SOME REMARKS ON ČERNIKOV p-GROUPS

BY

NORMAN BLACKBURN

Following Kurosh [9, vol. II, p. 230], we say that a p-group G is a *Černikov p-group* if G is locally finite and satisfies the minimal condition for subgroups. A subgroup or factor group of a Černikov p-group is a Černikov p-group, and if G is a group which has a normal subgroup N such that N and G/N are Černikov p-groups, then G is a Černikov p-group. An Abelian p-group is a Černikov p-group if and only if it is the direct product of a finite number of groups which are either cyclic or of type p^∞; this is equivalent to the condition that the group has a finite number of elements of order p.

This class of groups has been the subject of considerable investigation over the last twenty years, and the primary aim of this note is to give some improvements on the known results. Thus in §3 we give characterizations of normal Černikov p-subgroups of p-groups, and in §4 we generalize the theorem of Černikov on locally finite p-groups in which the Abelian subgroups satisfy the minimal condition. In §5 we continue a previous discussion of the simplest non-nilpotent Černikov p-groups. The proofs require the applications of known lemmas, which are described in §1 and §2. Our secondary aim is to present the theory of Černikov p-groups in a concise form, which is desirable since the existing literature on the subject is widely scattered. The proofs of the known results are therefore sketched.

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1. The hypercentre

For any group G we denote by $\bar{\zeta}(G)$ the *hypercentre* of G, defined to be the intersection of all normal subgroups N of G for which the centre of G/N is the unit subgroup. The *upper central series* of G is denoted by

$$1 = \zeta_0(G) \leqq \zeta_1(G) \leqq \zeta_2(G) \leqq \cdots ;$$

this is defined inductively by the property that for $n > 0$, $\zeta_n(G)/\zeta_{n-1}(G)$ is the centre of $G/\zeta_{n-1}(G)$. We also write

$$\zeta_\omega(G) = \bigcup_{n=0}^{\infty} \zeta_n(G).$$

It is easy to see that the centre of $G/\bar{\zeta}(G)$ is the unit group, and so if N is a normal subgroup of G contained in $\bar{\zeta}(G)$, then

$$\bar{\zeta}(G/N) = \bar{\zeta}(G)/N.$$

Clearly $\zeta_1(G) \leqq \bar{\zeta}(G)$, and so by induction on n, $\zeta_n(G) \leqq \bar{\zeta}(G)$. Hence

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$\zeta_\omega(G) \leqq \bar{\zeta}(G)$. $\bar{\zeta}(G)$ is in fact the terminal member of the transfinitely continued upper central series of G.

The following remarks on the hypercentre are well known.

LEMMA 1.1. *Let G be a group.*

(a) *Let M, N be normal subgroups of G such that $M < N \leqq \bar{\zeta}(G)$. Then there is an element x lying in N but not in M such that $xM \in \zeta_1(G/M)$ and $\{x, M\}$ is normal in G.*

(b) *If $H \leqq G$, then $\bar{\zeta}(H) \geqq \bar{\zeta}(G) \cap H$, and if $H \geqq \bar{\zeta}(G)$, then*

$$\bar{\zeta}(H)/\bar{\zeta}(G) = \bar{\zeta}(H/\bar{\zeta}(G)).$$

(c) *If N is a normal subgroup of G contained in $\bar{\zeta}(G)$, then either*

$$N \leqq \zeta_r(G) \quad \text{or} \quad N \cap \zeta_{r+1}(G) > N \cap \zeta_r(G) \quad \text{for each } r = 0, 1, 2, \cdots.$$

(d) *Let N be a normal subgroup of G, let A be a maximal normal Abelian subgroup of G contained in N, and let C be the centraliser of A in G. Then $C \cap \bar{\zeta}(G) \cap N = A \cap \bar{\zeta}(G)$.*

(e) *If N is a normal subgroup of G contained in $\bar{\zeta}(G)$, and if $(N:1) = p^r$ (p prime), then $N \leqq \zeta_r(G)$.*

All these statements except (b) are proved by virtually the same arguments as in the case when G is a finite p-group (see e.g. [12], Chap. IV, §3), the inductions being replaced by applications of Zorn's Lemma. The proof of (b) is most simply accomplished by using the subsidiary result that an element x of a group G lies in $\bar{\zeta}(G)$ if and only if given any infinite sequence $x_1, x_2, \cdots$ of elements of G there exists an integer n such that the simple commutator

$$[x, x_1, x_2, \cdots, x_n]$$

is 1; the proof of this is the same as in the case when $\bar{\zeta}(G) = G$ (see [9], vol. II, p. 219).

A group G for which $\bar{\zeta}(G) = G$ is called a *ZA-group*. Thus a group G is a ZA-group if and only if the centre of each nontrivial homomorphic image of G is nontrivial. We note that such a group G is equal to its derived group G' only if $G = 1$, for if $G = G'$, then

$$[G, \zeta_2(G)] = [G', \zeta_2(G)] = 1,$$

and so $\zeta_2(G) = \zeta_1(G)$, G is Abelian, and $G' = 1$. It was proved by Mal'cev that a ZA-group is locally nilpotent (see [9], vol. II, p. 223). The converse is not true, for there exist locally finite p-groups $G \neq 1$ for which $\zeta_1(G) = 1$. Several examples of such groups have been given; for instance Baumslag [4] has observed that if Γ is an infinite group and G is a nontrivial group, then the wreath product K of Γ and G has trivial centre, whereas of course if Γ and G are locally finite p-groups, so is K. In the same paper the following result is proved.

