda Y \in \mathcal{S}p_E$.

Similarly for a map $w : Y \to Z \in \mathcal{S}p_E$ we define a map $\Lambda w : \Lambda Y \to \Lambda Z$ by $\Lambda y \to \Lambda wy$ for all $y \in Y$. The function Λ so defined then is a functor $\Lambda : \mathcal{S}p_E \to \mathcal{S}p_E$, the *path functor.*

Then [2, §10] we have

2.2 Proposition. *Let $Y \in \mathcal{S}p_E$. Then $\pi_i \Lambda Y = 0$ for all i.*

This follows at once from the fact that for every simplex $\Lambda y \in \Lambda Y$ of degree i with $d_j \Lambda y = *$ for all j, we have $d_0 \Lambda s_0 y = y$ and $d_j \Lambda s_0 y = *$ for $j > 0$.

The remainder of this section will deal with some natural transformations between iterated path functors, which will be used in §5.

2.3 Definition. For every object $Y \in \mathcal{S}p_E$, simplex $y \in Y$, and map $w \in \mathcal{S}p_E$, let $\Lambda^0 Y = Y$, $\Lambda^0 y = y$, and $\Lambda^0 w = w$, and let

$$\Lambda^n Y = \Lambda\Lambda^{n-1} Y, \qquad \Lambda^n y = \Lambda\Lambda^{n-1} y, \quad \text{and} \quad \Lambda^n w = \Lambda\Lambda^{n-1} w$$

for $n > 0$. Then for every pair of integers (i, n) with $0 \leqq i \leqq n$, maps

$$\delta_i Y : \Lambda^{n+1} Y \to \Lambda^n Y \quad \text{and} \quad \sigma_i Y : \Lambda^{n+1} Y \to \Lambda^{n+2} Y$$

may be defined by the formulas

$$(\delta_i Y)\Lambda^{n+1} y = \Lambda^n d_{n-i}\, y \quad \text{and} \quad (\sigma_i Y)\Lambda^{n+1} y = \Lambda^{n+2} s_{n-i}\, y.$$

The functions δ_i and σ_i clearly are natural transformations $\delta_i : \Lambda^{n+1} \to \Lambda^n$ and $\sigma_i : \Lambda^{n+1} \to \Lambda^{n+2}$.

A straightforward calculation yields

2.4 Proposition. *The natural transformations δ_i and σ_i satisfy the identities*

$$\begin{aligned}
\delta_i \delta_j &= \delta_{j-1} \delta_i && \text{for } i < j, \\
\delta_i \sigma_j &= \sigma_{j-1} \delta_i && \text{for } i < j, \\
&= \text{identity} && \text{for } i = j, j+1, \\
&= \sigma_j \delta_{i-1} && \text{for } i > j+1, \\
\sigma_i \sigma_j &= \sigma_j \sigma_{i-1} && \text{for } i > j.
\end{aligned}$$

2.5 PROPOSITION. *Let $Y \in \mathcal{S}p_E$, and let $0 \leqq i \leqq n$. Then the maps*

$$\delta_i(\Lambda^j Y) : \Lambda^{n+1} Y \to \Lambda^n Y$$

coincide for all j with $0 \leqq j \leqq n - i$. The same holds for the maps

$$\sigma_i(\Lambda^j Y) : \Lambda^{n+1} Y \to \Lambda^{n+2} Y.$$

2.6 *Notational convention.* Instead of $\delta_i Y$ and $\sigma_i Y$ we will often write δ_i and σ_i. In view of Proposition 2.5 this will cause no confusion.

Another immediate consequence of the definitions is

2.7 PROPOSITION. *Let $Y \in \mathcal{S}p_E$, and let $0 \leqq i \leqq n$. Then the maps*

$$\Lambda(\delta_i Y) : \Lambda^{n+2} Y \to \Lambda^{n+1} Y \quad \text{and} \quad \delta_{i+1} Y : \Lambda^{n+2} Y \to \Lambda^{n+1} Y$$

coincide. The same holds for the maps $\Lambda(\sigma_i Y)$, $\sigma_{i+1} Y : \Lambda^{n+2} Y \to \Lambda^{n+3} Y$.

3. Loop spectra

We first consider for spectra the notions of fibre map and homotopy sequence of a fibre map.

3.1 DEFINITION. Let $p : E \to B \in \mathcal{S}p$ (the category of spectra), and let [2, §4] Ps $p = \{p_i\} : \{E_i\} \to \{B_i\} \in \mathcal{P}s$. Then p is called a *fibre map* if $p_i : E_i \to B_i \in \mathcal{S}_*$ is a fibre map for all i. If $\{*\} \subset B$ denotes the subspectrum consisting of the base points only, then the subspectrum $p^{-1}\{*\} \subset E$ is called the *fibre* of the fibre map $p : E \to B$.

