

JACOBIANS AND SYMMETRIC PRODUCTS

BY

R. L. E. SCHWARZENBERGER¹

The n -fold symmetric product $C(n)$ of a curve C is usually the starting point for the construction of the Jacobian variety J of C . Adopting this point of view, Mattuck [6] has determined the Chern classes of $C(n)$ regarded, for $n > 2g - 2$, as a projective fibre bundle over J . This determination led him to a set of intersection relations among the subvarieties of J which he conjectured should arise from an exact sequence of vector bundles on J .

In order to prove this conjecture, it is convenient to adopt a point of view exactly opposite to that mentioned above. Namely, we assume the existence, for a complete nonsingular algebraic curve C , of a Picard variety J satisfying the general properties used by Lang [5] to define the Picard variety. We then define certain sheaves on J which we call (following Mattuck [7]) Picard sheaves, and prove certain properties of exactness and duality which they satisfy. This is enough to obtain the intersection relations of Mattuck to which we alluded above. The advantage of this point of view is that it is not necessary to go over again any of the steps in the construction of the Jacobian, and hence that some (but not all) of the theory will extend to any Picard variety.

We shall make considerable use of certain constructions contained in the *Éléments* of Grothendieck [4], in particular, the construction which associates with any coherent sheaf \mathcal{E} a fibred variety $\mathbf{P}(\mathcal{E})$. It is this construction which enables us, finally, to reconstruct the symmetric products $C(n)$ from the Picard sheaves.

We give references to [4] whenever the relevant chapter is already available, but do not intend to imply that they cannot be found elsewhere. Nor do we, in giving yet another aspect of the link between Jacobians and symmetric products, wish to slight the rich literature which already exists on the subject, and to which references will be found in [5], [6], [7], [8]. We adopt the following notations involving coherent sheaves. If $f: Y \rightarrow X$ is a regular map and \mathcal{F} is a coherent sheaf on X , we write $\mathcal{F}^* = \text{Hom}(\mathcal{F}, \mathcal{O}_X)$ and $f^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_X}$. If \mathcal{G} is a coherent sheaf on Y , we write $f_r(\mathcal{G})$ for the sheaf with presheaf $f_r(\mathcal{G})(U) = H^r(f^{-1}(U), \mathcal{G})$. If V is a vector space, we write $\mathbf{P}(V)$ for the projective space whose points correspond to the hyperplanes of V . This construction is extended in [4, II, 4.1] to an arbitrary coherent sheaf.

1. Algebraic curves

Let C be a complete nonsingular curve of genus g defined over an algebraically closed field k , and $c \in C$ a fixed base point. For each integer s , define

Received November 25, 1961.

¹ Supported by a National Science Foundation grant.

the invertible sheaf \mathfrak{I}_s which corresponds to the divisor sc . There is an exact sequence of sheaves on C ,

$$(1) \quad 0 \rightarrow \mathfrak{I}_{s-1} \rightarrow \mathfrak{I}_s \rightarrow \mathfrak{I}_s/\mathfrak{I}_{s-1} \rightarrow 0$$

where the sheaf $\mathfrak{I}_s/\mathfrak{I}_{s-1}$ has support c and restriction \mathcal{O}_c to c . The invertible sheaves on C of degree zero form a group $\text{Pic}(C)$. Every invertible sheaf on C of degree n has the form $\mathfrak{M} \otimes \mathfrak{I}_n$ for some $\mathfrak{M} \in \text{Pic}(C)$. In particular, the canonical sheaf on C has the form $\mathfrak{K} \otimes \mathfrak{I}_{2g-2}$, for some $\mathfrak{K} \in \text{Pic}(C)$. The exact sequence (1) defines an exact cohomology sequence

$$0 \rightarrow H^0(C, \mathfrak{M} \otimes \mathfrak{I}_{s-1}) \rightarrow H^0(C, \mathfrak{M} \otimes \mathfrak{I}_s) \xrightarrow{\alpha} k \xrightarrow{\beta} H^1(C, \mathfrak{M} \otimes \mathfrak{I}_{s-1}) \\ \rightarrow H^1(C, \mathfrak{M} \otimes T_s) \rightarrow 0.$$

Taking first $\mathfrak{M} = \mathfrak{L}, s = n$, and second $\mathfrak{M} = \mathfrak{K} \otimes \mathfrak{L}^*, s = 2g - 2 - n$, this sequence defines homomorphisms

$$\alpha_n: H^0(C, \mathfrak{L} \otimes \mathfrak{I}_n) \rightarrow k, \\ \beta_n: k \rightarrow H^1(C, \mathfrak{K} \otimes \mathfrak{L}^* \otimes \mathfrak{I}_{2g-2-n}).$$

The duality theorem states that, under the duality between the vector spaces $H^0(C, \mathfrak{L} \otimes \mathfrak{I}_n), H^1(C, \mathfrak{K} \otimes \mathfrak{L}^* \otimes \mathfrak{I}_{2g-2-n})$, the homomorphism α_n corresponds to the element $\beta_n(1)$. The Riemann-Roch theorem for C states that

$$\dim H^0(C, \mathfrak{L} \otimes \mathfrak{I}_n) - \dim H^1(C, \mathfrak{L} \otimes \mathfrak{I}_n) = n - g + 1.$$

2. Picard sheaves

Let (J, \mathfrak{L}) be a Picard variety for C , as defined in [5, IV, §4]. In other words, J is an abelian variety defined over k . \mathfrak{L} is an invertible sheaf on the product variety $J \times C$. There are maps $p: J \times C \rightarrow J, q: J \times C \rightarrow C, i_y: J \rightarrow J \times C, j_x: C \rightarrow J \times C$, defined by $p(x, y) = x, q(x, y) = y, i_y(x) = j_x(y) = (x, y)$. The rule $\phi(x) = j_x^* \mathfrak{L}$ defines an isomorphism $\phi: J \rightarrow \text{Pic}(C)$. To make sure that (J, \mathfrak{L}) is defined uniquely [5, IV, §4], let x_0 be the zero of J , and demand that $i_c^* \mathfrak{L} = \mathcal{O}_J, j_{x_0}^* \mathfrak{L} = \mathcal{O}_C$.