LEMMA 1.2 (Baumslag). *If the p-group G has a normal nilpotent subgroup of finite index and finite exponent, then G is nilpotent.*

The following remark is of a similar nature.

LEMMA 1.3. *Let G be a locally nilpotent group generated by a normal subgroup N and a finite number of elements $x_1, x_2, \cdots, x_r$. Then $\bar{\zeta}(G) \geqq \bar{\zeta}(N)$. In particular if N is a ZA-group, so is G.*

Let $K = \bar{\zeta}(G) \cap \bar{\zeta}(N)$. If $K < \bar{\zeta}(N)$, then the centre L/K of N/K is not the unit group, and so $L > K$. L is a normal subgroup of G contained in $\bar{\zeta}(N)$. If a is an element of L which does not lie in K, let H be the group generated by $a, x_1, x_2, \cdots, x_r$ and K. Since G is locally nilpotent, H/K is nilpotent. But $L \cap H/K$ is a normal subgroup of H/K and is not the unit group since $a \in L \cap H$, $a \notin K$; hence by Lemma 1.1(a) there is an element b lying in $L \cap H$ but not in K such that $bK \in \zeta_1(H/K)$. Thus bK commutes with each $x_j K$ $(j = 1, 2, \cdots, r)$. But since $b \in L$, bK commutes with each element of N/K. Hence bK commutes with all generators of G/K, and $bK \in \zeta_1(G/K)$. Since $b \notin K$ but $b \in L \leqq \bar{\zeta}(N)$, it follows that if

$$\bar{\zeta}(N)/K \cap \zeta_1(G/K) = M/K,$$

then $M > K$. But since $K \leqq \bar{\zeta}(G)$, we have $M \leqq \bar{\zeta}(G)$: thus

$$M \leqq \bar{\zeta}(G) \cap \bar{\zeta}(N) = K,$$

which is a contradiction. Hence $K = \bar{\zeta}(N)$ and $\bar{\zeta}(N) \leqq \bar{\zeta}(G)$, as required. If N is a ZA-group, we obtain $\bar{\zeta}(G) \geqq N$. But $G/\bar{\zeta}(G)$ has trivial centre, and G/N is nilpotent: hence $G/\bar{\zeta}(G)$ is the unit group, and G is a ZA-group.

2. Lemmas on p-groups

In this section we give the lemmas which are fundamental for the theory of Černikov p-groups.

LEMMA 2.1 (McLain). *A minimal normal subgroup of a locally nilpotent group G is contained in the centre of G and is of prime order.*

The proof is given in [10].

If G is a p-group, we denote by $\lambda_r(G)$ the subgroup of G generated by all elements of order at most p^r $(r = 0, 1, 2, \cdots)$. $\lambda_r(G)$ is a characteristic subgroup of G, and if G is Abelian, $\lambda_r(G)$ consists precisely of the elements of G of order at most p^r. The following lemma is well known (cf. [3], p. 525).

LEMMA 2.2. *Let G be a p-group, and let A be a normal divisible Abelian subgroup of G. Then A and $\lambda_2(A)$ have the same centralisers in G.*

For if this is false, there is an element x which lies in the centraliser K of $\lambda_2(A)$, such that x^p lies in the centraliser C of A but $x \notin C$, since C, K are normal subgroups of G. Now if t is any element of A, then $[t, x] \in A$, and so

$$[t^p, x] = t^{-p}(t^p)^x = t^{-p}(t^x)^p = t^{-p}(t[t, x])^p = [t, x]^p.$$

Since $x \notin C$, we can choose an element y of A of the smallest possible order such that $[y, x] \neq 1$. Then $[y, x]^p = [y^p, x] = 1$. Since A is divisible, $y = z^p$ for some $z \in A$, and $[z, x]^p = [z^p, x] = [y, x]$. Hence $[z, x] \in \lambda_2(A)$ and $[z, x, x] = 1$. Thus

$$[x^p, z] = x^{-p}(x^p)^z = x^{-p}(x^z)^p = x^{-p}(x[x, z])^p = [x, z]^p = [x, y] \neq 1.$$

But $x^p \in C$, and so $[x^p, z] = 1$, which gives a contradiction.

COROLLARY. *Suppose that the p-group G has a normal Abelian subgroup A which is a Černikov p-group. Then the centraliser C of A has finite index in G.*

A is the direct product of a divisible subgroup D and a finite group F; moreover D is a characteristic subgroup of A and is therefore normal in G. By Lemma 2.2 the centraliser H of D is the same as that of $\lambda_2(D)$. Also $\lambda_2(D)$ is finite since A is a Černikov p-group. But G/H is isomorphic to a group of automorphisms of $\lambda_2(D)$ and is therefore finite. Now H/C is isomorphic to a group of automorphisms of A which leave invariant the elements of D. Such an automorphism α is completely determined by its effect on F and therefore by its effect on $\lambda_r(A)$, where $(F:1) = p^r$. Since $\lambda_r(A)$ is finite, it follows that H/C is finite. Hence G/C is finite.