As for set complexes [4] we have

3.2 PROPOSITION. *Let $p : E \to B \in \mathcal{S}p$ be a fibre map with fibre F. Then $F \in \mathcal{S}p_E$. Moreover $E \in \mathcal{S}p_E$ if and only if $B \in \mathcal{S}p_E$.*

3.3 DEFINITION. Let $p : E \to B \in \mathcal{S}p_E$ be a fibre map with fibre F, and let Ps $p = \{p_i\} : \{E_i\} \to \{B_i\}$. Then [2, §10]

$$\pi_n(F_i) = \pi_{n+1}(F_{i+1}), \qquad \pi_n(E_i) = \pi_{n+1}(E_{i+1}) \quad \text{and} \quad \pi_n(B_i) = \pi_{n+1}(B_{i+1})$$

for all $n > 0$ and i. Moreover the boundary maps $\partial_{n,i} : \pi_n(B_i) \to \pi_{n-1}(F_i)$ in the exact homotopy sequences of the fibre maps p_i [4] are such that $\partial_{n,i} = \partial_{n+1,\, i+1}$, i.e., commutativity holds in the diagram

$$\begin{tikzcd} \pi_n(B_i) \arrow[r, "\partial_{n,i}"] \arrow[d, "\mathrm{id}"] & \pi_{n-1}(F_i) \arrow[d, "\mathrm{id}"] \\ \pi_{n+1}(B_{i+1}) \arrow[r, "\partial_{n+1,i+1}"] & \pi_n(F_{i+1}). \end{tikzcd}$$

Hence the homotopy sequences of the fibre maps p_i give rise to a sequence

$$\cdots \to \pi_{n+1} B \xrightarrow{\partial} \pi_n F \xrightarrow{\pi_n j} \pi_n E \xrightarrow{\pi_n p} \pi_n B \xrightarrow{\partial} \cdots$$

where $j : F \rightarrow E$ is the inclusion map. It is called *the homotopy sequence* of the fibre map $p : E \rightarrow B$. Clearly we have

3.4 PROPOSITION. *The homotopy sequence of a fibre map in $\mathcal{S}p_E$ is exact.*

We now turn to loop spectra. As for set spectra one readily verifies

3.5 PROPOSITION. *Let $Y \in \mathcal{S}p_E$, and let (2.3) $\delta_0 : \Lambda Y \rightarrow Y \in \mathcal{S}p_E$ be the map given by $\delta_0 \Lambda y = d_0 y$ for all $y \in Y$. Then δ_0 is a fibre map.*

We therefore may state

3.6 DEFINITION. The *loop spectrum* of a spectrum $Y \in \mathcal{S}p_E$ is the fibre of the map $\delta_0 : \Lambda Y \rightarrow Y$. It will be denoted by ΩY. Similarly for a map $w : Y \rightarrow Z \in \mathcal{S}p_E$ we define a map $\Omega w : \Omega Y \rightarrow \Omega Z$ as the restriction of the map Λw. Clearly the function Ω so defined is a functor $\Omega : \mathcal{S}p_E \rightarrow \mathcal{S}p_E$, the *loop functor*.

Propositions 2.2 and 3.4 imply

3.7 PROPOSITION. *Let $Y \in \mathcal{S}p_E$. Then the map $\partial : \pi_{n+1} Y \rightarrow \pi_n \Omega Y$ in the homotopy sequence of the fibre map $\delta_0 : \Lambda Y \rightarrow Y$ is an isomorphism for all n.*

Another useful property of the loop functor is given by

3.8 PROPOSITION. *The functor $\Omega : \mathcal{S}p_E \rightarrow \mathcal{S}p_E$ preserves homotopies, i.e., maps homotopic maps into homotopic maps.*

Proof. By 3.7 and [2, 10.5], Ω maps homotopy equivalences into homotopy equivalences. The proposition now follows from

3.9 PROPOSITION. *Let $T : \mathcal{S}p_E \rightarrow \mathcal{S}p_E$ be a functor which maps homotopy equivalences into homotopy equivalences. Then T maps homotopic maps into homotopic maps.*

Proof. Let $\mathcal{S}p_H$ be the category obtained from $\mathcal{S}p_E$ by identifying two maps whenever they are homotopic, and let Q be the identification functor. The proposition now follows from [2, 9.1] and the fact that [2, 9.2] QT is a homotopy functor.

4. The functor Ω^{-1}

Given $Y \in \mathcal{S}p_E$, consider the problem of finding a $Z \in \mathcal{S}p_E$ such that $\Omega Z = Y$. It will be shown that for group spectra this can be done in a functorial manner, by using the analogue of the $\bar{W}$ construction of [3].

4.1 DEFINITION. For $B \in \mathcal{S}p_G$ (the category of group spectra [2, §5]) we denote by $\Omega^{-1}B$ the spectrum of which a simplex of degree q is any infinite sequence of simplices of B

$$c = (b_1, \cdots, b_i, \cdots)$$

such that

(i) degree $b_i = q - i$ for all i,

(ii) $b_i = *$ for all but a finite number of i's.