DEFINITION. The sheaves

$$E_n = p_0(\mathfrak{L} \otimes q^* \mathfrak{I}_n), \quad F_n = p_1(\mathfrak{L} \otimes q^* \mathfrak{I}_n)$$

are called *Picard sheaves* on J .

PROPOSITION 1. For each n there is an exact sequence

$$0 \rightarrow \mathfrak{E}_{n-1} \rightarrow \mathfrak{E}_n \rightarrow \mathcal{O}_J \rightarrow \mathfrak{F}_{n-1} \rightarrow \mathfrak{F}_n \rightarrow 0.$$

Proof. The exact sequence (1) induces on $J \times C$ an exact sequence

$$0 \rightarrow \mathfrak{L} \otimes q^* \mathfrak{I}_{n-1} \rightarrow \mathfrak{L} \otimes q^* \mathfrak{I}_n \rightarrow \mathfrak{M} \rightarrow 0,$$

where M has support $J \times c$ and restriction \mathcal{O}_J to $J \times c$. The exact cohomology sequence for sheaves now gives the sequence

$$(2) \quad 0 \rightarrow \mathcal{E}_{n-1} \rightarrow \mathcal{E}_n \rightarrow \mathcal{O}_J \rightarrow \mathcal{F}_{n-1} \rightarrow \mathcal{F}_n \rightarrow 0.$$

We shall use the following result of Grothendieck ([4, III]; see also [1, §7, Satz 7] and [3, §5]). Let $f: Y \rightarrow X$ be a proper map, \mathcal{F} an X -flat sheaf on Y , and suppose that $H^r(f^{-1}(x), \mathcal{F}_x) = 0$. Then $f_r(\mathcal{F})$ is zero in a neighborhood of x , and the natural homomorphism $f_{r-1}(\mathcal{F})_x \rightarrow H^{r-1}(f^{-1}(x), \mathcal{F}_x)$ is surjective. The cases which we shall use will have $X = J, f = p, \mathcal{F}$ locally free. Then certainly \mathcal{F} is J -flat, and we have

- (a) $p_r(\mathcal{F}) = 0$ for $r > 1$,
- (b) the natural homomorphism $p_1(\mathcal{F})_x \rightarrow H^1(p^{-1}(x), \mathcal{F}_x)$ is an epimorphism,
- (c) if $H^1(p^{-1}(x'), \mathcal{F}_x) = 0$, there is a neighborhood of x' in which $p_1(\mathcal{F})$ is zero and $p_0(\mathcal{F})$ is free,
- (d) if $H^0(p^{-1}(x'), \mathcal{F}_{x'}) = 0$, there is a neighborhood of x' in which $p_0(\mathcal{F})$ is zero and $p_1(\mathcal{F})$ is free,
- (e) if $\dim H^1(p^{-1}(x), \mathcal{F}_x)$ is constant in a neighborhood of x' , there is another neighborhood of x' in which $p_1(\mathcal{F})$ is free.

Statements (a) and (b) follow from the fact that $H^r(p^{-1}(x), \mathcal{F}_x) = 0$ for $r > 1$. The proofs of (c), (d), and (e) are similar, so we shall prove (d). Since $H^0(p^{-1}(x'), \mathcal{F}_{x'}) = 0$, there is a neighborhood U' of x' in which $p_0(\mathcal{F})$ is zero and in which (by the Riemann-Roch theorem) $\dim H^1(p^{-1}(x), \mathcal{F}_x)$ is constant. Choose a basis for $H^1(p^{-1}(x), \mathcal{F}_x)$. Since the homomorphism $p_1(\mathcal{F})_x \rightarrow H^1(p^{-1}(x), \mathcal{F}_x)$ is an epimorphism for $x \in U'$, the elements of this basis extend in a neighborhood U'' of x to sections of $p_1(\mathcal{F})$ which (by continuity) remain linearly independent. The sheaf $p_1(\mathcal{F})$ is therefore free in U'' .

PROPOSITION 2. \mathcal{E}_n is torsion-free for all n , zero for $n < 0$, locally free of rank $n - g + 1$ for $n > 2g - 2$.

PROPOSITION 3. \mathcal{F}_n is zero for $n > 2g - 2$, locally free of rank $g - n - 1$ for $n < 0$.

Proof. The fact that \mathcal{E}_n is torsion-free is a special case of [4, I, 7.4.5]. The other statements follow by (c) and (d) from the fact that

$$H^0(p^{-1}(x), j_x^* \mathcal{L} \otimes \mathcal{J}_n) = 0$$

for $n < 0$ and the duality theorem.

PROPOSITION 4. For $r > 2g - 2$ and $s < 0$ there is an exact sequence

$$0 \rightarrow E_r \rightarrow \mathfrak{M} \rightarrow \mathcal{F}_s \rightarrow 0,$$

where \mathfrak{M} is a successive extension of $(r - s)$ copies of \mathcal{O}_J .