It is easy to deduce the well-known fact that *a p-group G is a Černikov p-group if and only if G has a divisible Abelian normal subgroup D of finite index and D is the direct product of a finite number of groups of type p^∞*. For a group with this structure clearly satisfies the minimal condition for subgroups and is locally finite. Suppose conversely that G is a Černikov p-group. By Lemma 2.1, G is a ZA-group. Hence if A is a maximal normal Abelian subgroup of G, A is its own centraliser by Lemma 1.1(d). It follows from Lemma 2.2, Corollary that G/A is finite. Also the maximal divisible subgroup D of A is of finite index and is normal in G. Since D satisfies the minimal condition, D is the direct product of a finite number k of groups of type p^∞. We observe that D contains any divisible subgroup H of G, for HD/D, being both finite and divisible, must be the unit group. D is therefore a characteristic subgroup of G. We write $k = \delta(G)$; this is an invariant of G.

The following lemma was proved in [6].

LEMMA 2.3 (Černikov). *Let G be a p-group which has an Abelian normal subgroup A of finite index. If $\zeta_1(G)$ is a Černikov p-group, so is G.*

It is necessary to show that A satisfies the minimal condition for subgroups. If this is not so, there is a maximal normal subgroup K of G such that $A \leqq K \leqq G$ and $Z = \zeta_1(K)$ does not satisfy the minimal condition for subgroups. By hypothesis $K \neq G$, and so there is a normal subgroup H of G such that $H > K$ and H/K is of order p. Write $H = \{s, K\}$, $E = \lambda_1(Z)$,

$J = \{s, E\}$; note that E is an infinite normal subgroup of G. Hence by Lemma 1.2, J is nilpotent. Thus there is a smallest integer $n > 0$ such that $E \cap \zeta_n(J)$ is infinite. Let F be the finite group $E \cap \zeta_{n-1}(J)$, and let $t_1, t_2, \cdots$ be an infinite set of elements of $E \cap \zeta_n(J)$, no two of which lie in the same coset of F. Then $[t_r, s] \in F$, and so there exist indices i, j such that $[t_i, s] = [t_j, s]$. Hence $t_i^{-1}t_j$ commutes with s, and so $t_i^{-1}t_j \in E \cap \zeta_1(J)$. Since $t_i^{-1}t_j \notin F$, it follows that $n = 1$, and $E \cap \zeta_1(J)$ is infinite. But $E \cap \zeta_1(J) \leqq E \leqq \zeta_1(K)$ and $JK = H$; thus $E \cap \zeta_1(J) \leqq \zeta_1(H)$, and $\lambda_1\{\zeta_1(H)\}$ is infinite. This contradicts the definition of K.

For any p-group G we denote by $\rho_r(G)$ the group generated by all p^r-th powers of elements of G. This is a characteristic subgroup of G. The following useful lemma is proved in [7].

LEMMA 2.4 (Černikov). *Let G be a nilpotent p-group of class c.*
(a) *An element x of G of order p^n commutes with all elements of $\rho_{(c-1)n}(G)$.*
(b) $\bigcap_{r=1}^{\infty} \rho_r(G) \leqq \zeta_1(G)$.
(c) *Every divisible subgroup of G is contained in $\zeta_1(G)$.*

We prove (a) by induction on c. Thus we may assume that $c > 1$ and that if $y \in G$ and $t = y^{p^{(c-2)n}}$, then $[x, t] \in \zeta_1(G)$. Hence for any integers α, β,

$$[x^\alpha, t^\beta] = [x, t]^{\alpha\beta}.$$

In particular

$$[x, t^{p^n}] = [x, t]^{p^n} = [x^{p^n}, t] = 1,$$

as required. (b) and (c) are immediate consequences of (a).

The following is an easy consequence of this.

THEOREM 2.5. *If G is a Černikov p-group, then $\zeta_n(G)/\zeta_1(G)$ is finite* $(n = 1, 2, \cdots)$.

Let D be the maximal divisible subgroup of G, let $D_n = D \cap \zeta_n(G)$, and let D^* be the maximal divisible subgroup of D_n. Since $D^* \leqq \zeta_n(G)$, the group H generated by D^* and any element a of G is nilpotent. Hence by Lemma 2.4, $D^* \leqq \zeta_1(H)$. Thus $[D^*, a] = 1$ and $D^* \leqq \zeta_1(G)$. Hence

$$D^* \leqq D \cap \zeta_1(G) = D_1,$$

and D_n/D_1 is finite. And $\zeta_n(G)/D_n$, being isomorphic to $\zeta_n(G)D/D$, is finite. Thus $\zeta_n(G)/D_1$ and $\zeta_n(G)/\zeta_1(G)$ are finite.

COROLLARY. *If G is a nilpotent Černikov p-group, G has a finite characteristic subgroup K such that G/K is a divisible Abelian group.*

This follows almost at once from the Schur-Baer Theorem ([1], p. 396), but can also be simply deduced by induction on the class c of G. It is clear for $c = 1$. If $c > 1$, let $M = \zeta_2(G)$. By Theorem 2.5, $M/\zeta_1(M)$ is finite. Suppose that M is generated by $\zeta_1(M)$ and the elements $x_1, x_2, \cdots, x_r$. The finitely many commutators $[x_i, x_j]$ will then lie in a finite characteristic subgroup L of $\zeta_1(M)$. M/L is Abelian and therefore has a characteristic subgroup N/L of finite order such that M/N is divisible. By Lemma 2.4(c) $M/N \leqq \zeta_1(G/N)$, and so $[\zeta_2(G), G] \leqq N$. Hence G/N has class less than c, and the result follows from the inductive hypothesis.