Its faces and degeneracies are given by

$$d_i c = (d_{i-1} b_1, \cdots, (d_0 b_i) b_{i+1}, b_{i+2} \cdots),$$

$$s_i c = (s_{i-1} b_1, \cdots, s_0 b_i, *, b_{i+1}, \cdots).$$

For $i = 0$ this should be interpreted as $(b_2, \cdots)$ and $(*, b_1, \cdots)$. A straightforward calculation shows that $\Omega^{-1}B \in \mathcal{S}p_E$.

Similarly for a map $v : B \to B' \in \mathcal{S}p_G$ let $\Omega^{-1}v : \Omega^{-1}B \to \Omega^{-1}B'$ be the map given by $(b_1, \cdots) \to (vb_1, \cdots)$. Then clearly the function Ω^{-1} so defined is a functor $\Omega^{-1} : \mathcal{S}p_G \to \mathcal{S}p_E$. The use of the symbol Ω^{-1} for this functor is justified by

4.2 PROPOSITION. *Let $B \in \mathcal{S}p_G$. Then the map $B \to \Omega\Omega^{-1}B \in \mathcal{S}p_E$ given by $b \to \Lambda(b, *, \cdots, *, \cdots)$ for all $b \in B$ is natural and is an isomorphism.*

An argument similar to the one used in the proof of Proposition 3.8 yields

4.3 PROPOSITION. *The functor $\Omega^{-1} : \mathcal{S}p_G \to \mathcal{S}p_E$ maps homotopic maps into homotopic maps.*

We end by showing that the functors Ω and Ω^{-1} give rise to a functor $\Omega^n : \mathcal{S}p_A \to \mathcal{S}p_A$ ($\mathcal{S}p_A$ is the category of abelian group spectra [2, §5]) for every integer n.

4.4 DEFINITION. Let $B \in \mathcal{S}p_A$. Then the addition on B induces an addition on ΛB and ΩB turning them into abelian group spectra. Similarly $\Omega^{-1}B$ may be turned into an abelian group spectrum by coordinatewise addition. For a map $v : B \to B' \in \mathcal{S}p_A$ the maps Λv, Ωv, and $\Omega^{-1}v$ then become homomorphisms. The resulting functors $\mathcal{S}p_A \to \mathcal{S}p_A$ will also be denoted by Λ, Ω, and Ω^{-1}, as no confusion will arise from this.

A consequence of this definition is that for any $B \in \mathcal{S}p_A$ and pair of integers (i, n) with $0 \leqq i \leqq n$, the maps

$$\delta_i : \Lambda^{n+1}B \to \Lambda^n B \quad \text{and} \quad \sigma_i : \Lambda^{n+1}B \to \Lambda^{n+2}B$$

are also in $\mathcal{S}p_A$.

4.5 *Notational convention.* We will write $\Omega^0 : \mathcal{S}p_A \to \mathcal{S}p_A$ for the identity functor and $\Omega^n = \Omega\Omega^{n-1}$ and $\Omega^{-n} = \Omega^{-1}\Omega^{-(n-1)}$ for every integer $n > 0$. Also for every $B \in \mathcal{S}p_A$ and $b \in B$ we will identify b with the simplex

$$\Lambda(b, *, \cdots, *, \cdots) \in \Omega\Omega^{-1}B.$$

Then clearly we have

4.6 PROPOSITION. *Let i and j be integers $\geqq 0$. Then*

$$\Omega^i\Omega^{-j} = \Omega^{i-j} : \mathcal{S}p_A \to \mathcal{S}p_A \ .$$

5. Cochains on a spectrum

The cochain complex $C^*(Y; \pi)$ of a spectrum Y with coefficients in an abelian group π may be defined as for set complexes.

5.1 Definition. For $Y \in \mathcal{S}p$ let $\{C_q Y, \partial\}$ denote the *normalized chain complex* of Y, i.e., $C_q Y$ is the abelian group with a generator Cy for every $y \in Y_{(q)}$ and a relation $Cy = 0$ whenever y is degenerate; the boundary homomorphisms $\partial : C_q Y \to C_{q-1} Y$ are such that $\partial Cy = \sum_{i=0}^{\infty}(-1)^i Cd_i y$ for all y. The *normalized cochain complex* $C^*(Y; \pi)$ then is defined by $C^q(Y; \pi) = \operatorname{Hom}(C_q Y, \pi)$ and $\delta = \operatorname{Hom}(\partial, i_\pi)$.