Proof. The exact sequence (1) gives an exact sequence of sheaves on $J \times C$

$$0 \rightarrow \mathcal{L} \otimes q^*\mathcal{I}_s \rightarrow \mathcal{L} \otimes q^*\mathcal{I}_r \rightarrow \mathfrak{N} \rightarrow 0,$$

where \mathfrak{N} has support $J \times c$ and restriction to $J \times c$ a successive extension of copies of \mathcal{O}_J . The hypotheses imply that

$$p_0(\mathcal{L} \otimes q^*\mathcal{I}_s) = p_1(\mathcal{L} \otimes q^*\mathcal{I}_r) = 0,$$

and the exact cohomology sequence for sheaves then gives the result.

We remark that, in view of Corollary 2 of Proposition 6, this proposition is actually true whenever $s < g$.

The automorphism θ of $\text{Pic}(C)$ for which $\theta(\mathcal{L}) = \mathcal{K} \otimes \mathcal{L}^*$ induces automorphisms, still denoted by θ , of J and $J \times C$. The definition of θ then implies

LEMMA 1. $\theta^*\mathcal{E}_n = p_0(\theta^*\mathcal{L} \otimes q^*\mathcal{I}_n) = p_0(\mathcal{K} \otimes \mathcal{L}^* \otimes q^*\mathcal{I}_n)$, and
 $\theta^*\mathcal{F}_n = p_1(\theta^*\mathcal{L} \otimes q^*\mathcal{I}_n) = p_1(\mathcal{K} \otimes \mathcal{L}^* \otimes q^*\mathcal{I}_n)$.

In particular we observe that the exact sequence (2) remains exact when θ^* is applied to each term. There is a duality between the sheaves $\mathcal{E}_n, \mathcal{F}_n$ which is made explicit in Theorem 1 below. We shall first give an example to illustrate two points raised by this duality: that the sheaves \mathcal{F}_n carry more information than the sheaves \mathcal{E}_n , and that results true in the locally free case (such as $\mathcal{E}_n = \mathcal{E}_n^{**}$) are not true in general.

The sheaf \mathcal{E}_0 is clearly zero on $J - x_0$ and is torsion-free (Proposition 2). Therefore $\mathcal{E}_0 = 0$. Similarly, the sheaf $\theta^*\mathcal{F}_{2g-2}$ is zero on $J - x_0$. Since the sheaf $\theta^*\mathcal{F}_{2g-1} = 0$ (Proposition 3), the sequence (2) implies that $\theta^*\mathcal{F}_{2g-2, x_0}$ has rank at most one. On the other hand, there is an epimorphism

$$\theta^*\mathcal{F}_{2g-2, x_0} \rightarrow H^1(C, \mathcal{K} \otimes \mathcal{I}_{2g-2}),$$

and so $\theta^*\mathcal{F}_{2g-2, x_0}$ has rank at least one. Thus $\theta^*\mathcal{F}_{2g-2}$ is a torsion sheaf with support x_0 and restriction \mathcal{O}_{x_0} to x_0 . In particular, $\theta^*\mathcal{F}_{2g-2}^* = 0$.

We shall define homomorphisms

$$\lambda_n: \mathcal{E}_n \rightarrow \theta^*\mathcal{F}_{2g-2-n}^*, \quad \mu_n: \theta^*\mathcal{F}_{2g-2-n} \rightarrow \mathcal{E}_n^*,$$

and we have just seen that λ_0 must be an isomorphism and μ_0 an epimorphism (since both map onto zero sheaves). This is the general situation. It is convenient for the sequel to state the duality in a slightly more general form than we need for Theorem 1. A map $h: X \rightarrow J$ defines a diagram of maps

$$\begin{array}{ccc} X \times C & \xrightarrow{\bar{h}} & J \times C \\ \downarrow \bar{p} & & \downarrow p \\ X & \xrightarrow{h} & J. \end{array}$$

Define on X the sheaves

$$\bar{h}\mathcal{E}_n = \bar{p}_0(\bar{h}^*\mathcal{L} \otimes q^*\mathcal{I}_n), \quad \bar{h}\theta^*\mathcal{F}_n = \bar{p}_1(\bar{h}^*\theta^*\mathcal{L} \otimes q^*\mathcal{I}_n).$$

Note that $\bar{h}\mathcal{E}_n$ is not in general isomorphic to $h^*\mathcal{E}_n$; this is however true if $n > 2g - 2$, for then $\bar{h}\mathcal{E}_n$ and \mathcal{E}_n are both locally free.

PROPOSITION 5. $\bar{h}\mathcal{E}_n = (\bar{h}\theta^*\mathcal{F}_{2g-2-n})^*$.