A number of other properties of Černikov p-groups are special cases of results in [2]. For instance if G is a Černikov p-group, $\zeta_\omega(G)$ consists precisely of those elements of G which have only a finite number of conjugates, and $\zeta_\omega(G)$ contains its centraliser in G.

The well-known fact that *a locally finite p-group is a Černikov p-group if and only if it satisfies the minimal condition for normal subgroups* is proved in [10], and is easily deduced from more general results in [2]. It also follows immediately from Lemma 2.1 and the following theorem.

THEOREM 2.6. *Let $\mathfrak{A}$ be a p-group of automorphisms of the p-group G. Suppose that G and $\mathfrak{A}$ are ZA-groups and that G satisfies the minimal condition for $\mathfrak{A}$-invariant subgroups. Then G is a Černikov p-group.*

Suppose that this is false. Then there is a minimal $\mathfrak{A}$-invariant subgroup M of G which is not a Černikov p-group. Since G is a ZA-group, $M' < M$, and so M' is a Černikov p-group, on account of the definition of M. Now $\mathfrak{A}$ induces a p-group $\mathfrak{B}$ of automorphisms of M/M'. Also $\mathfrak{B}$ is a ZA-group, and M/M' satisfies the minimal condition for $\mathfrak{B}$-invariant subgroups. Thus $\mathfrak{B} \neq 1$, since otherwise M/M' is a Černikov p-group and hence so is M.

Let $A = M/M'$, and let B be the subgroup of the holomorph of A generated by A and $\mathfrak{B}$. Since $\mathfrak{B} \neq 1$ and $\mathfrak{B}$ is a ZA-group, there is a normal subgroup C/A of B/A of order p. By Lemma 2.3, $\zeta_1(C)$ is not a Černikov p-group. Hence if $K/M' = \zeta_1(C) \cap A$, K/M' is not a Černikov p-group, but since $\zeta_1(C)$ is a normal subgroup of B, K is an invariant subgroup of M. It follows that K is not a Černikov p-group, and so by the definition of M, $K = M$. Hence $A \leqq \zeta_1(C)$. But A is its own centraliser in its holomorph, and so we have a contradiction. Thus G is a Černikov p-group.

3. Normal Černikov p-subgroups

THEOREM 3.1. *Let G be a p-group, let N be a normal subgroup of G contained in $\bar{\zeta}(G)$, and let C be the centraliser of N in G. Then N is a Černikov p-group if and only if G/C and $N \cap \zeta_1(G)$ are Černikov p-groups.*

We deal first with the case when G/C and $N \cap \zeta_1(G)$ are Černikov p-groups. This is a generalization of Lemma 2.3, for by Lemma 1.3 the group of Lemma 2.3 is a ZA-group. We observe that $N/(N \cap C)$, being isomorphic to NC/C, is a Černikov p-group, and it is therefore only necessary to show that the Abelian group $N \cap C$ is a Černikov p-group. Let $A = \lambda_1(N \cap C)$; this is normal in G. Let D/C be the maximal divisible subgroup of the Černikov p-group G/C; D is a normal subgroup of finite index in G. If $A_n = A \cap \zeta_n(G)$, we show by induction on n that $[A_n, D] = 1$. This is clear for $n = 1$, and for $n > 1$ we have $[A_{n-1}, D] = 1$. If $b \in D$, there exist elements $x \in D$, $y \in C$ such that $b = x^p y$, since K/C is divisible. For any element $a \in A_n$, $[x, a]$ lies in A_{n-1} and therefore commutes with x. Hence

$$[x^p, a] = x^{-p}(x^a)^p = x^{-p}(x[x, a])^p = [x, a]^p = 1.$$

But since $a \in N$ and $y \in C$, a and y commute; hence a commutes with $x^p y = b$, and $[A_n, D] = 1$ as required. Thus if $A_\omega = \bigcup_{n=0}^{\infty} A_n$, $[A_\omega, D] = 1$.

Since G/D is finite, there is a finite set of elements $x_1, x_2, \cdots, x_r$ which together with D generate G. Let H be the group generated by $x_1, x_2, \cdots, x_r$ and A_ω. Since $[A_\omega, D] = 1$, $\zeta_1(H) \cap A_\omega \leqq \zeta_1(G) \cap N$; hence $\zeta_1(H) \cap A_\omega$ is finite, since it is a Černikov p-group of exponent p. But H/A_ω is finite; hence $\zeta_1(H)$ is finite. Thus by Lemma 2.3, H is a Černikov p-group, and so A_ω is finite. Hence $A_\omega = A_r$ for some integer r, and so $A \cap \zeta_{r+1}(G) = A \cap \zeta_r(G)$. Thus by Lemma 1.1(c), $A \leqq \zeta_r(G)$, since $A \leqq \bar{\zeta}(G)$. This shows that $A = A_r$ is finite, and since $A = \lambda_1(N \cap C)$, $N \cap C$ is a Černikov p-group, as required.