This definition will be generalized in two respects. In the first place one may, as for set complexes [4] establish a one-to-one correspondence between the elements of $C^q(Y; \pi)$ and the maps $Y \to \Lambda\Omega^{-q-1}K\pi$ where $K\pi$ denotes a suitably defined "Eilenberg-Mac Lane spectrum of π". This suggests considering maps $Y \to \Lambda\Omega^{-q-1}B$ for an arbitrary abelian group spectrum B. Furthermore $C^*(Y; \pi)$ may be considered as a chain complex (if the degree of a q-cochain is taken to be $-q$), which in view of [2, §5], determines an essentially unique abelian group spectrum containing it. The cocycles of $C^*(Y; \pi)$ then correspond exactly with the simplices of which all faces are $= *$, and the cochains with those of which all faces, except possibly the 0-face, are $= *$. This suggests that the other simplices might equally well be worth considering. We will call them cochains too and therefore state

5.2 Definition. Let $Y \in \mathcal{S}p$ and $B \in \mathcal{S}p_A$, and denote by $\operatorname{Hom}(Y, B)$ the abelian group of the maps $Y \to B \in \mathcal{S}p$, where the addition is induced by the addition on B. For every integer q the inclusions (see 4.6)

$$\Omega^{-q}B \subset \cdots \subset \Lambda^j\Omega^{-q-j}B \subset \cdots$$

induce inclusions

$$\operatorname{Hom}(Y, \Omega^{-q}B) \subset \cdots \subset \operatorname{Hom}(Y, \Lambda^j\Omega^{-q-j}B) \subset \cdots.$$

The union of these groups will be denoted by $C^q(Y; B)$; its elements will be called *q-cochains of Y with coefficients in B*. Furthermore for every q-cochain $c \in C^q(Y; B)$ and integer $i \geqq 0$ we define a $(q+1)$-cochain $\delta_i c$ and a $(q-1)$-cochain $\sigma_i c$ as the compositions

$$Y \xrightarrow{c} \Lambda^j\Omega^{-q-j}B \xrightarrow{\delta_i} \Lambda^{j-1}\Omega^{-q-1}B,$$

$$Y \xrightarrow{c} \Lambda^j\Omega^{-q-j}B \xrightarrow{\sigma_i} \Lambda^{j+1}\Omega^{-q-1}B$$

for suitably large j. That these are well defined, i.e., independent of j, follows from 2.5 and the naturality of δ_i and σ_i. The groups $C^q(Y; B)$ together with the operators δ_i and σ_i will be denoted by $C^*(Y; B)$. An immediate consequence of Proposition 2.4 then is

5.3 Proposition. *$C^*(Y; B)$ is an abelian group spectrum if the δ_i and σ_i are considered as the face and degeneracy operators respectively, and the degree of a q-cochain is taken as $-q$.*

The following proposition is also readily verified.

5.4 PROPOSITION. *Let* $c \in C^q(Y; B)$. *Then* $c \in \mathrm{Hom}(Y, \Lambda^j\Omega^{-q-j}B)$ *if and only if* $\delta_i c = 0$ *for* $i \geqq j$.

In view of this proposition we may make the following notational convention.

5.5 *Notational convention.* Let $c \in C^q(Y; B)$ be such that $q \leqq 0$ and $\delta_i c = 0$ for $i \geqq -q$, and let $c' \in C^r(B; B')$. Then c may be represented by a map $c : Y \to \Lambda^{-q}B$, and c' by a map $c' : B \to \Lambda^j\Omega^{-r-j}B'$ for suitably large j. We then denote by $c'c \in C^{q+r}(Y; B')$ the composition

$$Y \xrightarrow{c} \Lambda^{-q}B \xrightarrow{\Lambda^{-q}c'} \Lambda^{-q+j}\Omega^{-r-j}B'.$$

We end with defining the Eilenberg-Mac Lane spectrum $K\pi$ of an abelian group π and relating cochains with coefficients in $K\pi$ with normalized cochains with coefficients in π.

5.6 DEFINITION. Let π be an abelian group. For every integer $i \geqq 0$ and element $\alpha \in \pi$, identify α with the corresponding i-simplex of the Eilenberg-Mac Lane complex $K(\pi, i) \in \mathcal{S}_*$ [4]. Let $\{K(\pi, i), k_i\}$ be the prespectrum [2, §3] such that $k_i(\alpha, \phi_0) = \alpha \in K(\pi, i+1)$ for every i-simplex $\alpha \in K(\pi, i)$. Then *the Eilenberg-Mac Lane spectrum of* π is the spectrum $K\pi$ defined by $K\pi =$ Sp $\{K(\pi, i), k_i\}$ [2, §4]. It is not difficult to verify that the addition of the $K(\pi, i)$ induces an addition on $K\pi$, turning it into an abelian group spectrum. In fact, as for the $K(\pi, i)$, this is the only way in which $K\pi$ can be turned into a group spectrum.