Proof. Let U be an open set of X , and write

$$\begin{aligned} V_n &= H^0(U \times C, \bar{h}^*\mathcal{L} \otimes q^*\mathcal{I}_n), & V_{n,x} &= H^0(C, \bar{h}^*j_x^*\mathcal{L} \otimes q^*\mathcal{I}_n), \\ W_n &= H^1(U \times C, \bar{h}^*\mathcal{L}^* \otimes \mathcal{K} \otimes q^*\mathcal{I}_{2g-2-n}), \\ W_{n,x} &= H^1(C, \bar{h}^*j_x^*\mathcal{L}^* \otimes \mathcal{K} \otimes q^*\mathcal{I}_{2g-2-n}). \end{aligned}$$

The duality theorem defines an isomorphism

$$\delta: V_{n,x} \rightarrow \text{Hom}(W_{n,x}, k),$$

and there are restriction homomorphisms

$$\rho_x: V_n \rightarrow V_{n,x}, \quad \sigma_x: \text{Hom}(W_{n,x}, k) \rightarrow \text{Hom}(W_n, k).$$

The set of composite homomorphisms

$$\sigma_x \delta \rho_x: V_n \rightarrow \text{Hom}(W_n, k)$$

for $x \in U$ defines a homomorphism

$$\lambda_n(U): V_n \rightarrow \text{Hom}(W_n, \Gamma(U, \mathcal{O}_U))$$

which is compatible with inclusions of open sets. It defines

$$\lambda_n: \bar{h}\mathcal{E}_n \rightarrow (\bar{h}\theta^*\mathcal{F}_{2g-2-n})^*.$$

For $n > 2g - 2$, λ_n is an isomorphism since the sheaves involved are locally free. Consider the diagram, obtained analogously to (2),

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{h}\mathcal{E}_{n-1} & \longrightarrow & \bar{h}\mathcal{E}_n & \longrightarrow & \mathcal{O}_x \\ & & \downarrow \lambda_{n-1} & & \downarrow \lambda_n & & \downarrow \iota \\ 0 & \longrightarrow & (\bar{h}\theta^*\mathcal{F}_{2g-1-n})^* & \longrightarrow & (\bar{h}\theta^*\mathcal{F}_{2g-2-n})^* & \longrightarrow & \mathcal{O}_x \end{array}$$

where ι is the identity homomorphism. The bottom line is exact because $(\)^*$ is left exact. The first square commutes by the definition of λ_n , the second by the statement of the duality theorem in §1. If λ_n is an isomorphism, then the diagram implies that λ_{n-1} is an isomorphism. By downward induction, λ_n is an isomorphism for all n . This proves Proposition 4.

THEOREM 1. For all n there is an isomorphism $\lambda_n: \mathcal{E}_n \rightarrow \theta^*\mathcal{F}_{2g-2-n}^*$ and an epimorphism $\mu_n: \theta^*\mathcal{F}_{2g-2-n} \rightarrow \mathcal{E}_n^*$. μ_n is an isomorphism if $n > 2g - 2$.

Proof. Define λ_n by Proposition 5, taking h to be the identity map. There is an isomorphism (dual to λ_n) $\lambda_n^*: \theta^* \mathfrak{F}_{2g-2-n}^{**} \rightarrow \mathfrak{E}_n^*$ which, combined with the epimorphism $\theta^* \mathfrak{F}_{2g-2-n} \rightarrow \theta^* \mathfrak{F}_{2g-2-n}^{**}$, defines an epimorphism $\mu_n: \theta^* \mathfrak{F}_{2g-2-n} \rightarrow \mathfrak{E}_n^*$. The latter is an isomorphism whenever $\theta^* \mathfrak{F}_{2g-2-n} = \theta^* \mathfrak{F}_{2g-2-n}^{**}$.

COROLLARY. *For $r > 2g - 2, s > 2g - 2$, there is an exact sequence*

$$0 \rightarrow \mathfrak{E}_r \rightarrow \mathfrak{M} \rightarrow \theta^* \mathfrak{E}_s^* \rightarrow 0,$$

where \mathfrak{M} is a successive extension of $(r + s - 2g + 2)$ copies of \mathcal{O}_J .

Proof. Combine Proposition 4 and Theorem 1.

3. Varieties associated to Picard sheaves

In this section we shall use a construction due to Grothendieck, referring to the appropriate paragraph of [4] for its detailed properties. A coherent sheaf \mathfrak{E} on a variety J defines a J -variety $P = \mathbf{P}(\mathfrak{E})$ called the projective fibred variety associated to \mathfrak{E} and an invertible sheaf $\mathcal{O}_P(1)$ called the fundamental sheaf on P [4, II, 4.1.1].

LEMMA 2. *If $h: X \rightarrow J$ is a map, then $\mathbf{P}(h^* \mathfrak{E}) = \mathbf{P}(\mathfrak{E}) \times_J X$.*

For the proof, see [4, II, 4.1.3].

LEMMA 3. *An epimorphism $u: \mathfrak{E} \rightarrow \mathfrak{F}$ induces a closed immersion*

$$q: \mathbf{P}(\mathfrak{F}) \rightarrow \mathbf{P}(\mathfrak{E}),$$

and $q^* \mathcal{O}_P(1) = \mathcal{O}_P(1)$.

For the proof, see [4, II, 4.1.2].

LEMMA 4. *If $u: \mathfrak{E} \rightarrow \mathfrak{F}$ is an epimorphism with $\ker u = \mathcal{O}_J$, then the immersion $q: \mathbf{P}(\mathfrak{F}) \rightarrow \mathbf{P}(\mathfrak{E})$ is represented by the sheaf of ideals $\mathcal{O}_P(-1)$ dual to $\mathcal{O}_P(1)$.*

Proof. Let $\mathbf{S}(\mathfrak{E})$ be the symmetric \mathcal{O}_J -algebra of \mathfrak{E} [4, II, 1.7.1]. $\mathbf{S}(\mathfrak{E})$ is graded; we denote by $\mathbf{S}_k(\mathfrak{E})$ the set of linear combinations of k elements, and write $\mathbf{S}(\mathfrak{E})(n)$ for the graded \mathcal{O}_J -algebra \mathbf{S} with $\mathbf{S}_k = \mathbf{S}_{n+k}(E)$. An exact sequence

$$0 \rightarrow \mathcal{O}_J \rightarrow \mathfrak{E} \rightarrow \mathfrak{F} \rightarrow 0$$

defines an exact sequence

$$0 \rightarrow \mathbf{S}' \rightarrow \mathbf{S}(\mathfrak{E}) \rightarrow \mathbf{S}(\mathfrak{F}) \rightarrow 0$$

in which $\mathbf{S}'_n = \mathbf{S}_{n-1}(\mathfrak{E})$, so that $\mathbf{S}' = \mathbf{S}(\mathfrak{E})(-1)$. Now [4, II, 3.6.2] and the definition of $\mathcal{O}_P(-1)$ [4, II, 2.5.10] give the required result.