Now suppose conversely that N is a Černikov p-group. Clearly $N \cap \zeta_1(G)$ is a Černikov p-group. Let D be the maximal divisible Abelian subgroup of N, and let H be the centraliser of D in G. D and H are normal in G, and by Lemma 2.2, H is the centraliser of $\lambda_2(D)$. But $\lambda_2(D)$ is finite, and so G/H is finite. Next let K/D be the centraliser of N/D in H/D. Since N/D is finite, H/K is finite. Now by Hall's Three Subgroup Theorem (see e.g. [12], Chap. II, Theorem 14)

$$[K', N] \leqq [K, N, K] \leqq [D, K] = 1,$$

and so $K' \leqq C$. Hence G/C has an Abelian subgroup K/C of finite index, and it is only necessary to show that K/C has only a finite number of elements of order p. To do this we prove that the number of automorphisms α of N of order p which centralise N/D and D is finite. If N is generated by elements $x_1, x_2, \cdots, x_r$ together with D, α is determined by its effect on each x_j, and $x_j^\alpha = x_j y_j$ $(j = 1, 2, \cdots, r)$, where y_j is an element of D of order p. Hence the number of automorphisms α is at most $p^{r\delta(G)}$. This completes the proof of Theorem 3.1.

We also prove the following generalization of a theorem of Muhammedžan [11].

THEOREM 3.2. *Suppose that the p-group G is a ZA-group and that N is a normal subgroup of G. Each of the following conditions is necessary and sufficient for N to be a Černikov p-group.*

(1) *$N \cap \zeta_n(G)$ is a Černikov p-group for each $n = 1, 2, \cdots$.*

(2) *For each finite elementary Abelian normal subgroup K of G contained in N, $\zeta_1(G/K) \cap N/K$ is a Černikov p-group.*

It is clear that if N is a Černikov p-group, (1) is satisfied. To show that (1) implies (2), we observe that if K is a normal subgroup of G of order p^r

and $K \leqq N$, then by Lemma 1.1(e), $K \leqq \zeta_r(G)$, since G is a ZA-group. Hence

$$\zeta_1(G/K) \cap N/K \leqq (\zeta_{r+1}(G) \cap N)K/K$$
$$\cong \zeta_{r+1}(G) \cap N/\zeta_{r+1}(G) \cap K,$$

and so $\zeta_1(G/K) \cap N/K$ is a Černikov p-group.

It therefore remains to show that if N satisfies (2), then N is a Černikov p-group. Under this assumption let K be a maximal normal Abelian subgroup of G contained in N, let $E = \lambda_1(K)$, and let $E_n = E \cap \zeta_n(G)$ $(n = 0, 1, \cdots)$. The groups E, E_n are normal in G. We shall assume that E is infinite and obtain a contradiction. First we show that E_n is finite by induction on n. This is clear for $n = 0$, and for $n > 0$, E_{n-1} is finite. The condition (2) shows that $\zeta_1(G/E_{n-1}) \cap N/E_{n-1}$ is a Černikov p-group. Since E_n/E_{n-1} is contained in this group and is of exponent p, it follows that E_n/E_{n-1} is finite, as required.

Now let Θ be a set of suffixes, and let $(E_\theta)_{\theta\epsilon\Theta}$ be a chain of infinite normal subgroups of G contained in E. Put $D = \bigcap_{\theta\epsilon\Theta} E_\theta$. Since E_n is finite, E_θ cannot be contained in E_n, and hence cannot be contained in $\zeta_n(G)$. Hence by Lemma 1.1(c), $E_\theta \cap \zeta_{n+1}(G) > E_\theta \cap \zeta_n(G)$, that is, $E_\theta \cap E_{n+1} > E_\theta \cap E_n$. But now we may choose $\mu \epsilon \Theta$ such that $E_\mu \cap E_{n+1}$ is minimal. Since (E_θ) is a chain, it follows that for each $\theta \epsilon \Theta$, $E_\theta \cap E_{n+1} \geqq E_\mu \cap E_{n+1}$. Hence

$$D \cap E_{n+1} = E_\mu \cap E_{n+1} > E_\mu \cap E_n \geqq D \cap E_n .$$

Thus D is infinite.

Hence by Zorn's lemma there is a minimal infinite Abelian normal subgroup A of G contained in E. Let C be the centraliser of A in G. Since A is infinite and E_1 is finite, $C < G$. And since G is a ZA-group, there is a normal subgroup B of G such that $B \geqq C$ and $(B:C) = p$. Then $\zeta_1(B) \cap A$ is a normal subgroup of G properly contained in A, and so by the definition of A, $\zeta_1(B) \cap A$ is finite. By Theorem 3.1, A is therefore a Černikov p-group, and we have a contradiction.

It follows that E is finite and K is a Černikov p-group. By Lemma 1.1(d), K is its own centraliser in N. Hence by Lemma 2.2, Corollary, N/K is finite and N is a Černikov p-group.

4. Maximal Abelian subgroups

In this section we prove the following theorem.

THEOREM 4.1. *Suppose that G is a locally finite p-group, and that there is a maximal Abelian subgroup A of exponent p of G which is finite of order p^r. Then G is a Černikov p-group, and $\delta(G) \leqq p^{r-1}$. This bound is best possible.*

This is a generalization of a theorem of Černikov [7], which asserts that if G is a locally finite p-group and all Abelian subgroups of G are Černikov p-groups, then G is a Černikov p-group. It is of course not sufficient to assume that G is locally finite and one maximal Abelian subgroup is a Černikov p-group, as the wreath product of a group of type p^∞ with a cyclic group of order p shows. Under this hypothesis, however, Černikov has shown [8] that the hypercentre of G is a Černikov p-group. We shall prove this first of all.