Now for $Y \in \mathcal{S}p$ and $c \in C^q(Y; \pi)$ denote by $hc : Y \to \Lambda\Omega^{-q-1}K\pi \in \mathcal{S}p$ the (unique) map such that for every $y \in Y$ of degree q

$$\Lambda^q(hc)y = (-1)^q[c(Cy)] \qquad \text{if } q \geqq 0,$$

or

$$(hc)y = (-1)^q\Lambda^{-q}[c(Cy)] \qquad \text{if } q \leqq 0.$$

Then we have

5.7 PROPOSITION. *If* $C^*(Y; \pi)$ *is considered as a chain complex, and* $C^*(Y; K\pi)$ *as an abelian group spectrum, then the function* h *is an isomorphism*

$$h : (C^*(Y; \pi)) \approx MC^*(Y; K\pi)$$

where the chain complex $MC^*(Y; K\pi)$ *is as in* [2, §5].

It is easily verified that $hc \in MC^*(Y; K\pi)$ for all $c \in C^*(Y; \pi)$, i.e., $\delta_i hc = 0$ for $i > 0$, and that degreewise h is an isomorphism. It thus remains to show that h is a chain map, i.e., that $\delta_0(hc)y = h(\delta c)y$ for every $c \in C^q(Y; \pi)$ and $y \in Y_{(q+1)}$. If $q \geqq 0$, let $z \in \Omega^{-q-1}K\pi$ be such that $(hc)y = \Lambda z$. Then

$$\Lambda^{q+1} d_{i+1}\, z = \Lambda^q d_i\, \Lambda z = \Lambda^q d_i (hc) y = (-1)^q [c(Cd_i\, y)]$$

for $i \geqq 0$. Moreover $\sum \Lambda^{q+1} d_i\, z = 0$. Consequently

$$\begin{aligned}\Lambda^{q+1}\delta_0(hc)y = \Lambda^{q+1}\delta_0\, \Lambda z &= \Lambda^{q+1} d_0\, z = -\sum \Lambda^{q+1} d_{i+1}\, z \\ &= (-1)^{q+1}[\sum c(Cd_i\, y)] = (-1)^{q+1}[(\delta c)(Cy)] = \Lambda^{q+1} h(\delta c) y,\end{aligned}$$

and hence $\delta_0(hc)y = h(\delta c)y$. The proof for $q < 0$ is similar and will be omitted.

6. Decompositions of spectra and cochain systems

One often tries to reduce a problem on spectra to one on abelian group spectra (which one then hopes to be able to solve) by decomposing one or more of the spectra involved into abelian group spectra. It is this kind of decompositions which will be studied in more detail in this section and the next. It turns out that one can associate with such a decomposition a collection of cochains (in the sense of §5) and that, under suitable conditions, these cochains completely determine the decomposition.

6.1 Definition. A *decomposition* is a commutative (infinite) diagram

$$\begin{tikzcd}
{} \arrow[dr] & & \Lambda Y_i \arrow[dr, "\delta_0"] & & \Lambda Y_{i+1} \arrow[dr, "\delta_0"] & & {} \\
& \cdots\ Y_{i-1} \arrow[ur, "h_{i-1}"] \arrow[dr, "f_{i-1}"'] & & Y_i \arrow[ur, "h_i"] \arrow[dr, "f_i"'] & & Y_{i+1}\ \cdots \arrow[ur] \arrow[dr] & \\
{} \arrow[ur] & & A_{i-1} \arrow[ur, "g_{i-1}"'] & & A_i \arrow[ur, "g_i"'] & & {}
\end{tikzcd}$$

such that for every integer i

(i) $Y_i \in \mathcal{S}p_E$,

(ii) $A_i \in \mathcal{S}p_A$,

(iii) f_i is induced by g_i, i.e., for every $a \in A_i$ and $v \in \Lambda Y_{i+1}$ with $g_i\, a = \delta_0\, v$, there is exactly one simplex $y \in Y_i$ such that $f_i\, y = a$ and $h_i\, y = v$. Such a decomposition will be denoted by $(Y_i, A_i, f_i, g_i, h_i)$.

6.2 Definition. Let $\cdots, A_{i-1}, A_i, \cdots$ be a collection of abelian group spectra indexed by the integers, and for every pair of integers (i, j) with $0 \leqq i \leqq j$ let be given a cochain $k_i^j \in C^{i+1-j}(A_i\, ; A_j)$. Then the set $\{k_i^j\}$ will be called a *cochain system* if (see Notational Convention 5.5)

$$\begin{aligned} \delta_n\, k_i^j &= 0 && \text{for } n \geqq j - i - 1, \\ \delta_n\, k_i^j &= k_{i+1+n}^j\, k_i^{i+1+n} && \text{for } n < j - i - 1. \end{aligned}$$

6.3 Definition. *Let $(Y_i, A_i, f_i, g_i, h_i)$ be a decomposition. For every pair of integers (i, j) with $0 \leqq i \leqq j$ denote by m_i^j the composition*

$$Y_i \xrightarrow{h_i} \Lambda Y_{i+1} \xrightarrow{\Lambda h_{i+1}} \cdots \xrightarrow{\Lambda^{j-i-1} h_{j-1}} \Lambda^{j-i} Y_j \xrightarrow{\Lambda^{j-i} f_j} \Lambda^{j-i} A_j,$$

and by k_i^j the composition (now $i < j$)

$$A_i \xrightarrow{g_i} Y_{i+1} \xrightarrow{m_{i+1}^j} \Lambda^{j-i-1} A_j .$$

Clearly $m_i^j \in C^{i-j}(Y_i\,;\,A_j)$ and $k_i^j \in C^{i+1-j}(A_i\,;\,A_j)$. The set $\{k_i^j\}$ then will be called *the cochain system of the decomposition* $(Y_i, A_i, f_i, g_i, h_i)$.