We write C_n for $\mathbf{P}(\theta^* \mathfrak{F}_{2g-2-n})$, $\pi_n: C_n \rightarrow J$ for the projection map, $\mathcal{O}_n(1)$ for the fundamental sheaf on C_n , and $\mathcal{O}_n(-1)$ for its dual. By Lemma 3, there

are closed immersions $q_n:C_{n-1} \rightarrow C_n$ such that $q_n^*\mathcal{O}_n(1) = \mathcal{O}_{n-1}(1)$ and $\pi_n q_n = \pi_{n-1}$.

PROPOSITION 6. *For $n < 0$, C_n is empty; for $n \geq 0$, $\dim C_n = n$; for $n > 2g - 2$, C_n is a projective fibre bundle over J with fibre P_{n-g} .*

Proof. C_n is empty for $n < 0$ and a projective fibre bundle for $n > 2g - 2$ by Proposition 3. The example which follows Proposition 4 shows that C_0 consists of a single point, mapped by π_0 onto x_0 . Therefore, for $n = 0$ and $n > 2g - 2$, C_n is irreducible of dimension n .

Now consider the sequence

$$(3) \quad h^*\mathcal{O}_J \xrightarrow{\phi} h^*\theta^*\mathcal{F}_{2g-2-n} \rightarrow h^*\theta^*\mathcal{F}_{2g-1-n} \rightarrow 0,$$

where $h:X \rightarrow J$ is the inclusion map of a subvariety of J . If $\dim C_n > n$, let X be the support of an irreducible component of C_n of maximum dimension. If ϕ is a monomorphism, $\dim C_{n-1} > n - 1$. If ϕ is not a monomorphism, $\dim C_{n-1} > n$. Either way we may continue the argument to prove ultimately that $\dim C_0 > 0$, which is a contradiction. Therefore $\dim C_n \leq n$. On the other hand, $\dim C_{2g-1} = 2g - 1$, and so the same argument shows that every component of C_n is of dimension at least n . We conclude that $\dim C_n = n$.

COROLLARY 1. *If $X = \text{supp } \theta^*\mathcal{F}_{2g-2-n}$, then ϕ is a monomorphism.*

Proof. Otherwise, C_{n-1} would have a component of dimension n .

COROLLARY 2. *If $n < g$, $\mathcal{E}_n = 0$.*

Proof. Otherwise, since \mathcal{E}_n is torsion-free, Theorem 1 would imply

$$\text{supp } \theta^*\mathcal{F}_{2g-2-n} = J \quad \text{and} \quad \dim C_n \geq g.$$

THEOREM 2. *The closed immersion $q_n:C_{n-1} \rightarrow C_n$ is associated to the sheaf of ideals $\mathcal{O}_n(-1)$. If $\xi \in A^1(C_n)$ is the element of the ring of rational equivalence of C_n which represents C_{n-1} , then $\xi^r \in A^r(C_n)$ represents the subvariety C_{n-r} .*

Proof. Corollary 1 of Proposition 6 shows that the hypotheses of Lemma 4 are satisfied, and therefore that $q_n:C_{n-1} \rightarrow C_n$ is associated to the sheaf of ideals $\mathcal{O}_n(-1)$. Since $q_n^*\mathcal{O}_n(-1) = \mathcal{O}_{n-1}(-1)$, the element $q_n^*\xi \in A^1(C_{n-1})$ represents the subvariety C_{n-2} of C_{n-1} . The theorem now follows by induction.

4. Chern classes

Following Mattuck, [6], write

$$W_n = \pi_n(C_n), \quad U_n = \theta(W_n)$$

for $0 \leq n \leq g$, and denote the classes in the ring of rational equivalence of J which correspond to U_{g-i} , W_{g-i} by u_i , w_i respectively. For $n > 2g - 2$,

C_n is the dual projective bundle of $\theta^*\mathcal{F}_{2g-2-n}$. The formal properties of the dual projective bundle [2] imply that, if $\sum c_i$ is the (total) Chern class of $\theta^*\mathcal{F}_{2g-2-n}$, then, by Theorem 2

$$\left(\sum (-1)^i c_i\right) \left(\sum w_i\right) = 1.$$

On the other hand, Theorem 1 implies that, for $n > 2g - 2$,

$$c(E_n) = \sum (-1)^i c_i,$$

and therefore, by the corollary of Theorem 1,

$$c(\theta^*\mathcal{E}_n^*) = c(\mathcal{E}_n)^{-1} = \sum w_i.$$

We conclude that

$$\begin{aligned} c(\mathcal{E}_n) &= \sum (-1)^i u_i \quad \text{for } n > 2g - 2, \\ c(\mathcal{F}_n) &= \sum w_i \quad \text{for } n < g, \end{aligned}$$

and that there is in $A(J)$ a relation

$$\left(\sum (-1)^i u_i\right) \left(\sum w_i\right) = 1.$$

This relation and the above values of the Chern classes are due to Mattuck, who also conjectured the corollary of Theorem 1 [6].

5. Further properties of the Picard sheaves

We return to the situation of Proposition 5, in which a map $h: X \rightarrow J$ defined a diagram of maps

$$\begin{array}{ccc} X \times C & \xrightarrow{\bar{h}} & J \times C \\ \downarrow \bar{p} & & \downarrow p \\ X & \xrightarrow{h} & J \end{array}$$

and sheaves $\bar{h}\mathcal{E}_n = \bar{p}_0(\bar{h}^*\mathcal{L} \otimes q^*\mathcal{F}_n)$, $\bar{h}\theta^*\mathcal{F}_n = \bar{p}_1(\bar{h}^*\theta^*\mathcal{L} \otimes q^*\mathcal{F}_n)$. Although in general the sheaves $\bar{h}\mathcal{E}_n$ and $h^*\mathcal{E}_n$ are unequal, there is an isomorphism

$$u_n: h^*\theta^*\mathcal{F}_n \rightarrow \bar{h}\theta^*\mathcal{F}_n$$

for all n . To construct u_n we resort to the device used in [4, II, 1.5.2]. Namely, consider the canonical homomorphism [4, 0, 4.4.3]

$$\rho: (\theta^*\mathcal{L} \otimes q^*\mathcal{F}_n) \rightarrow \bar{h}_0(\bar{h}^*\theta^*\mathcal{L} \otimes q^*\mathcal{F}_n).$$

Applying h^*p_1 gives a homomorphism

$$h^*p_1 \rho: h^*\theta^*\mathcal{F}_n \rightarrow h^*p_1 \bar{h}_0(\bar{h}^*\theta^*\mathcal{L} \otimes q^*\mathcal{F}_n) = h^*h_0 \bar{h}\theta^*\mathcal{F}_n.$$

Finally, composing $h^*p_1 \rho$ with the canonical homomorphism [4, 0, 4.4.3]

$$\sigma: h^*h_0 \bar{h}\theta^*\mathcal{F}_n \rightarrow \bar{h}\theta^*\mathcal{F}_n,$$

we obtain a homomorphism

$$u_n: h^*\theta^*\mathcal{F}_n \rightarrow \bar{h}\theta^*\mathcal{F}_n$$

which is an isomorphism in the locally free case ($n < 0$).

PROPOSITION 7. For all n , u_n is an isomorphism.

Proof. Just as in Proposition 5, consider the diagram

$$\begin{array}{ccccccc}
 \mathcal{O}_x & \rightarrow & h^*\theta^*\mathcal{F}_{n-1} & \rightarrow & h^*\theta^*\mathcal{F}_n & \rightarrow & 0 \\
 \downarrow \iota & & \downarrow u_{n-1} & & \downarrow u_n & & \\
 \mathcal{O}_x & \rightarrow & \bar{h}\theta^*\mathcal{F}_{n-1} & \rightarrow & \bar{h}\theta^*\mathcal{F}_n & \rightarrow & 0
 \end{array}$$

where ι is the identity homomorphism. The top line is exact because $h^*(\)$ is right exact, and the squares commute by the definition of u_n . By upward induction, u_n is an isomorphism for all n .

Several properties of the fibred varieties associated to Picard sheaves are consequences of Propositions 5 and 7. Consider the fibred variety

$$C_n = \mathbf{P}(\theta^*\mathcal{F}_{2g-2-n}).$$

If $\pi_n:C_n \rightarrow J$ is the projection map, and $h:X \rightarrow J$ is the injection map of a subvariety of J , Proposition 7 and Lemma 2 imply

PROPOSITION 8. $\pi_n^{-1}(X) = \mathbf{P}(\bar{h}\theta^*\mathcal{F}_{2g-2-n})$, and the closed immersion $\pi_{n-1}^{-1}(X) \rightarrow \pi_n^{-1}(X)$ is induced by the epimorphism $\bar{h}\theta^*\mathcal{F}_{2g-2-n} \rightarrow \bar{h}\theta^*\mathcal{F}_{2g-1-n}$. In particular, if $x \in J$, $\pi_n^{-1}(x)$ is isomorphic to the projective space associated to $H^0(C, j_x^*\mathcal{L} \otimes \mathcal{I}_n)$; $\pi_{n-1}^{-1}(x)$ to the subspace corresponding to sections zero at c .

It is now possible to give partial answers to the two classical questions: For which X is $\pi_n^{-1}(X)$ a projective fibre bundle? For which X is $\pi_n^{-1}(X)$ obtained from X by dilatations?

Write $d_n(X) = \inf_{x \in X} \dim \pi_n^{-1}(x)$, and define $U \subset X$ by

$$U = \{x \in X; d_n(x) = d_n(X)\}.$$

Proposition 7 and the methods of §2 imply that U is open in X and $\bar{h}\theta^*\mathcal{F}_{2g-2-n}$ is locally free over U . Therefore, by Proposition 8, $\pi_n^{-1}(U)$ is a projective fibre bundle over U .

Let X be an irreducible subvariety of J for which $\bar{h}\theta^*\mathcal{F}_{2g-2-n}$ is torsion-free. Then $\pi_n^{-1}(X)$ is irreducible [4, II, 3.1.14]. If in addition $\bar{h}\theta^*\mathcal{F}_{2g-2-n}$ has rank one, it is an \mathcal{O}_X -submodule of $\mathcal{R}(X)$, the sheaf of rational functions on X [4, I, 7.4.3]. Then $\pi_n^{-1}(X)$ is the variety obtained by blowing up X along the sheaf $\bar{h}\theta^*\mathcal{F}_{2g-2-n}$ ([4, II, 8.1.4] and [4, III, 2.3.8]).

Consider the fibred variety $B_n = \mathbf{P}(\mathcal{E}_n)$. Again let $\pi_n:B_n \rightarrow J$ be the projection map, and $\mathcal{O}_n(1)$ the fundamental sheaf on B_n . The definition of \mathcal{E}_n implies that there is a canonical homomorphism $\sigma:p^*\mathcal{E}_n \rightarrow \mathcal{L} \otimes q^*\mathcal{I}_n$. If σ is an epimorphism, there is a map $r:J \times C \rightarrow B_n$ such that

$$r^*\mathcal{O}_n(1) = \mathcal{L} \otimes q^*\mathcal{I}_n$$

[4, II, 4.2.3]. Then, for each $x \in J$, the map $rj_x:C \rightarrow \pi_n^{-1}(x)$ defines $j_x^*\mathcal{L} \otimes \mathcal{I}_n$ as a projectively induced sheaf. Standard arguments show that, for

$n > 2g - 1$, σ is always an epimorphism, and that for $n > 2g$ the map rj_x is always an embedding (in these cases $\pi_n^{-1}(x) = \mathbf{P}(H^0(C, j_x^* \mathcal{L} \otimes \mathcal{I}_n))$ by Proposition 2).

THEOREM 3. *For $n > 2g - 2$, B_n is a projective fibre bundle over J , $\pi_n^{-1}(\theta(x))$ being the projective space associated to the vector space $H^1(C, j_x^* \mathcal{L} \otimes \mathcal{I}_{2g-2-n})$. For $n > 2g - 1$, there is a map $r: J \times C \rightarrow B_n$ such that $r^* \mathcal{O}_n(1) = \mathcal{L} \otimes q^* \mathcal{I}_n$. For $n > 2g$, r is an embedding.*

Theorem 3 has two (actually related) applications to variation of algebraic structure which will be studied in detail elsewhere: (i) invariants of projective embeddings of C , (ii) reducible vector bundles with base C . In each case, B_n is used in a natural manner to construct an algebraic parameter variety.

6. Symmetric products

This section establishes the relation between Picard sheaves and symmetric products, under the assumption of §1: that C is a complete nonsingular curve of genus g with a fixed base point $c \in C$. I wish to thank the referee for pointing out a mistake in the original version of this section.

Let $C(n)$ be the n -fold symmetric product of C . A point of $C(n)$ is an (un-ordered) set $y_1 + \dots + y_n$ of points $y_i \in C$. There are maps

$$f: C(n - 1) \times C \rightarrow C(n)$$

defined by $f(y_1 + \dots + y_{n-1}, y_n) = y_1 + \dots + y_n$ and

$$f': C(n - 1) \times C \rightarrow C(n) \times C$$

defined by $f'(y_1 + \dots + y_{n-1}, y_n) = (y_1 + \dots + y_n, y_n)$. Now define:

$$X_1 = \text{image of } C(n - 1) \times c \text{ under } f,$$

$$X'_1 = \text{image of } C(n - 1) \times C \text{ under } f'.$$

These are partly the notations of [6, Part II]. Both subvarieties are of codimension one, and each therefore defines a divisor and a torsion-free sheaf of rank one.

The map $h: C(n) \rightarrow J$, defined by

$$h(y_1 + \dots + y_n) = y_1 + \dots + y_n - nc \in \text{Pic}(C),$$

gives a diagram of maps

$$\begin{array}{ccc} C(n) \times C & \xrightarrow{\bar{h}} & J \times C \\ \downarrow \bar{p} & & \downarrow p \\ C(n) & \xrightarrow{h} & J. \end{array}$$

Let $\mathfrak{K}_n, \mathfrak{K}'_n$ be the sheaves on $C(n), C(n) \times C$ defined by the divisors X_1, X'_1 , and let $\mathcal{L} \otimes q^* \mathcal{I}_n$ be the sheaf on $J \times C$ defined in §2.

PROPOSITION 9. $\mathfrak{N}'_n = \bar{p}^*\mathfrak{N}_n \otimes \bar{h}^*(\mathcal{L} \otimes q^*\mathcal{J}_n)$.

Proof. Consider the sheaf $\mathfrak{N}'_n \otimes \bar{p}^*\mathfrak{N}_n^* \otimes \bar{h}^*q^*\mathcal{J}_n^*$. It is defined by a divisor with restriction $y_1 + \dots + y_n - nc$ to $\bar{p}^{-1}(y_1 + \dots + y_n)$ and restriction zero to $\bar{h}^{-1}(x, y)$ for $x \in J, y \in C$. Therefore

$$\mathfrak{N}'_n \otimes \bar{p}^*\mathfrak{N}_n^* \otimes \bar{h}^*q^*\mathcal{J}_n = \bar{h}^*\mathcal{L},$$

where \mathcal{L} is a sheaf on $J \times C$ with the properties claimed in §2.

PROPOSITION 10. *There is an epimorphism $h^*\theta^*\mathfrak{F}_{2g-2-n} \rightarrow \mathfrak{N}_n$.*

Proof. It is sufficient to construct a section of $\mathfrak{N}_n \otimes (h^*\theta^*\mathfrak{F}_{2g-2-n})^*$ which never determines the germ of a zero-valued function. Consider the subvariety X'_1 of $C(n) \times C$. Its local equations define, by Proposition 9, a section of the sheaf $\bar{p}^*\mathfrak{N}_n \otimes \bar{h}^*(\mathcal{L} \otimes q^*\mathcal{J}_n)$. This section determines germs of zero-valued functions only along X'_1 . The corresponding section of the sheaf

$$\bar{p}_0(\bar{p}^*\mathfrak{N}_n \otimes \bar{h}^*(\mathcal{L} \otimes q^*\mathcal{J}_n)) = \mathfrak{N}_n \otimes \bar{h}\mathcal{E}_n$$

determines the germ of a zero-valued function at $z \in C(n)$ if and only if X'_1 contains $\bar{p}^{-1}(z)$. (Here we are using [4, 0, 5.4.10].) By Propositions 5 and 7, $\mathfrak{N}_n \otimes \bar{h}\mathcal{E}_n = \mathfrak{N}_n \otimes (h^*\theta^*\mathfrak{F}_{2g-2-n})^*$, which completes the proof.

PROPOSITION 11. *There is a map $r:C(n) \rightarrow C_n$ such that $\pi_n r = h$ and $r^*\mathcal{O}_n(1) = \mathfrak{N}_n$.*

Proof. Recall that $C_n = \mathbf{P}(\theta^*\mathfrak{F}_{2g-2-n})$. According to [4, II, 4.2.3] the result of Proposition 10 is precisely the sufficient condition for the existence of a map r with $\pi_n r = h$ and $r^*\mathcal{O}_n(1) = \mathfrak{N}_n$. It is easy to trace through Propositions 9, 10, 11 the effect of r on a fibre $h^{-1}(x), x \in J$, by using the diagram

$$\begin{array}{ccc} h^{-1}(x) \times C & \rightarrow & x \times C \\ \downarrow & & \downarrow \\ h^{-1}(x) & \rightarrow & x \end{array}$$

and Propositions 7 and 8. Let P be the projective space associated to the space $H^0(C, j_x^*\mathcal{L} \otimes \mathcal{J}_n)$.

One obtains the following description of r . Identify P with $\pi_n^{-1}(x)$ by the isomorphism of Proposition 8, and let $y_1 + \dots + y_n$ be a point of $h^{-1}(x)$. Then $r(y_1 + \dots + y_n)$ is defined by the section $s \in H^0(C, j_x^*\mathcal{L} \otimes \mathcal{J}_n)$ with zeros precisely at y_1, \dots, y_n (s is determined uniquely up to scalar multiples).

It follows at once that r is surjective, and that (again using Proposition 8) r maps the subvariety X_1 of $C(n)$ on to C_{n-1} . Therefore r commutes with the closed immersions $C(n-1) \rightarrow C(n), C_{n-1} \rightarrow C_n$.

We have not proved that C_n is nonsingular or even irreducible. This is however clear for $n > 2g - 2$, when C_n and $C(n)$ are both projective fibre bundles over J [7]. Therefore, $r_n = r|C(n)$ is an isomorphism for

$n > 2g - 2$. Now suppose that r_n is an isomorphism. Then r_{n-1} maps $C(n-1)$ onto C_{n-1} , and is a surjective closed immersion. To prove r_{n-1} an isomorphism, it remains to show that it induces an isomorphism of local rings.

Proposition 11 implies that $r_n^* \mathcal{O}_n(-1) = \mathfrak{N}_n^*$ [4, 0, 5.4.5]. Since r_n induces an isomorphism of local rings, it induces an isomorphism $\mathfrak{N}_n^* \rightarrow \mathcal{O}_n(-1)$. But \mathfrak{N}_n^* , $\mathcal{O}_n(-1)$ are the sheaves of ideals defining $C(n-1)$, C_{n-1} , and therefore $r_n|_{C(n-1)} = r_{n-1}: C(n-1) \rightarrow C_{n-1}$ is an isomorphism [4, I, 4.1.3]. We have proved, by downward induction,

THEOREM 4. *The fibred variety C_n is isomorphic to the n -fold symmetric product $C(n)$.*

REFERENCES

1. H. GRAUERT, *Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques, Paris, No. 5, 1960.
2. A. GROTHENDIECK, *La théorie des classes de Chern*, Bull. Soc. Math. France, vol. 86 (1958), pp. 137-154.
3. ———, *Géométrie formelle et géométrie algébrique*, Séminaire Bourbaki, vol. 11 (1958-1959), exposé 182.
4. A. GROTHENDIECK AND J. DIEUDONNÉ, *Éléments de géométrie algébrique*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques, Paris, (Chap. I) No. 4, 1960; (Chap. II) No. 8, 1961; (Chap. III) No. 11, 1962.
5. S. LANG, *Abelian varieties*, New York, Interscience Publishers, 1959.
6. A. MATFUCK, *Symmetric products and Jacobians*, Amer. J. Math., vol. 83 (1961), pp. 189-206.
7. ———, *Picard bundles*, Illinois J. Math., vol. 5 (1961), pp. 550-564.
8. J-P. SERRE, *Groupes algébriques et corps de classes*, Paris, Hermann, 1959.

INSTITUTE FOR ADVANCED STUDY
 PRINCETON, NEW JERSEY
 UNIVERSITY OF LIVERPOOL
 LIVERPOOL, ENGLAND