THEOREM 4.2 (Černikov). *Suppose that the p-group G is a ZA-group and has a maximal Abelian subgroup A which is a Černikov p-group. Then G is a Černikov p-group.*

Let N be a maximal normal Abelian subgroup of G, and let $H = NA$. Then A is a maximal Abelian subgroup of H, and so $A \geqq \zeta_1(H)$; hence $N \cap \zeta_1(H)$ is a Černikov p-group. Since also H is a ZA-group and H/N is a Černikov p-group, we deduce from Theorem 3.1 that N is a Černikov p-group. By Lemma 1.1(d), N is its own centraliser, and so by Lemma 2.2, Corollary, G/N is finite. Hence G is a Černikov p-group.

To prove Theorem 4.1 let H be a maximal subgroup such that $H \geqq A\bar{\zeta}(G)$ and $H/\bar{\zeta}(G)$ is Abelian of exponent p. By Lemma 1.1(b), H is a ZA-group. If B is a maximal Abelian subgroup of H containing A, then B has only a finite number of elements of order p, since such an element must lie in A. Hence B is a Černikov p-group, and by Theorem 4.2, H is a Černikov p-group. Thus $\bar{H} = H/\bar{\zeta}(G)$ is finite.

Put $\bar{G} = G/\bar{\zeta}(G)$. If $\bar{G} \neq 1$, then $\bar{H} \neq 1$; let $x_1, x_2, \cdots, x_k$ be the elements of $\bar{H}$ other than 1. Since $\zeta_1(\bar{G}) = 1$, there exist elements $y_1, y_2, \cdots, y_k$ such that $[x_i, y_i] \neq 1$. Let $\bar{K}$ be the group generated by $x_1, \cdots, x_k, y_1, \cdots, y_k$. Since G is locally finite, $\bar{K}$ is finite. Also $\bar{K} \neq 1$, and so $\zeta_1(\bar{K})$ contains an element z of order p. By the definition of $\bar{K}$, $z \notin \bar{H}$. But $\{z, \bar{H}\}$ is Abelian of exponent p, which contradicts the definition of H. Hence $G = \bar{\zeta}(G) = H$ is a Černikov p-group.

To show that $\delta(G) \leqq p^{r-1}$, let D be the maximal divisible subgroup of G. We need only apply the following lemma to $A\lambda_1(D)$.

LEMMA 4.3. *Let G be a finite p-group of the form $G = AN$, where N is a normal elementary Abelian p-group and A is a maximal elementary Abelian subgroup. If $(A:1) = p^r$, then $(N:1) \leqq p^{p^{r-1}}$.*

The proof is by induction on r. The result is trivial if $r = 1$ or if $A \leqq \zeta_1(G)$. Otherwise if $x \in \zeta_1(G)$, write $x = yz$, where $y \in A$, $z \in N$. For each element $t \in A$, $x = x^t = yz^t$, and so $z^t = z$. Hence $z \in \zeta_1(G)$, and $\{z, A\}$ is elementary Abelian. Thus $z \in A$ and $x \in A$. Hence $\zeta_1(G) < A$. Let A_1 be a maximal subgroup of A containing $\zeta_1(G)$, and choose an element a of A which does not lie in A_1. The centraliser C_1 of a in N is normal in G. Let $G_1 = C_1 A_1$, and let B_1 be the centraliser of A_1 in G_1. Then since G_1 is contained in the centraliser of a, $B_1 \leqq A$. But $A \cap G_1 \leqq (N \cap A)A_1 \leqq \zeta_1(G)A_1 = A_1$, and so $B_1 = A_1$. Thus A_1 is a maximal elementary Abelian subgroup of G_1, and by the inductive hypothesis $(C_1:1) \leqq p^{p^{r-2}}$.

We now define a series of subgroups $C_1 \leqq C_2 \leqq C_3 \leqq \cdots$ of N inductively, C_i being the set of elements x of N for which $[x, a] \in C_{i-1}$ $(i = 2, 3, \cdots)$. Now suppose that for some $i \geqq 1$, $e_1, e_2, \cdots, e_n$ are elements of G whose cosets modulo C_i form a minimal basis for C_{i+1}. If $b_i = [e_i, a]$ and

$$b_1^{\alpha_1} b_2^{\alpha_2} \cdots b_n^{\alpha_n} \in C_{i-1} \qquad (0 \leqq \alpha_i < p),$$

then $[e_1^{\alpha_1} e_2^{\alpha_2} \cdots e_n^{\alpha_n}, a] \in C_{i-1}$, so that $e_1^{\alpha_1} e_2^{\alpha_2} \cdots e_n^{\alpha_n} \in C_i$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Hence $(C_i : C_{i-1}) \geqq (C_{i+1} : C_i)$ and $(C_{i+1} : C_i) \leqq (C_1 : 1) \leqq p^{p^{r-2}}$. But $C_p = N$, for if $x \in N$, since $a^p = 1$, $a^p = (a^x)^p = (a[a, x])^p$. Thus

$$a^p = a^p a_1^p a_2^{\binom{p}{2}} \cdots a_p,$$

where $a_1 = [a, x]$ and $a_n = [a_{n-1}, a]$ for $n > 1$. Thus $a_p = 1$, and $x \in C_p$, as required. Hence $(N:1) \leqq p^{p^{r-1}}$.