This terminology is justified by

6.4 Proposition. *The cochain system of a decomposition is a cochain system.*

Proof. In view of 2.7 and the naturality of δ_n, the following diagram is commutative

$$\begin{tikzcd}
\Lambda^n Y_{i+1+n} \arrow[r, "\Lambda^n h_{i+1+n}"] \arrow[d, "\Lambda^n f_{i+1+n}"] & \Lambda^{n+1} Y_{i+2+n} \arrow[r, "\Lambda^{n+1} m_{i+2+n}^{j-i-2-n}"] \arrow[d, "\Lambda^n(\delta_0 Y_{i+2+n}) = \delta_n"] & \Lambda^{j-i-1} A_j \arrow[d, "\delta_n"] \\
\Lambda^n A_{i+1+n} \arrow[r, "\Lambda^n g_{i+1+n}"] & \Lambda^n Y_{i+2+n} \arrow[r, "\Lambda^n m_{i+2+n}^{j-i-2-n}"] & \Lambda^{j-i-2} A_j .
\end{tikzcd}$$

This readily implies that $\delta_n k_i^j = k_{i+1+n}^j k_i^{i+1+n}$ for $n < j - i - 1$. That $\delta_n k_i^j = 0$ for $n \geqq j - i - 1$ is a direct consequence of 5.4.

7. Locally finite decompositions and cochain systems

7.1 Definition. A decomposition $(Y_i, A_i, f_i, g_i, h_i)$ will be called *locally finite* if for every integer i and every $y \in Y_i$ there is an integer s (depending on y) such that $m_i^j y = *$ for $j > s$. Similarly a cochain system $\{k_i^j\}$, where $k_i^j \in C^{i+1-j}(A_i\,;\,A_j)$, is called *locally finite* if for every i and every $a \in A_i$ there is an integer s (depending on a) such that $k_i^j a = *$ for $j > s$.

Clearly we have

7.2 Proposition. *The cochain system of a locally finite decomposition is locally finite.*

The following two propositions now essentially assert that there is a one-to-one correspondence between locally finite decompositions and locally finite cochain systems.

7.3 Proposition. *Every locally finite cochain system is the cochain system of a locally finite decomposition.*

7.4 Proposition. *Let* $(Y_i, A_i, f_i, g_i, h_i)$ *and* $(Y_i', A_i, f_i', g_i', h_i')$ *be locally finite decompositions which have the same cochain system. Then there are unique isomorphisms* $v_i : Y_i \to Y_1'$ *such that* $f_i' v_i = f_i$, $g_i' = v_{i+1} g_i$, *and* $h_i' v_i = (\Lambda v_{i+1}) h_i$ *for all* i.

Proof of Proposition 7.3. Let $k_i^j \in C^{i+1-j}(A_i\,;\,A_j)$ be a cochain system. For every integer i define a spectrum Y_i' as follows. A simplex of Y_i' of

degree q is any infinite sequence $(a^i, a^{i+1}, \cdots)$ such that
(i) $a^j \in \Lambda^{j-i} A_j$ for all $j \geqq i$,
(ii) $a^j = *$ for all but a finite number of j's,
(iii) degree $a^j = q$ for all $j \geqq i$,
(iv) $\delta_n a^j = k^j_{i+n} a^{i+n}$ for all $n < j - i$.
Its faces and degeneracies are

$$d_n(a^i, a^{i+1}, \cdots) = (d_n a^i, d_n a^{i+1}, \cdots),$$

$$s_n(a^i, a^{i+1}, \cdots) = (s_n a^i, s_n a^{i+1}, \cdots).$$

They clearly are also in Y'_i, and hence Y'_i is a spectrum.

Define maps $f'_i : Y'_i \to A_i$ and $g'_i : A_i \to Y'_{i+1}$ by

$$f'_i(a^i, a^{i+1}, \cdots) = a^i,$$

$$g'_i a^i = (k^{i+1}_i a^i, k^{i+2}_i a^i, \cdots).$$

If we denote a simplex $\Lambda^n(a^i, a^{i+1}, \cdots) \in \Lambda^n Y'_i$ also by $(\Lambda^n a^i, \Lambda^n a^{i+1}, \cdots)$, then we can define a map $h'_i : Y'_i \to \Lambda Y'_{i+1}$ by the formula

$$h'_i(a^i, a^{i+1}, \cdots) = (a^{i+1}, a^{i+2}, \cdots).$$

A simple computation then yields that f'_i, g'_i, and h'_i are indeed maps of spectra, that $\delta_0 h'_i = g'_i f'_i$ for all i, and that f'_i is induced by g'_i (Definition 6.1(iii)). The proof that $Y'_i \in \mathcal{S}p_E$ is also straightforward although somewhat longer and uses the fact that the A_i satisfy the extension condition [2, §5]. The details are left to the reader.

That the decomposition $(Y'_i, A'_i, f'_i, g'_i, h'_i)$ is locally finite and has $\{k^j_i\}$ as cochain system now follows from the fact that

$$m^j_i(a^i, a^{i+1}, \cdots) = a^j$$

for all $(a^i, a^{i+1}, \cdots) \in Y'_i$ and $j \geqq i$.

Proof of Proposition 7.4. Let $\{k^j_i\}$ be the cochain system of $(Y_i, A_i, f_i, g_i, h_i)$. Then it suffices to prove the proposition for the case that $(Y'_i, A_i, f'_i, g'_i, h'_i)$ is the locally finite decomposition from the proof of Proposition 7.3.

For every integer i and $y \in Y_i$ let

$$v_i\, y = (m^i_i\, y, m^{i+1}_i y, \cdots).$$

Then a simple calculation yields that $f'_i v_i = f_i$, $g'_i = v_{i+1} g_i$, and $h'_i v_i = (\Lambda v_{i+1}) h_i$, while iterated application of 6.1(iii) yields that the v_i are isomorphisms $v_i : Y_i \to Y'_i$.

In order to prove the uniqueness of the v_i, assume that $w_i : Y_i \to Y'_i$ are maps such that $f'_i w_i = f_i$, $g'_i = w_{i+1} g_i$ and $h'_i w_i = (\Lambda w_{i+1}) h_i$ for all i. These conditions clearly imply that $w_i\, y = (m^i_i\, y, m^{i+1}_i y, \cdots)$ for all i and $y \in Y_i$, i.e., $w_i = v_i$ for all i.

7.5 *Remark*. It should be noted that the only property of the A_i used in

§§6 and 7 is that they satisfy the extension condition. No use was made of their addition.

8. Application to homotopy types

8.1 DEFINITION. A cochain system $\{k_i^j\}$ where $k_i^j \in C^{i+1-j}(A_i\,;A_j)$ is called an *elementary cochain system* if $A_i = \Omega^{-2i}K(\pi_{2i}A_i)$ for every i.

Clearly we have

8.2 PROPOSITION. *Every elementary cochain system is locally finite.*

We shall now associate with every locally finite cochain system $\{k_i^j\}$ a spectrum $L\{k_i^j\} \in \mathcal{S}p_E$ and show that every homotopy type in $\mathcal{S}p_E$ can be obtained in this manner. In fact for every spectrum $X \in \mathcal{S}p_E$ there is an *elementary* cochain system $\{k_i^j\}$ such that $L\{k_i^j\}$ has the same homotopy type as X. Thus *every homotopy type of spectra may be "described" by means of an* (in general not unique) *elementary cochain system.*

First we state

8.3 DEFINITION. Let P be as in [2, §2]. Then for $Y \in \mathcal{S}p$, its *suspension* is the spectrum of which the simplices of degree q are the base point and all pairs (ϕ, y) such that $\phi \in P$, $y \in Y$, $y \neq *$, and $\dim \phi + \text{degree } y = q - 1$; the face and degeneracy operators are given by

$$\begin{aligned} d_i(\phi, y) &= (d_i\,\phi, y), & s_i(\phi, y) &= (s_i\,\phi, y), & i &\leqq p, \\ &= (\phi, d_{i-p-1}\,y), & &= (\phi, s_{i-p-1}\,y), & i &> p \end{aligned}$$

(where $p = \dim \phi$) whenever this has a meaning, and $d_i(\phi, y) = *$ otherwise. Similarly the suspension of a map $w : Y \to Z \in \mathcal{S}p$ is the map $Sw : SY \to SZ$ given by $(\phi, y) \to (\phi, wy)$ whenever this has a meaning, and $(\phi, y) \to *$ otherwise. The function S so defined is a functor $S : \mathcal{S}p \to \mathcal{S}p$, the *suspension functor*. We shall sometimes denote by S^0 the identity functor of $\mathcal{S}p$, and for every integer $n > 0$ by S^n the composite functor $SS^{n-1} : \mathcal{S}p \to \mathcal{S}p$.