The fact that the bound in Theorem 4.1 is best possible is seen by observing that the wreath product of an elementary Abelian group of order p^{r-1} with a group of type p^∞ has a maximal elementary Abelian subgroup of order p^r.

5. Some special Černikov p-groups

In [5] it was shown that if G is a p-group and

$$(G : \zeta_\omega(G)) = (\zeta_{n+1}(G) : \zeta_n(G)) = p \quad (n = 0, 1, 2, \cdots),$$

then G is isomorphic to one of two groups, which have generators $s, s_1, s_2, \cdots$ and defining relations

$$s_n^p s_{n-1}^{\binom{p}{2}} \cdots s_{n-p+1} = 1 \qquad (n = p, p+1, \cdots),$$

$$s_1^p = s_2^p = \cdots = s_{p-1}^p = 1, \qquad [s_i, s_j] = 1,$$

$$[s_1, s] = 1, \qquad [s_n, s] = s_{n-1} \quad (n = 2, 3, \cdots),$$

$$s^p = 1 \quad \text{or} \quad s_1 .$$

These groups are precisely the (nontrivial) direct limits of p-groups of maximal class and will be referred to as such. Such a group G is a Černikov p-group, and $\delta(G) = p - 1$. On the other hand Černikov has shown [4] that if G is a non-nilpotent Černikov p-group, then $\delta(G) \geqq p - 1$. In this section we investigate the case $\delta(G) = p - 1$ and give some characterizations of the direct limits of p-groups of maximal class.

THEOREM 5.1. *Let G be a Černikov p-group for which $\delta(G) \leqq p - 1$. Then either $G/\zeta_1(G)$ is finite, or G has a finite normal subgroup N such that G/N is a direct limit of p-groups of maximal class.*

To prove this we need the following lemma.

LEMMA 5.2. *Let G be a non-Abelian p-group which has a normal subgroup N of index p, and suppose that N is the direct product of at most $p - 1$ groups of type p^∞. Then G is a direct limit of p-groups of maximal class.*

Since G is non-Abelian, $N \nleqq \zeta_1(G)$, and so by Lemma 2.4, G is not nilpotent. Now G satisfies the minimal condition for subgroups, and so the lower central series of G terminates after a finite number of steps in a group K for which $[K, G] = K$. G/K is nilpotent, but G is not nilpotent; hence $K \nleqq \zeta_n(G)$ for any $n = 0, 1, 2, \cdots$. Let L be the maximal divisible subgroup of K. L is normal in G, and K/L is a finite normal subgroup of G/L. Hence by Lemma 1.1(e), $K/L \leqq \zeta_r(G/L)$ for some integer r. Thus $[K, G, \cdots, G] \leqq L$. Since $[K, G] = K$, it follows that $K = L$ is divisible.

Let s be an element which together with N generates G. We show next that each element x of K can be written in the form $x = [y, s]$ for some $y \in K$. The normaliser of $[K, s]$ contains s and N and is therefore G. Again the centraliser of $K/[K, s]$ contains s and N and is therefore G. Thus $K = [K, G] \leqq [K, s]$. Hence

$$x = [y_1, s]^{\varepsilon_1}[y_2, s]^{\varepsilon_2} \cdots [y_r, s]^{\varepsilon_r}$$

for some $y_1, y_2, \cdots, y_r \in K$ and $\varepsilon_i = \pm 1$. Since K is Abelian,

$$x = [y_1^{\varepsilon_1}y_2^{\varepsilon_2} \cdots y_r^{\varepsilon_r}, s],$$

as required.

Now choose an element $s_1 \neq 1$ in $K \cap \zeta_1(G)$. We may define inductively elements $s_2, s_3, \cdots$ of K by the rule

$$s_{n-1} = [s_n, s] \qquad (n = 2, 3, \cdots).$$

Since $s^p \in N$, we have

$$s^p = (s^p)^{s_n} = (s^{s_n})^p = (ss_{n-1}^{-1})^p = s^p(s_{n-1}^p s_{n-2}^{\binom{p}{2}} \cdots s_{n-p})^{-1},$$

where we interpret $s_0 = s_{-1} = s_{-2} = \cdots = 1$. Thus

$$s_{n-1}^p s_{n-2}^{\binom{p}{2}} \cdots s_{n-p} = 1.$$

In particular

$$s_1^p = s_2^p = \cdots = s_{p-1}^p = 1, \qquad s_p^p = s_1^{-1}.$$

Since $s_1^p = 1$, $K \cap \zeta_1(G)$ is of exponent p.

Let $E = \lambda_1(K)$. E is a normal subgroup of N, and by hypothesis $(E:1) \leqq p^{p-1}$. But $s_{p-1} \in E$ and $s_{p-1} \notin \zeta_{p-2}(G)$. Hence $E \nleqq \zeta_{p-2}(G)$, and so by Lemma 1.1(c),

$$E \cap \zeta_{p-1}(G) > E \cap \zeta_{p-2}(G) > \cdots > E \cap \zeta_1(G) > 1.$$

Hence $(E \cap \zeta_1(G):1) = p$ and $(E:1) = p^{p-1}$. K therefore contains all elements of N of order p, and since K is divisible, it follows that $K = N$ by a well-known theorem on Abelian groups [9, vol. I, p. 163]. Hence $N \cap \zeta_1(G)$ is of exponent p and is thus contained in $E \cap \zeta_1(G)$. But obviously $\zeta_1(G) \leqq N$, and so $\zeta_1(G)$ is of order p. It follows by induction that $\zeta_n(G) \leqq N$ and $\zeta_n(G)/\zeta_{n-1}(G)$ is of order p $(n = 1, 2, \cdots)$. For each $r = 1, 2, \cdots, \lambda_r(N)$ is finite, and so by Lemma 1.1(e), $\lambda_r(N) \leqq \zeta_\omega(G)$. Thus $N \leqq \zeta_\omega(G)$.

Since $\zeta_n(G) \leqq N$ for each $n = 1, 2, \cdots$, we have $N = \zeta_\omega(G)$, and $G/\zeta_\omega(G)$ is of order p. Thus G is a direct limit of p-groups of maximal class.

To prove Theorem 5.1 let G be a Černikov p-group for which $\delta(G) \leqq p - 1$, and let D be the maximal divisible subgroup of G. By Theorem 2.6 we may assume that G is not nilpotent, and so the centraliser C of D is a proper subgroup of G. C is nilpotent, and so by Theorem 2.5, Corollary there is a finite characteristic subgroup K of C such that C/K is divisible. If C^*/K is the centraliser of C/K, then C^*/K is nilpotent. Hence C^* is nilpotent, and by Lemma 2.4, $D \leqq \zeta_1(C^*)$. Hence $C^* = C$, and C/K is its own centraliser in G/K.

We may assume without loss of generality that $K = 1$ and C is its own centraliser in G. Thus if x is any element of order p modulo C, the group X generated by x and C is not nilpotent, and so by Lemma 5.2, X is a direct limit of p-groups of maximal class. Now for $k > 0$

$$\zeta_k(G) \cap C \leqq \zeta_k(G) \cap X \leqq \zeta_k(X),$$

and $(\zeta_k(X):1) = p^k$. By Lemma 1.1, $(C \cap \zeta_k(G):1) \geqq p^k$ since $C \nleqq \zeta_k(G)$. Hence $\zeta_k(X) = \zeta_k(G) \cap C$, and $\zeta_k(X)$ is a normal subgroup of G. Suppose that $\zeta_2(X)$ is generated by $\zeta_1(X)$ and s_2, and put $[x, s_2] = s_1$. Then given $y \in G$, $[y, s_2] \in \zeta_1(X)$, and so $[y, s_2] = s_1^\xi$ for some integer ξ. Hence $z = yx^{-\xi}$ commutes with both s_2 and s_1. This implies that $z \in C$, for otherwise some power of z would generate together with C a direct limit of p-groups of maximal class, and this could not have both s_2 and s_1 in its centre. Hence $y \in X$, and so $G = X$, as required.

THEOREM 5.3. *Suppose that the locally finite p-group G has a normal subgroup N which is the direct product of at most $p - 1$ groups of type p^∞. Then the centraliser of N has index at most p in G.*

If this is false, let C be the centraliser of N, and let J/C be a subgroup of G/C of order p^2. If J is generated by x, y, and C, let H be the group generated by x, y, and N. Then H is a non-nilpotent Černikov p-group and $\delta(H) \leqq p - 1$. By Theorem 5.1 there is a finite normal subgroup K of H such that H/K is a direct limit of p-groups of maximal class. Since NK/K is Abelian, NK is nilpotent, and so by Lemma 2.4, $NK \leqq C$. But NK/K is the maximal divisible subgroup of H/K, and so $(H:NK) = p$. Thus $(H:C \cap H) \leqq p$, and since $CH = J$, $(J:C) \leqq p$, a contradiction.

THEOREM 5.4. *If G is a locally finite p-group, and for some integer $r \geqq p$ all subgroups of G of order p^r can be generated by fewer than p elements, then either $G/\zeta_1(G)$ is finite, or G is a direct limit of p-groups of maximal class.*

Since $r \geqq p$, G has no elementary Abelian subgroup of order p^r, and so by Theorem 4.1, G is a Černikov p-group. If G is nilpotent, $G/\zeta_1(G)$ is finite. Otherwise let D be the maximal divisible subgroup of G, and let K be a normal subgroup of G of order p^{r-p} contained in D. Let $\bar{G} = G/K$, $\bar{D} = D/K$.

Then $\bar{G}$ is not nilpotent and has no elementary Abelian subgroup of order p^p. Hence by Theorem 5.1, $\delta(G) = \delta(\bar{G}) = p - 1$. Now $\bar{D}$ is its own centraliser in $\bar{G}$, for if $\bar{x} \notin \bar{D}$ and $[\bar{x}, \bar{D}] = 1$, then $\bar{H} = \{\bar{x}, \bar{D}\}$ is Abelian and is thus the direct product of the divisible group $\bar{D}$ and a nontrivial group, so that $(\lambda_1(\bar{H}):1) = p^p$. Thus by Theorem 5.3, $(G:D) = (\bar{G}:\bar{D}) = p$. Since $\delta(G) = p - 1$, it follows from Lemma 5.2 that G is a direct limit of p-groups of maximal class.

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University of Chicago
Chicago, Illinois