8.4 PROPOSITION. *Let $Y \in \mathcal{S}p_E$, let $\phi_0 \in P$ be as in* [2, §2], *let i be an integer > 0, and let $\Omega^i : \mathcal{S}p_E \to \mathcal{S}p_E$ be the i-fold loop functor. Then there is a unique map $t : S^i\Omega^i Y \to Y$ such that $t(\phi_0, \cdots(\phi_0, \Lambda^i y)\cdots) = y$ for all $\Lambda^i y \in \Omega^i Y$. Moreover this map is natural and is a weak homotopy equivalence.*

Proof. Existence, uniqueness, and naturality may be verified by a straightforward computation. That t is a weak homotopy equivalence is proved just as in the proof of [2, 5.3].

8.5 DEFINITION. Let the cochains $k_i^j \in C^{i+1-j}(A_i\,; A_j)$ form a locally finite cochain system, and let $(Y_i, A_i, f_i, g_i, h_i)$ be the essentially unique decomposition associated with it (§7). The maps $f_i : Y_i \to A_i$ are easily verified to be fibre maps. Denote the fibre of f_i by F_i, and let $L'\{k_i^j\}$ be the spectrum obtained from the union of the spectra Y_0, SY_{-1}, S^2Y_{-2}, $\cdots$ by identifying, for every integer $i > 0$, S^iF_{-i} with its image under the composite map

$$S^iF_{-i} \xrightarrow{S^ig_{-i}} S^i\Omega Y_{-i+1} \xrightarrow{S^{i-1}t} S^{i-1}Y_{-i+1}.$$

Then we define the spectrum $L\{k_i^j\} \in \mathcal{S}p_E$ by $L\{k_i^j\} = FL'\{k_i^j\}$.

8.6 PROPOSITION. *Let $X \in \mathcal{S}p_E$, and for every integer i let $A_i = \Omega^{-2i}K(\pi_i X)$. Then there exists a (not necessarily unique) elementary cochain system $\{k_i^j\}$ with $k_i^j \in C^{i+1-j}(A_i\ ; A_j)$ such that $L\{k_i^j\}$ has the homotopy type of X.*

Proof. We will outline the proof of Proposition 8.6 but leave to the reader many details which are merely analogues for spectra of "well known" results on set complexes [4].

For every integer i let $E_i X$ denote the i-Eilenberg subspectrum of X, i.e., the largest subspectrum which has no simplices of degree $< i$ except the base points, and let $P_i X$ be the i-Postnikov quotient spectrum, i.e., the spectrum obtained from X by identifying two simplices whenever they have the same iterated faces of degree i. In view of [2, §8], X may supposed to be minimal. For every integer $i \leqq 0$ let $Y_i = \Omega^{-i}E_i X$, and for every integer $i \geqq 0$ let Y_i be a minimal spectrum such that $\Omega^i Y_i = E_i X$. The existence of the latter follows readily from §4 and [2, §5 and §8]. For every integer i let $A_i = P_{2i} Y_i$, and let $f_i : Y_i \to A_i$ be the projection. Clearly A_i is minimal, $\pi_{2i} A_i = \pi_i X$, and $\pi_j A_i = 0$ for $j \neq 2i$, i.e., $A_i = \Omega^{-2i}K(\pi_i X)$. Also f_i is a fibre map with ΩY_{i+1} as fibre. Let Y_i' be the spectrum obtained from Y_i by identifying all simplices of ΩY_{i+1} with the appropriate base point, and let $h_i' : Y_i \to \Lambda Y_{i+1}$ be a map which is the identity on ΩY_{i+1}. Such a map exists in view of the contractibility of ΛY_{i+1} (§2). Then the composition $\delta_0 h_i' : Y_i \to Y_{i+1}$ induces a map $h_i'' : Y_i' \to Y_{i+1}$, and the map $f_i : Y_i \to A_i$ induces a map $f_i' : Y_i' \to A_i$. The latter is a weak homotopy equivalence (by the argument used in the proof of [2, 5.3]). As $Y_{i+1} \in \mathcal{S}p_E$, this readily implies the existence of a map $g_i : A_i \to Y_{i+1}$ such that $g_i f_i' \sim h_i''$ and hence $g_i f_i = \delta_0 h_i'$. In exactly the same manner as for set complexes [1] one may prove a homotopy lifting theorem for fibre maps of spectra. Applying this we get a map $h_i : Y_i \to \Lambda Y_{i+1}$ such that $g_i f_i = \delta_0 h_i$, and a straightforward calculation yields that $(Y_i, A_i, f_i, g_i, h_i)$ is a locally finite decomposition, and that its cochain system $\{k_i^j\}$ is an elementary cochain system.

Finally let $w : L'\{k_i^j\} \to X$ be the unique map such that for every integer $i \leqq 0$ and simplex $y \in S^{-i}Y_i = S^{-i}\Omega^{-i}E_i X$ we have $wy = ty \in E_i X \subset X$. Then it is not difficult to verify that w is a weak homotopy equivalence, and that hence $L\{k_i^j\}$ and X have the same homotopy type.

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS