COHOMOLOGY OF ALGEBRAIC LINEAR GROUPS

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1. Introduction

The theory of rational representations of algebraic linear groups over fields of characteristic 0 has, for some time, been in a sufficiently well developed state to call for an adaptation of homological algebra to the requisite category of "rational modules". The most elementary portion of this program is carried out in Section 2 below, and this sets the stage for what follows.

In the later results, a vital role is played by the decomposition of an algebraic linear group into a semidirect product of the maximum unipotent normal subgroup by a fully reducible subgroup. This was established, for not necessarily irreducible algebraic linear groups over fields of characteristic 0, by Mostow in [7]. In the irreducible case, the group decomposition follows easily from the corresponding decomposition of the Lie algebra. However, the proof of the general case seems to require Mostow's result on the conjugacy of the maximal fully reducible subgroups. For this reason, and also by way of illustration, we apply (in Section 3) the elementary theory of rational modules to obtain a simple direct proof of the conjugacy theorem. At the same time, we sketch the resulting simplification in the proof of the decomposition theorem, and we discuss the decomposition with reference to representations and group extensions.

Sectons 4 and 5 contain the main results. The suggestion for these comes from the results of van Est [9] on the differentiable cohomology of Lie groups. They concern the relations between the rational group cohomology, the ordinary Lie algebra cohomology, and the cohomology of the differential forms. It is due to the semidirect product decomposition that the results for algebraic linear groups are more precise than van Est's results for Lie groups, which concern a situation that is somewhat more general than the straight analogue of what is considered here.

Van Est's theory has been rounded out and strengthened by Mostow (in [8]), and it has become clear from Mostow's approach that a cohomology theory of groups that takes account of additional structure (topological, differentiable, or algebraic) must be based on *injective* resolutions in the requisite category of modules while, contrary to the case of discrete groups, the *projective* part of the machinery of homological algebra is inapplicable. This realization was the point of departure for the present investigation.

In Section 6, we apply the results on the rational group cohomology to obtain the expected interpretation of the 2-dimensional rational cohomology groups as groups of equivalence classes of rational group extensions. This

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may be regarded as a test case which illustrates how the passage from group to Lie algebra extends in a natural and smooth way to the superstructure of rational cohomology.

2. Rational modules

Let G be an algebraic linear group over a field F and suppose that G acts as a group of linear automorphisms on some vector space M over F. We shall say that M is a *rational G-module* if it is the sum of finite-dimensional G-stable subspaces V such that the representation of G on each V is a rational representation of G in the usual sense.

A rational G-module M is said to be *rationally injective* if, whenever U is a rational G-module, and φ is a G-module homomorphism of a G-submodule V of U into M, φ can be extended to a G-module homomorphism of U into M. Dually, M is said to be *rationally projective* if, with U and V as above, every G-module homomorphism of M into U/V can be "lifted" to a G-module homomorphism of M into U.

Let $F[G]$ denote the ordinary group algebra of G over F. If A is any unitary $F[G]$-module, then the sum of all finite-dimensional $F[G]$-submodules V of A such that the representation of G on V is a rational representation is evidently the unique maximum rational submodule of A; we shall denote it A^*. Clearly, every G-module homomorphism of a rational G-module M into A sends M into A^*.

PROPOSITION 2.1. *Every rational G-module can be G-monomorphically imbedded in a rationally injective G-module.*

Proof. Let M be a rational G-module. By the theory of ordinary modules over a ring, there exists an injective unitary $F[G]$-module A and an $F[G]$-monomorphism $\varphi : M \rightarrow A$. Now, by our above remark, $\varphi(M) \subset A^*$, and the same remark shows that A^* is rationally injective.

On the other hand, it is *not* generally true that every rational G-module is a G-homomorphic image of a rationally projective G-module. This is shown by the following example. Let F be a field of characteristic 0, and let G be the additive group of F, with its natural structure of an algebraic (linear) group, so that the polynomial functions on G are the polynomials in the identity map of F onto itself. Regard F as a rational G-module, with G operating trivially. Let $F(t)$ be the ring of all integral power series in the variable t with coefficients in F. Operate with G on $F(t)$ by associating with each element x of G the multiplication by $\exp(xt)$. For every positive integer n, let F_n be the G-module $F(t)/F(t)t^n$. Evidently, F_n is a rational G-module, and the map $f(t) \rightarrow f(0)$ induces a G-module epimorphism $\varphi_n : F_n \rightarrow F$. Now suppose that there is a rationally projective G-module A and a G-module epimorphism $\alpha : A \rightarrow F$. Then there is a G-module homomorphism $\alpha_n : A \rightarrow F_n$ such that $\varphi_n \circ \alpha_n = \alpha$. Let a be an element of A such that $\alpha(a) = 1$. Let ρ be the representation of G on A. Since every rational representation of G is unipotent (by [7, Prop. 3.2]), there is a positive integer q such that $(\rho(x) - 1)^q(a) = 0$, for every $x \in G$, where 1 stands for the identity map. Let σ_n be the representation of G on F_n. Then we must have $(\sigma_n(x) - 1)^q(\alpha_n(a)) = 0$, for every $x \in G$. Now a representative of $\alpha_n(a)$ in $F(t)$ is of the form $1 + tg(t)$, with $g(t) \in F(t)$. Our last result therefore means that $(\exp(xt) - 1)^q(1 + tg(t))$ lies in $F(t)t^n$, for every $x \in G$. This means that $q \geqq n$, so that, for large enough n, we have reached a contradiction. Thus (A, α) cannot exist.

In the category of the rational G-modules, the derived functors Ext_G^n of Hom_G must therefore be defined from *injective* resolutions; these exist, in the appropriate sense, in virtue of Proposition 2.1. By a *rationally injective resolution* of the rational G-module M we mean an exact sequence of G-module homomorphisms: $(0) \to M \to X_0 \to X_1 \to \cdots$, where the X_i are rationally injective G-modules. If A is any other rational G-module, we define $\mathrm{Ext}_G^n(A, M)$ as the n^{th} cohomology group of the complex:

$$(0) \to \mathrm{Hom}_G(A, X_0) \to \mathrm{Hom}_G(A, X_1) \to \cdots,$$

noting that, as in the ordinary case, these cohomology groups are, to within natural isomorphisms, independent of the particular choice of the rationally injective resolution X. It is clear that $\mathrm{Ext}_G^0(A, M)$ may be identified with $\mathrm{Hom}_G(A, M)$.

A familiar development shows that $\mathrm{Ext}_G^1(A, M)$ may be identified with the F-space of the equivalence classes of the rational G-module extensions

$$(0) \to M \xrightarrow{i} E \xrightarrow{j} A \to (0)$$

of M by A. In fact, such an extension gives rise to an element of $\mathrm{Ext}_G^1(A, M)$ as follows: Let X be a rationally injective resolution of M, and let $\gamma : M \to X_0$ be its "augmentation". Since X_0 is rationally injective, there exists a G-module homomorphism $\varphi : E \to X_0$ such that $\varphi \circ i = \gamma$. Given $a \in A$, the elements $\varphi(e)$, with $e \in E$ such that $j(e) = a$, constitute precisely a coset $\rho(a) \in X_0/\gamma(M)$, and $\rho \in \mathrm{Hom}_G(A, X_0/\gamma(M))$. The image of ρ in $\mathrm{Hom}_G(A, X_1)$ (by the map induced from $X_0 \to X_1$) is the representative of an element of $\mathrm{Ext}_G^1(A, M)$, which depends only on the equivalence class of the given extension of M by A. Conversely, an element of $\mathrm{Ext}_G^1(A, M)$ is represented by an element $h \in \mathrm{Hom}_G(A, X_0/\gamma(M))$, and we define E_h as the submodule of $A \oplus X_0$ consisting of all pairs (a, x) such that $h(a) = x + \gamma(M)$. Evidently, E_h is a rational G-module, and we have an exact sequence of G-module homomorphisms:

$$(0) \to M \xrightarrow{i} E_h \xrightarrow{j} A \to (0),$$

where $i(m) = (0, \gamma(m))$ and $j(a, x) = a$. The equivalence class of this rational extension of M by A depends only on the element of $\mathrm{Ext}_G^1(A, M)$ represented by h, and not on the particular choice of h. Now it is simple routine to check that the above constructions give mutually inverse maps establishing a natural isomorphism between $\mathrm{Ext}^1_G(A, M)$ and the space of the equivalence classes of the rational extensions of M by A.

For our later applications, a certain standard rationally injective resolution will be important, which we proceed to develop now. An F-valued function on the algebraic linear group G is called a *rational representative function* if it is a rational function defined at every point of G and if its translates by elements of G span only a finite-dimensional space of functions. As is well known (see the proof of Lemma 10.1 in [5]), the second requirement is a consequence of the first whenever F is algebraically closed. In general, the rational representative functions are precisely the composites of the rational representations of G with the linear functionals on the endomorphism algebras of the corresponding representation spaces (cf. [5, Section 2]), and every rational representative function on G is of the form $d^{-n}p$, where p is a polynomial function on G, n is a nonnegative integer, and d is the determinant function on G [5, Lemma 10.1]. If H is any other algebraic linear group over F, a map $\varphi : G \to H$ is called a *rational representative map* if, for every rational representative function f on H, $f \circ \varphi$ is a rational representative function on G.

The rational representative functions on G constitute an F-algebra, which we shall denote by R, or by $R(G)$ if the group is to be mentioned. If $f \in R$ and $x \in G$, the left and right translates, $x \cdot f$ and $f \cdot x$, of f by x are defined by $(x \cdot f)(y) = f(yx)$, $(f \cdot x)(y) = f(xy)$. Let M be a rational G-module. We make the tensor product $R \otimes M$ (taken relative to F) into a G-module such that $x \cdot (f \otimes m) = (f \cdot x^{-1}) \otimes (x \cdot m)$. As a tensor product of the two rational G-modules R and M, $R \otimes M$ is evidently a rational G-module. We wish to prove that $R \otimes M$ is rationally injective. However, for later use, we shall actually prove the following more general fact.

PROPOSITION 2.2. *Let G be an algebraic linear group over a field F, and let H be an algebraic subgroup of G. Suppose that there is a rational representative map $\rho : G \to H$ such that $\rho(yx) = y\rho(x)$, for every $y \in H$ and every $x \in G$. Let M be a rational H-module, and let $R \otimes M$ be the rational H-module with $y \cdot (f \otimes m) = (f \cdot y^{-1}) \otimes (y \cdot m)$. Then $R \otimes M$ is rationally injective. If A is any rationally injective G-module, then $A \otimes M$ is rationally injective as an H-module.*

Proof. Let B be a rational H-module, and let γ be an H-module homomorphism of a submodule C of B into $R \otimes M$. Let φ be any F-linear projection of B onto C. Whenever convenient, we shall identify elements of $R \otimes M$ with the naturally corresponding maps of G into M. For $b \in B$, define the map $\beta(b) : G \to M$ by

$$\beta(b)(x) = \rho(x) \cdot [\gamma(\varphi(\rho(x)^{-1} \cdot b))(\rho(x)^{-1}x)].$$

We show first that $\beta(b) \in R \otimes M$. Let $(b_1, \cdots, b_n)$ be a basis for the finite-dimensional rational H-module generated by b. Then we may write

$$\varphi(\rho(x)^{-1}\cdot b) = \sum_{i=1}^{n} g_i(\rho(x))\varphi(b_i),$$

where the g_i's are rational representative functions on H. If $f_i = g_i \circ \rho$, then $f_i \in R$, and we have

$$\gamma(\varphi(\rho(x)^{-1}\cdot b)) = \sum_{i=1}^{n} f_i(x)u_i ,$$

where $u_i = \gamma(\varphi(b_i)) \in R \otimes M$. Write $u_i = \sum_j g_{ij} \otimes m_{ij}$, with $g_{ij} \in R$ and $m_{ij} \in M$. Then we have

$$\beta(b)(x) = \sum_{i,j} f_i(x)g_{ij}(\rho(x)^{-1}x)\rho(x)\cdot m_{ij} .$$

Finally, let $(m_1, \cdots, m_q)$ be a basis for the finite-dimensional rational H-module that is generated by the m_{ij}'s. Then we may write

$$\beta(b)(x) = \sum_{i,j,k} f_i(x)g_{ij}(\rho(x)^{-1}x)h_{ijk}(\rho(x))m_k .$$

Since ρ is a rational representative map, $h_{ijk} \circ \rho \in R$, and the map $x \to \rho(x)^{-1}x$ is a rational representative map of G into G. Hence it is clear that $\beta(b) \in R \otimes M$.

If $y \in H$, we have $(y\cdot\beta(b))(x) = y\cdot\beta(b)(y^{-1}x)$, and if we substitute this in the definition of $\beta(b)$ and use $\rho(y^{-1}x) = y^{-1}\rho(x)$ we find that $(y\cdot\beta(b))(x) = \beta(y\cdot b)(x)$. Thus β is an H-module homomorphism of B into $R \otimes M$. Finally, if $b \in C$, we have

$$\beta(b)(x) = \rho(x)\cdot[\gamma(\rho(x)^{-1}\cdot b)(\rho(x)^{-1}x)] = (\rho(x)\cdot\gamma(\rho(x)^{-1}\cdot b))(x) = \gamma(b)(x),$$

i.e., β coincides with γ on C. Thus $R \otimes M$ is rationally injective.

Now the rationally injective G-module A becomes identified with a direct G-module summand of $R \otimes A$, by the map $a \to 1 \otimes a$. Hence $A \otimes M$ is isomorphic, as an H-module, with a direct H-module summand of $R \otimes A \otimes M$. By the above, applied to $A \otimes M$ in the place of M, it follows that $A \otimes M$ is rationally injective. This completes the proof of Proposition 2.2.

Let φ_{-1} be the natural G-module monomorphism $M \to R \otimes M$, where $\varphi_{-1}(m) = 1 \otimes m$. Define $C^{-1}(M) = M$, $C^{q+1}(M) = R \otimes C^q(M)$, and suppose that we have already defined a G-module homomorphism

$$\varphi_q : C^q(M) \to C^{q+1}(M).$$

Then we define φ_{q+1} such that $\varphi_{q+1}(f \otimes u) = 1 \otimes f \otimes u - f \otimes \varphi_q(u)$, where $f \in R$ and $u \in C^q(M)$. Then $\varphi_{q+1} \circ \varphi_q = 0$, and we have a G-module complex $(0) \to M \to C^0(M) \to C^1(M) \to \cdots$. By Proposition 2.2, $C^q(M)$ is rationally injective, for every $q \geqq 0$. Define $\psi_q : C^q(M) \to C^{q-1}(M)$ such that $\psi_q(f \otimes u) = f(1)u$ (and $\psi_{-1} = 0$). Then one verifies directly that $\varphi_{q-1} \circ \psi_q + \psi_{q+1} \circ \varphi_q$ is the identity map on each $C^q(M)$. Hence our complex is a rationally injective resolution of M.

In particular, the cohomology groups $H^n(G, M) = \mathrm{Ext}_G^n (F, M)$ are the cohomology groups of the complex formed with the G-fixed parts $C^q(M)^G$ of the $C^q(M)$, with $q \geqq 0$. Now, for $q \geqq 0$, $C^q(M)^G$ is isomorphic, as an F-space, with $C^{q-1}(M)$; in the functional notation, such an isomorphism is given by $g \to g'$, where $g'(x_1, \cdots, x_q) = g(1, x_1, x_1 x_2, \cdots, x_1 \cdots x_q)$, its inverse being given by $f \to f^*$, where

$$f^*(x_0, \cdots, x_q) = x_0 \cdot f(x_0^{-1} x_1, \cdots, x_{q-1}^{-1} x_q).$$

The coboundary, for these *nonhomogeneous rational representative cochains* becomes $f \to \delta f$, where

$$(\delta f)(x_1, \cdots, x_q) = x_1 \cdot f(x_2, \cdots, x_q) + \textstyle\sum_{i=1}^{q-1} (-1)^i f(x_1, \cdots, x_i x_{i+1}, \cdots, x_q) + (-1)^q f(x_1, \cdots, x_{q-1}).$$

3. Decompositions and factor groups

Let F be a field of characteristic 0, and let G be an algebraic group of linear automorphisms of a finite-dimensional F-space V. Let V' denote the semisimple rational G-module associated with V, i.e., take V' to be the direct sum of the factor modules formed with successive terms in a composition series for the rational G-module V. Let σ denote the rational representation of G on V', and let N be the kernel of σ. Then N is evidently the unique maximum unipotent normal algebraic subgroup of G. Let $\mathfrak{G}$ denote the Lie algebra of G, $\mathfrak{N}$ the Lie algebra of N. Then $\mathfrak{N}$ is the unique maximum ideal of nilpotent elements of $\mathfrak{G}$. By [2, Prop. 5, p. 144], there is a fully reducible subalgebra $\mathfrak{K}$ of $\mathfrak{G}$ such that $\mathfrak{G}$ is the semidirect sum $\mathfrak{N} + \mathfrak{K}$. Using that an ideal of a fully reducible Lie algebra of linear transformations is fully reducible and that $\mathfrak{N}$ has no nonzero fully reducible subalgebras, we see that $\mathfrak{K}$ is a *maximal* fully reducible subalgebra of $\mathfrak{G}$. Hence $\mathfrak{K}$ is the Lie algebra of an irreducible algebraic subgroup K_1 of G, and K_1 is fully reducible as a group of linear transformations. Let G_1 be the irreducible component of the identity in G. Since N is unipotent, it is irreducible. Since $\mathfrak{N} \subset \mathfrak{G}$, which is also the Lie algebra of G_1, we must have $N \subset G_1$. Similarly, $K_1 \subset G_1$. Consider $K_1 \cap N$. As a subgroup of N, this is unipotent, and, as a normal subgroup of K_1, it is fully reducible. Hence $K_1 \cap N = (1)$. Now the argument on pp. 131–132 of [2], which deals with this situation in the case where G_1 is solvable, shows also in our general case that G_1 is the semidirect product $N \cdot K_1$.

Since K_1 is fully reducible, there is a K_1-module isomorphism between V and V'. This yields a rational isomorphism $\tau : \sigma(K_1) \to K_1$ such that $\tau \circ \sigma$ is the identity map on K_1. Clearly, $\sigma(K_1) = \sigma(G_1)$. Hence, if ρ denotes the composite of the restriction of σ to G_1 with τ, then ρ is a rational group epimorphism of G_1 onto K_1 with kernel N, and ρ is the identity map on K_1.

Now let us assume, which has already been proved in the case $G = G_1$, that G is a semidirect product $N \cdot K$, where K is a fully reducible algebraic subgroup of G, and that the corresponding projection ρ of G onto K is rational. Let $[G, N]$ denote the subgroup of N that is generated by the commutators $xyx^{-1}y^{-1}$, with $x \in G$ and $y \in N$. Let L be any fully reducible algebraic subgroup of G. We wish to prove the result due to Mostow [7, Th. 7.1] according to which *there is an element* $t \in [G, N]$ *such that* $tLt^{-1} \subset K$. For every $x \in L$, write $\varphi(x) = x\rho(x)^{-1} \in N$. Then $\varphi(xy) = \varphi(x)\rho(x)\varphi(y)\rho(x)^{-1}$, for all x and y in L. Since N is unipotent, the exponential map, $\exp : \mathfrak{N} \to N$, and the logarithm map, $\log : N \to \mathfrak{N}$, are defined as polynomial maps and are mutually inverse. Let $f(x)$ denote the coset mod $[\mathfrak{N}, \mathfrak{N}]$ of $\log(\varphi(x))$. The adjoint representation of G induces the structure of a rational L-module on $\mathfrak{N}$ and hence on $\mathfrak{N}/[\mathfrak{N}, \mathfrak{N}]$. For $x \in L$ and $u \in \mathfrak{N}/[\mathfrak{N}, \mathfrak{N}]$, $x \cdot u$ denotes the corresponding transform of u by x. The above identity for φ yields:

$$f(xy) = x \cdot f(y) + f(x), \qquad \text{for all } x \text{ and } y \text{ in } L.$$

Now we define the structure of a rational L-module on the direct sum $F \oplus \mathfrak{N}/[\mathfrak{N}, \mathfrak{N}]$ such that $x \cdot (a, u) = (a, af(x) + x \cdot u)$. Since L is fully reducible, this rational L-module is semisimple [7, Prop. 3.2]. Hence $\mathfrak{N}/[\mathfrak{N}, \mathfrak{N}]$ has a module complement in $F \oplus \mathfrak{N}/[\mathfrak{N}, \mathfrak{N}]$. This complement contains one and only one element of the form $(1, u)$. Operating with $x \in L$, we see that we must have $f(x) = u - x \cdot u$. Moreover, since $\mathfrak{N}/[\mathfrak{N}, \mathfrak{N}]$ is semisimple as an L-module, u is the sum of an L-fixed element and an element v that is a sum of elements of the form $w - y \cdot w$, with $w \in \mathfrak{N}/[\mathfrak{N}, \mathfrak{N}]$ and $y \in L$. We have $f(x) = v - x \cdot v$. Now v has a representative v' in $\mathfrak{N}$ that is of a form analogous to that of v, and it is clear from the properties of the exponential map that then $\exp(v') \in [G, N]$. Put $s = \exp(v')$. Now, denoting the commutator subgroup of N by N', we have $\varphi(x) \in \exp(v' - x \cdot v')N'$, whence $\varphi(x) \in sxs^{-1}x^{-1}N'$. Also $\varphi(x)s\varphi(x)^{-1}s^{-1} \in N'$, so that, with the last result, we get $sxs^{-1}x^{-1}s\varphi(x)^{-1}s^{-1} \in N'$, whence also $xs^{-1}x^{-1}s\varphi(x)^{-1} \in N'$. This gives $s^{-1}x^{-1}s \in x^{-1}\varphi(x)N' = \rho(x)^{-1}N' \subset KN'$. Thus we have $s^{-1}Ls \subset KN'$. Replacing L by $s^{-1}Ls$, we may therefore suppose now that $L \subset KN'$. We may now repeat the above argument with N' in the place of N to conclude that L can be conjugated into KN'', etc. Finally, we reach the desired result.

With this conjugacy result established, for G_1, the arguments supplied in [7, Section 6] prove that *there is a fully reducible algebraic subgroup K of G such that G is the semidirect product $N \cdot K$.* Exactly as above for $G_1 = N \cdot K_1$, we see now that *the projection of G onto K with kernel N is a rational group epimorphism*, so that the conjugacy result holds generally, by the proof above.

Now we can apply [5, Prop. 2.4] to conclude that the algebra R of the rational representative functions on G is the tensor product of the subalgebra R^N of its N-fixed elements by another subalgebra S whose elements satisfy $f(xy) = f(y)$, for all $x \in K$ and all $y \in N$. Moreover, this implies that the restrictions of the elements of S to N make up the algebra of all rational representative functions, i.e., the algebra of all polynomial functions on N. On the other hand, if ρ is our projection $G \to K$, the map $f \to f \circ \rho$ is an isomorphism of the algebra of all rational representative functions on K onto R^N.

Lemma 3.1. *Let G be an algebraic group of linear automorphisms of the* *finite-dimensional F-space V. Then every proper automorphism of R, i.e., every automorphism leaving the constants fixed and commuting with the right translations, is a left translation.*

Proof. Let α be a proper automorphism of R. Then α defines an F-homomorphism $\alpha' : R \to F$, where $\alpha'(f) = \alpha(f)(1)$, for every $f \in R$. Now $R = P[d^{-1}]$, where P is the algebra of all polynomial functions on G, and d is the determinant function on G. Hence it is clear that α' determines an element $x \in G$ such that $\alpha'(f) = f(x)$, for every $f \in R$. Now, for every $y \in G$,

$$\alpha(f)(y) = (\alpha(f)\cdot y)(1) = \alpha(f\cdot y)(1) = \alpha'(f\cdot y) = (f\cdot y)(x) = f(yx).$$

Thus $\alpha(f) = x\cdot f$, Q.E.D.

THEOREM 3.1. *Let U be a unipotent normal algebraic subgroup of the algebraic linear group G over the field F of characteristic 0. Then G/U can be given the structure of an algebraic linear group such that $R(G/U)$ is canonically isomorphic with $R(G)^U$. Moreover, there exists a rational representative map $\varphi : G/U \to G$ such that, if τ is the rational epimorphism $G \to G/U$, $\tau \circ \varphi$ is the identity map on G/U.*

Proof. Let N be the maximum normal unipotent subgroup of G, so that $U \subset N$. We shall first deal with the case where $G = N$. By [2, p. 119, Prop. 11], there exists a rational representation ρ of N whose kernel is precisely U. Let $\rho^\bullet$ denote the Lie algebra homomorphism on the Lie algebra $\mathfrak{N}$ of N that is induced by ρ. Then $\rho^\bullet(\mathfrak{N})$ is a Lie algebra consisting of nilpotent linear endomorphisms. By [2, p. 123, Prop. 14], the exponentials of the elements of $\rho^\bullet(\mathfrak{N})$ constitute a unipotent algebraic linear group. For every $u \in \mathfrak{N}$, we have $\rho(\exp(u)) = \exp(\rho^\bullet(u))$. Hence the group consisting of the exponentials of the elements of $\rho^\bullet(\mathfrak{N})$ coincides with $\rho(N)$, so that $\rho(N)$ is an algebraic linear group. Now let γ be a linear map of $\rho^\bullet(\mathfrak{N})$ into $\mathfrak{N}$ such that $\rho^\bullet \circ \gamma$ is the identity map on $\rho^\bullet(\mathfrak{N})$. Define the polynomial map $\psi : \rho(N) \to N$ as the composite $\exp_{\mathfrak{N}} \circ \gamma \circ \log_{\rho(N)}$. Then $\rho \circ \psi$ is the identity map on $\rho(N)$. Now it is easily checked that the map $f \to f \circ \rho$ is an isomorphism of $R(\rho(N))$ onto $R(N)^U$ whose inverse is the map $g \to g \circ \psi$.

We have $R(G) = R(G)^N \otimes R(N)^+$, where $R(N)^+$ is the image of $R(N)$ in $R(G)$ by the map $f \to f^+$, with $f^+(xy) = f(y)$, for every $x \in K$ and every $y \in N$. It follows that $R(G)^U = R(G)^N \otimes (R(N)^+)^U$. Let α be a proper automorphism of $R(G)^U$. We wish to prove that α is the left translation by an element of G. Since $R(G)^N$ may be identified with $R(K)$, it follows from Lemma 3.1 that there is an element $x \in K$ such that α coincides on $R(G)^N$ with the left translation by x. Hence we may now suppose that α leaves the elements of $R(G)^N$ fixed. Now consider the homomorphism $\alpha' : R(N)^U \to F$ defined by $\alpha'(f) = \alpha(f^+)(1)$. By [5, Prop. 2.5], there is a proper automorphism β of $R(N)^U$ such that $\alpha'(f) = \beta(f)(1)$, for every $f \in R(N)^U$. Now we have shown above that $R(N)^U$ is isomorphic with $R(\rho(N))$. Applying

Lemma 3.1 to the proper automorphism of $R(\rho(N))$ that is induced by β, we see that β is the left translation by an element $y \epsilon N$. Hence we have $\alpha(f^+)(1) = f(y) = f^+(y)$, for every $f \epsilon R(N)^U$. Since $(R(N)^+)^U = (R(N)^U)^+$, and since $\alpha(g)(1) = g(y)$, for every $g \epsilon R(G)^N$, it follows that we have $\alpha(h)(1) = h(y)$, for every $h \epsilon R(G)^U$. But this means that α is the left translation by y. Thus we have proved that every proper automorphism of $R(G)^U$ is the left translation by an element of G.

Since $R(G)^N$ is isomorphic with $R(K)$, and $(R(N)^+)^U$ is isomorphic with $R(\rho(N))$, it is clear that $R(G)^U$ is finitely generated as an F-algebra. Let T be a finite-dimensional two-sidedly G-stable subspace of $R(G)^U$ such that $R(G)^U = F[T]$. Let σ be the representation of G by left translations on T. Then the kernel of σ is precisely U. It follows from [5, Prop. 2.6] and our above result on the proper automorphisms that $\sigma(G)$ is an algebraic linear group. Clearly, $R(\sigma(G)) = R(G)^U$. Thus the first part of Theorem 3.1 is proved.

We have $\sigma(G) = \sigma(N)\sigma(K)$. Since $\sigma(K)$ is fully reducible and $\sigma(N)$ unipotent, it is clear that $\sigma(N)$ is the kernel of the semisimple representation associated with the identity representation of $\sigma(G)$, and that $\sigma(K)$ is a maximal fully reducible subgroup of $\sigma(G)$. Hence $\sigma(K)$ is an algebraic linear group, and $\sigma(G) = \sigma(N)\cdot\sigma(K)$ is a standard decomposition of $\sigma(G)$. Hence $R(\sigma(K)) = R(\sigma(G))^{\sigma(N)}$, which may be identified with $R(G)^N$, and hence with $R(K)$. The restriction of σ to K is a rational isomorphism, and our result on the representative functions means that its inverse σ_K^{-1} is a rational isomorphism of $\sigma(K)$ onto K. On the other hand, we have already shown in the beginning of this proof that there is a polynomial map ψ of $\sigma(N)$ into N such that $\sigma \circ \psi$ is the identity map on $\sigma(N)$. Let α and β be the projections of $\sigma(G)$ onto $\sigma(N)$ and $\sigma(K)$, respectively, corresponding to our decomposition $\sigma(G) = \sigma(N)\cdot\sigma(K)$. We know from the decomposition theory that they are rational representative maps. Now define the map $\varphi : \sigma(G) \to G$ by $\varphi(x) = \psi(\alpha(x))\sigma_K^{-1}(\beta(x))$. If we identify $\sigma(G)$ with G/U, then φ satisfies the requirements of Theorem 3.1. This completes the proof of Theorem 3.1.

If the base field F is algebraically closed, the first part of Theorem 3.1 holds for any (not necessarily unipotent) normal algebraic subgroup U of G, because then the image of a rational group homomorphism is always an algebraic group, and the inverse of a rational isomorphism is always rational [5, Lemma 10.2]. However, the second part of the theorem may fail to hold if U is not unipotent. This is shown by the following example. Let G be the group of all linear automorphisms of a 2-dimensional vector space over the field C of the complex numbers. Let U be the subgroup consisting of the scalar multiplications. We claim that, in this case, there is not even a continuous map of G/U into G whose composite with the canonical map $G \to G/U$ is the identity map. Indeed, if such a map existed, it would follow that the underlying topological space of G is homeomorphic with the Cartesian product of the spaces underlying U and G/U. Let E^n denote real n-space, and let S^n denote the n-sphere. Then U is homeomorphic with $E^1 \times S^1$, and G/U is homeomorphic with $E^3 \times P^3$, where P^3 is projective 3-space, so that $U \times (G/U)$ is homeomorphic with $E^4 \times P^3 \times S^1$. On the other hand, G is homeomorphic with $E^4 \times S^3 \times S^1$. Hence the fundamental group of G is isomorphic with the additive group Z of the integers, while the fundamental group of $U \times (G/U)$ is isomorphic with the direct product of Z by a group of order 2. Thus G is not homeomorphic with $U \times (G/U)$.

If F is not algebraically closed and U is not unipotent, it can happen that G/U cannot be given an algebraic structure such that there is a rational epimorphism $G \to G/U$. This is shown by the following example. Let F be the field of the real numbers, and let G be the group of all matrices $x = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, with $a^2 + b^2 \neq 0$. Let U be the subgroup of all elements of determinant 1. Define the functions α and β on G by $\alpha(x) = a$, $\beta(x) = b$. Then $R(G) = F[\alpha, \beta, d^{-1}]$, where d is the determinant function $\alpha^2 + \beta^2$ on G. It is straightforward to verify that $R(G)^U = F[d, d^{-1}]$. Now the group of the G-translations on $F[d, d^{-1}]$ is the group of automorphisms whose elements send d onto cd, where c ranges over the *positive* real numbers. Now suppose that there exists a rational representation σ of G such that $\sigma(G)$ is an algebraic linear group and the kernel of σ is precisely U. Then it follows from Lemma 3.1 that there is a two-sidedly G-stable subalgebra S of $R(G)^U$, namely the subalgebra generated by the constants and the representative functions associated with σ, such that the representation of G/U by left translations on S is faithful and sends G/U onto the group of *all* proper automorphisms of S. For every nonzero real number s, there is a proper automorphism of $F[d, d^{-1}]$ that sends d onto sd. Since S is stable under the group of all proper automorphisms of $F[d, d^{-1}]$, we deduce easily that S is generated by a set of powers of d (with positive or negative exponents). Now let q denote the highest power of 2 such that $S \subset F[d^q, d^{-q}]$. The automorphism of $F[d^q, d^{-q}]$ that sends d^q onto $-d^q$ induces a proper automorphism of S. By the maximal property of q, S contains an odd power of d^q, whence we see that this proper automorphism of S is not a translation. Hence there can be no rational representation σ of G which has U for its kernel and is such that $\sigma(G)$ is algebraic.

The rational representative map $\varphi : G/U \to G$ of Theorem 3.1 yields rational representative 2-dimensional cocycles (factor sets) for G/U in U/U' and thus determines an element of $H^2(G/U, U/U')$, which is trivial if and only if, with the algebraic structures of the factor groups as in Theorem 3.1, G/U' is rationally a semidirect product $(U/U')\cdot(G/U)$. We shall give a full discussion of the connection between rational group extensions and 2-dimensional rational cohomology classes in Section 6.

4. Differential forms

Let G be an irreducible algebraic linear group over a field F of characteristic 0, and let R be the algebra of all rational representative functions on G. Let T denote the R-module of all F-derivations of R. We define the structure of a G-module on T by putting, for $x \in G$, $\tau \in T$, $f \in R$,

$$(x \cdot \tau)(f) = \tau(f \cdot x) \cdot x^{-1}.$$

The Lie algebra $\mathfrak{G}$ of G may be identified with the Lie subalgebra (over F) T^G of T that consists of all G-fixed elements of T.

LEMMA 4.1. *The canonical map of $R \otimes \mathfrak{G}$ into T is an isomorphism.*

Proof. Let Q denote the field of quotients of R. Then Q is the field of the rational functions on G. By [1, p. 132, Prop. 4], the canonical map of $Q \otimes \mathfrak{G}$ into the space of all F-derivations of Q is an isomorphism. This implies already that the canonical map $R \otimes \mathfrak{G} \rightarrow T$ is a monomorphism. Furthermore, if $\tau \in T$, then its canonical extension (still denoted τ) to a derivation of Q belongs to $Q\mathfrak{G}$. Hence there are elements $f \neq 0$ and f_i in R such that $f\tau = \sum_{i=1}^{n} f_i \tau_i$, where $(\tau_1, \cdots, \tau_n)$ is an F-basis for $\mathfrak{G}$. Operating with $x \in G$, we obtain

$$(f \cdot x^{-1})(x \cdot \tau) = \textstyle\sum_{i=1}^{n} (f_i \cdot x^{-1}) \tau_i .$$

There are elements g_j in R such that $\tau_i(g_j)(1) = \delta_{ij}$. Applying the derivation just written to g_j and evaluating the resulting element of R at 1, we obtain $f(x^{-1})\tau(g_j \cdot x)(x^{-1}) = f_j(x^{-1})$. Clearly, if $h_j(x) = \tau(g_j \cdot x^{-1})(x)$, then $h_j \in R$, and $f_j = fh_j$. Hence we have $\tau = \sum_{i=1}^{n} h_i \tau_i \in R\mathfrak{G}$, and the proof of Lemma 4.1 is complete.

The isomorphism between T and $R \otimes \mathfrak{G}$ shows immediately that T is a rational G-module. We put $T^* = \mathrm{Hom}_R(T, R)$, and we make T^* into a G-module such that, for $x \in G$, $\alpha \in T^*$ and $\tau \in T$,

$$(x \cdot \alpha)(\tau) = \alpha(x^{-1} \cdot \tau) \cdot x^{-1}.$$

Clearly, this G-module T^* may be identified with $\mathrm{Hom}_F(\mathfrak{G}, R)$, with G acting such that $(x \cdot h)(\tau) = h(\tau) \cdot x^{-1}$, where $h \in \mathrm{Hom}_F(\mathfrak{G}, R)$ and $\tau \in \mathfrak{G}$. Hence T^* is a rational G-module. Let $E(T^*)$ denote the exterior R-algebra built over T^*. The rational G-module structure on T^* extends canonically to a rational G-module structure on $E(T^*)$. Put $\mathfrak{G}^* = \mathrm{Hom}_F(\mathfrak{G}, F)$, and let $E(\mathfrak{G}^*)$ be the exterior F-algebra built over $\mathfrak{G}^*$. Then it is clear that the G-module $E(T^*)$ may be identified with $E(\mathfrak{G}^*) \otimes R$, where the action of G is given by $x \cdot (u \otimes f) = u \otimes (f \cdot x^{-1})$.

For $\tau \in T$, we denote by c_τ the homogeneous R-linear antiderivation of degree -1 on $E(T^*)$ that is characterized by: $c_\tau(R) = (0)$, and $c_\tau(\alpha) = \alpha(\tau)$, for every $\alpha \in T^*$. Also, we define the homogeneous F-linear derivation t_τ of degree 0 on $E(T^*)$ such that t_τ coincides with τ on R while, for $\alpha \in T^*$, $t_\tau(\alpha)(\sigma) = \tau(\alpha(\sigma)) + \alpha([\sigma, \tau])$, for all $\sigma \in T$. One verifies inductively on the degree that $c_\tau^2 = 0$, $[t_\sigma, t_\tau] = t_{[\sigma,\tau]}$, and $[t_\sigma, c_\tau] = c_{[\sigma,\tau]}$. Finally, one shows, again inductively on the degree, that there is one and only one homogeneous F-linear derivation d of degree 1 on $E(T^*)$ such that $dc_\tau + c_\tau d = t_\tau$. Then one shows inductively that $dt_\tau = t_\tau d$ and $d^2 = 0$. All these results are familiar, and we do not carry out the details of their proofs. For the general technique of derivations that is involved here, see [1, Ch. 1]. The complex $(E(T^*), d)$ is called the *complex of the differential forms on G.*

For $x \in G$, we denote by s_x the corresponding F-linear automorphism of $E(T^*)$, for the rational G-module structure defined above. One verifies easily that $s_x c_\tau s_x^{-1} = c_{x\cdot\tau}$, and that $s_x t_\tau s_x^{-1} = t_{x\cdot\tau}$. It follows at once from this and from the above characterization of d that d commutes with s_x. We wish to show that the action of G on the cohomology group $H(E(T^*), d)$ that is induced by the $(s_x)_{x\in G}$ is trivial.

In order to prove this, we consider a certain involution φ of $E(T^*)$, and conjugate the G-module structure $x \to s_x$ with φ. For $f \in R$, we define $\varphi(f) \in R$ by $\varphi(f)(x) = f(x^{-1})$. This induces an involution $\tau \to \tau' = \varphi\tau\varphi$ of T, so that, on R, we have $\varphi t_\tau = t_{\tau'}\varphi$. Now we extend φ to an involution of $R + T^*$ so as to satisfy $\varphi c_\tau = c_{\tau'}\varphi$. Thus, for $\alpha \in T^*$, we define $\varphi(\alpha) \in T^*$ by $\varphi(\alpha)(\tau) = \varphi(\alpha(\tau'))$. Using that $[\sigma, \tau]' = [\sigma', \tau']$, one verifies directly that $\varphi t_\tau = t_{\tau'}\varphi$ also on T^*. Now we extend φ canonically to a homogeneous F-algebra automorphism of degree 0 on $E(T^*)$. Then we have $\varphi^2 =$ identity map, $\varphi c_\tau = c_{\tau'}\varphi$, and $\varphi t_\tau = t_{\tau'}\varphi$, on all of $E(T^*)$. Now we define a new G-module structure $x \to t_x$ on $E(T^*)$ by setting $t_x = \varphi s_x \varphi$. Clearly, this is again the structure of a *rational* G-module on $E(T^*)$.

One sees immediately that $t_x(f) = x\cdot f$, for every $f \in R$. Now let $\alpha \in T^*$. Then we have

$$\begin{aligned} t_x(\alpha)(\tau) = \varphi s_x \varphi(\alpha)(\tau) = \varphi(s_x \varphi(\alpha)(\tau')) &= \varphi(\varphi(\alpha)(x^{-1}\cdot\tau')\cdot x^{-1}) \\ &= x\cdot(\varphi(\varphi(\alpha)(x^{-1}\cdot\tau'))) = x\cdot\alpha((x^{-1}\cdot\tau')'). \end{aligned}$$

Now

$$\begin{aligned} (x^{-1}\cdot\tau')'(f) = \varphi((x^{-1}\cdot\tau')(\varphi(f))) &= \varphi(\tau'(\varphi(f)\cdot x^{-1})\cdot x) \\ &= \varphi(\tau'(\varphi(x\cdot f))\cdot x) = x^{-1}\cdot\varphi(\tau'(\varphi(x\cdot f))) = x^{-1}\cdot\tau(x\cdot f). \end{aligned}$$

Define $\tau\cdot x \in T$ by $(\tau\cdot x)(f) = x^{-1}\cdot\tau(x\cdot f)$. Then our result is that

$$t_x(\alpha)(\tau) = x\cdot\alpha(\tau\cdot x).$$

We see from this that the differential of the rational representation $x \to t_x$ is the map $\tau \to t_\tau$ on $\mathfrak{G}$.

Since $dt_\tau = t_\tau d$, and since G is irreducible, it follows that d commutes with each t_x. Now let $u \in E(T^*)$, and suppose that $d(u) = 0$. Then $t_\tau(u) = d(c_\tau(u)) \in d(E(T^*))$. Since G is irreducible, and since $E(T^*)$ is a rational G-module under the representation $x \to t_x$, it follows from this (applied with $\tau \in \mathfrak{G}$) that $t_x(u) - u \in d(E(T^*))$, for every $x \in G$. Now we claim that d

commutes with φ; indeed, we have $\varphi\, d\varphi c_\tau + c_\tau\, \varphi\, d\varphi = \varphi\, dc_{\tau'}\, \varphi + \varphi c_{\tau'}\, d\varphi = \varphi t_{\tau'}\, \varphi = t_\tau$, whence $\varphi\, d\varphi = d$, or $\varphi d = d\varphi$. Hence our above result gives $\varphi(t_x(u) - u) \in d(E(T^*))$, i.e., $s_x(\varphi(u)) - \varphi(u) \in d(E(T^*))$. Since $d(\varphi(u)) = \varphi(d(u)) = 0$, all this applies also to $\varphi(u)$ in the place of u, and we conclude that $s_x(u) - u \in d(E(T^*))$. Thus we have shown that the action of G on $H(E(T^*), d)$ that is induced by the representation $x \to s_x$ is trivial.

THEOREM 4.1. *Let G be an irreducible algebraic linear group over a field F of characteristic* 0. *Let $\mathfrak{G}$ be the Lie algebra of G, and let $\mathfrak{N}$ be the maximum ideal of $\mathfrak{G}$ consisting of nilpotent endomorphisms. Let ρ be the linear monomorphism $E((\mathfrak{G}/\mathfrak{N})^*) \to E(T^*)$ that is obtained as the canonical extension of the dual of the natural epimorphism $\mathfrak{G} \to \mathfrak{G}/\mathfrak{N}$. Then ρ induces an isomorphism of the ordinary Lie algebra cohomology group $H(\mathfrak{G}/\mathfrak{N}, F)$ onto $H(E(T^*), d)$.*

Proof. We use the semidirect product decomposition $G = N \cdot K$, where N is the maximum unipotent normal subgroup of G, and K is a fully reducible algebraic subgroup of G. The Lie algebra of N is $\mathfrak{N}$. We recall that $H(\mathfrak{G}/\mathfrak{N}, F)$ may be identified with the cohomology group $H(E((\mathfrak{G}/\mathfrak{N})^*), \delta)$, where δ is defined in a manner strictly analogous to that in which we defined d above; i.e., we have a homogeneous antiderivation c_σ of degree -1 on $E((\mathfrak{G}/\mathfrak{N})^*)$, for each $\sigma \in \mathfrak{G}/\mathfrak{N}$, and a homogeneous derivation t_σ of degree 0, and δ is characterized by $c_\sigma\, \delta + \delta c_\sigma = t_\sigma$. The definitions of the c_σ and t_σ are obtained from the corresponding definitions above simply by putting $R = F$. Let $\tau \to \tau'$ indicate the natural epimorphism $\mathfrak{G} \to \mathfrak{G}/\mathfrak{N}$. Then we have evidently $c_\tau\, \rho = \rho c_{\tau'}$, and it is easily verified that $t_\tau\, \rho = \rho t_{\tau'}$. Hence we have

$$c_\tau(d\rho - \rho\delta) = (t_\tau - dc_\tau)\rho - c_\tau\, \rho\delta = \rho t_{\tau'} - d\rho c_{\tau'} - \rho c_{\tau'}\, \delta = (\rho\delta - d\rho)c_{\tau'},$$

and this enables us to see by induction on the degree that $\rho\delta = d\rho$.

Since $E(T^*)$ is a rational G-module, under the representation $x \to s_x$, and since K is fully reducible, it follows that $E(T^*)$ is semisimple as a K-module. We have seen that the action of G (hence of K) on $H(E(T^*), d)$ is trivial. Hence we conclude that $H(E(T^*)) = H(E(T^*)^K)$. We have $E(T^*)^K = E(\mathfrak{G}^*) \otimes R^K$, where R^K denotes the subalgebra of R consisting of the elements left fixed by the *right* translations with the elements of K. Since $\rho\delta = d\rho$, and since $\rho(E((\mathfrak{G}/\mathfrak{N})^*)) \subset E(\mathfrak{G}^*) \subset E(T^*)^K$, we may identify $E((\mathfrak{G}/\mathfrak{N})^*)$ with a subcomplex of $E(T^*)^K$. Hence Theorem 4.1 will be proved as soon as we have shown that the cohomology groups of the factor complex are all trivial. Let P denote this factor complex.

Let J denote the ideal of $E(T^*)^K$ that is generated by $(\mathfrak{G}/\mathfrak{N})^*$, where $(\mathfrak{G}/\mathfrak{N})^*$ is now identified with its natural image in $\mathfrak{G}^*$. We define a decreasing sequence of subcomplexes of P by setting

$$P_q = (E((\mathfrak{G}/\mathfrak{N})^*) + J^q)/E((\mathfrak{G}/\mathfrak{N})^*),$$

agreeing that $P_q = P$ for $q \leqq 0$. We have $P_q = (0)$ when q exceeds the dimension of $\mathfrak{G}/\mathfrak{N}$. Hence, in order to show that $H(P) = (0)$, it suffices to prove that $H(P_q/P_{q+1}) = (0)$, for all q. We may write

$$P_q/P_{q+1} = E^q((\mathfrak{G}/\mathfrak{N})^*)E(\mathfrak{G}^*)R^K/(E^q((\mathfrak{G}/\mathfrak{N})^*) + J^{q+1}).$$

In the notation of Section 3, R^K is the isomorphic image $R(N)^+$ of $R(N)$ in $R(G)$. The natural map $\mathfrak{G}^* \to \mathfrak{N}^*$ and the restriction isomorphism $R^K \to R(N)$ combine to give an epimorphism $E(\mathfrak{G}^*)R^K \to E(\mathfrak{N}^*)R(N)$, whose kernel is evidently our ideal J. The image $E(\mathfrak{N}^*)R(N)$ is the complex of the differential forms on N, and it is easy to verify that our epimorphism is compatible with the differential operators, i.e., that it is an epimorphism of complexes. Hence we see from the above form of P_q/P_{q+1} that $H(P_q/P_{q+1})$ is isomorphic with $E^q((\mathfrak{G}/\mathfrak{N})^*) \otimes H(E(\mathfrak{N}^*)R(N)/F)$. Since N is unipotent, $R(N)$ is a polynomial algebra over F [2, p. 123, Prop. 14]. Now it follows from the algebraic version of the "Poincaré Lemma" (see [3, Th. 2.2], with the remark at the end of its proof) that the cohomology groups of the complex $E(N^*)R(N)/F$ are 0 in all degrees. This completes the proof of Theorem 4.1.

5. The rational Ext functor and cohomology

Let G be an algebraic linear group over an infinite field F. Let A and B be rational G-modules, and assume that A is of finite dimension over F. Let $(0) \to B \to X_0 \to X_1 \to \cdots$ be a rationally injective resolution of B. We make $\mathrm{Hom}_F(A, X_i)$ into a G-module such that $(x \cdot h)(a) = x \cdot h(x^{-1} \cdot a)$. Then $\mathrm{Ext}_G(A, B)$ is the cohomology group of the complex formed with the $(\mathrm{Hom}_F(A, X_i))^G$. Now let $A^* = \mathrm{Hom}_F(A, F)$. Since A is finite-dimensional, the G-modules $\mathrm{Hom}_F(A, X_i)$ and $A^* \otimes X_i$ are naturally isomorphic, and each is a rational G-module. By Proposition 2.2, $A^* \otimes X_i$ is rationally injective. Hence the sequence

$$(0) \to A^* \otimes B \to A^* \otimes X_0 \to A^* \otimes X_1 \to \cdots$$

is a rationally injective resolution of the rational G-module $A^* \otimes B$. Thus we have identified $\mathrm{Ext}_G(A, B)$ with the cohomology group of a complex formed with the G-fixed parts of a rationally injective resolution of $A^* \otimes B$, i.e., *we have identified* $\mathrm{Ext}_G(A, B)$ *with* $H(G, A^* \otimes B)$.

On the other hand, let $\mathfrak{G}$ denote the Lie algebra of G, and regard A and B as $\mathfrak{G}$-modules, or as unitary $U(\mathfrak{G})$-modules, where $U(\mathfrak{G})$ is the universal enveloping algebra of $\mathfrak{G}$. It is known that $\mathrm{Ext}_{U(\mathfrak{G})}(A, B)$ *may be identified with the ordinary Lie algebra cohomology group* $H(\mathfrak{G}, \mathrm{Hom}_F(A, B))$, and this holds quite generally for an arbitrary Lie algebra $\mathfrak{G}$ and arbitrary $\mathfrak{G}$-modules A and B. We shall give a short direct proof which is analogous to the proof above.

Observe first that, if X is any injective $\mathfrak{G}$-module, then the $\mathfrak{G}$-module $\mathrm{Hom}_F(A, X)$, where $(u \cdot h)(a) = u \cdot h(a) - h(u \cdot a)$, is also injective. Indeed, let C be a $\mathfrak{G}$-module, and let δ be a $\mathfrak{G}$-module homomorphism of a submodule D of C into $\mathrm{Hom}_F(A, X)$. Then δ defines an element $\delta^* \in \mathrm{Hom}_F(D \otimes A, X)$ such that $\delta^*(d \otimes a) = \delta(d)(a)$. One verifies immediately that δ^* is actually a $\mathfrak{G}$-module homomorphism. Since X is injective, δ^* is the restriction to $D \otimes A$ of a $\mathfrak{G}$-module homomorphism γ^* of $C \otimes A$ into X. Define $\gamma \in \mathrm{Hom}_F(C, \mathrm{Hom}_F(A, X))$ by $\gamma(c)(a) = \gamma^*(c \otimes a)$. Then γ is a $\mathfrak{G}$-module homomorphism of C into $\mathrm{Hom}_F(A, X)$, and γ coincides with δ on D. This proves that $\mathrm{Hom}_F(A, X)$ is injective.

Now let $(0) \to B \to X_0 \to X_1 \to \cdots$ be a $U(\mathfrak{G})$-injective resolution of the $\mathfrak{G}$-module B. Then $\mathrm{Ext}_{U(\mathfrak{G})}(A, B)$ is the cohomology group of the complex formed with the $\mathrm{Hom}_{U(\mathfrak{G})}(A, X_i)$, which is the $\mathfrak{G}$-annihilated part of $\mathrm{Hom}_F(A, X_i)$. By the above, the sequence

$$(0) \to \mathrm{Hom}_F(A, B) \to \mathrm{Hom}_F(A, X_0) \to \mathrm{Hom}_F(A, X_1) \to \cdots$$

is an injective resolution of the $\mathfrak{G}$-module $\mathrm{Hom}_F(A, B)$. Hence we have identified $\mathrm{Ext}_{U(\mathfrak{G})}(A, B)$ with $H(\mathfrak{G}, \mathrm{Hom}_F(A, B))$. Finally, if A is finite-dimensional, we may identify $\mathrm{Hom}_F(A, B)$ with $A^* \otimes B$.

As a consequence of these identifications, the relations we shall establish for the cohomology groups of G and $\mathfrak{G}$ will extend immediately to the more general Ext functors when the first variable A is restricted to be finite-dimensional.

Let $E(\mathfrak{G})$ denote the exterior algebra built over the F-space $\mathfrak{G}$. The cohomology of $\mathfrak{G}$ is usually computed from the $U(\mathfrak{G})$-projective resolution $U(\mathfrak{G}) \otimes E(\mathfrak{G})$ of F. In this resolution, $U(\mathfrak{G}) \otimes E(\mathfrak{G})$ is regarded as a $U(\mathfrak{G})$-module such that $v \cdot (u \otimes e) = (vu) \otimes e$. The boundary map $\gamma : U(\mathfrak{G}) \otimes E^k(\mathfrak{G}) \to U(\mathfrak{G}) \otimes E^{k-1}(\mathfrak{G})$ can be characterized as follows. For $\tau \in \mathfrak{G}$, define the endomorphism ε_τ of $U(\mathfrak{G}) \otimes E(\mathfrak{G})$ by $\varepsilon_\tau(u \otimes e) = u \otimes (\tau e)$. Define a right $\mathfrak{G}$-module structure on $U(\mathfrak{G}) \otimes E(\mathfrak{G})$, $\tau \to \rho_\tau$, such that $\rho_\tau(u \otimes e) = (u \otimes e) \cdot \tau = (u\tau) \otimes e + u \otimes (e \cdot \tau)$, where the action $e \to e \cdot \tau$ on $E(\mathfrak{G})$ is the homogeneous derivation of degree 0 for which $\sigma \cdot \tau = [\sigma, \tau]$. Then γ is characterized, as a homogeneous $U(\mathfrak{G})$-linear endomorphism of degree -1, by $\gamma\varepsilon_\tau + \varepsilon_\tau \gamma = \rho_\tau$. The $U(\mathfrak{G})$-projective resolution of F is obtained by augmenting the complex $(U(\mathfrak{G}) \otimes E(\mathfrak{G}), \gamma)$ with the canonical projection $U(\mathfrak{G}) \otimes E^0(\mathfrak{G}) = U(\mathfrak{G}) \to F$ (see [4, Section 5]).

In order to compare the rational cohomology of the irreducible algebraic linear group G with the cohomology of its Lie algebra $\mathfrak{G}$, we shall define a map of the complex $(E(T^*) \otimes A, d \otimes 1)$ into the complex

$$(\mathrm{Hom}_F(U(\mathfrak{G}) \otimes E(\mathfrak{G}), A), \gamma^*),$$

where A is a rational G-module, and γ^* denotes the dual $\mathrm{Hom}_F(\gamma, A)$ of γ.

We have $E(T^*) = R(G) \otimes E(\mathfrak{G}^*)$. With each element $\alpha \in E^k(\mathfrak{G}^*)$ we associate the element $\alpha^* \in (E^k(\mathfrak{G}))^*$ that is defined by $\alpha^*(\tau_1 \cdots \tau_k) = c_{\tau_k} \cdots c_{\tau_1}(\alpha)$. With each element $f \in R(G)$ we associate the element $f^* \in U(\mathfrak{G})^*$ that is defined by $f^*(u) = (u \cdot f)(1)$, where the action $f \to u \cdot f$

of $U(\mathfrak{G})$ on $R(G)$ is obtained from the $\mathfrak{G}$-module structure of $R(G)$ that is derived from the action of G by left translations $f \to x \cdot f$ on $R(G)$, i.e., for $\tau \in \mathfrak{G}$, $\tau \cdot f = \tau(f)$. Now we define

$$\psi : R(G) \otimes E^k(\mathfrak{G}^*) \otimes A \to \mathrm{Hom}_F(U(\mathfrak{G}) \otimes E^k(\mathfrak{G}), A)$$

such that

$$\psi(f \otimes \alpha \otimes a)(u \otimes e) = f^*(u)\alpha^*(e)a.$$

Evidently, $(c_\tau(\alpha))^* = \alpha^*\varepsilon_\tau$, whence it is clear that $\psi(c_\tau \otimes 1) = (\varepsilon_\tau)^*\psi$. We have $(\tau \cdot f)^*(u) = f^*(u\tau)$, and one verifies directly from the definitions that $(t_\tau(\alpha))^*(e) = \alpha^*(e \cdot \tau)$. It follows at once from these relations that $\psi(t_\tau \otimes 1) = (\rho_\tau)^*\psi$. Hence we obtain

$$\psi(d \otimes 1)(c_\tau \otimes 1) + \psi(c_\tau \otimes 1)(d \otimes 1) = \psi(t_\tau \otimes 1) = (\rho_\tau)^*\psi,$$

i.e.,

$$\begin{aligned}\psi(d \otimes 1)(c_\tau \otimes 1) + (\varepsilon_\tau)^*\psi(d \otimes 1) &= (\varepsilon_\tau)^*\gamma^*\psi + \gamma^*(\varepsilon_\tau)^*\psi \\ &= (\varepsilon_\tau)^*\gamma^*\psi + \gamma^*\psi(c_\tau \otimes 1),\end{aligned}$$

whence $(\varepsilon_\tau)^*(\psi(d \otimes 1) - \gamma^*\psi) + (\psi(d \otimes 1) - \gamma^*\psi)(c_\tau \otimes 1) = 0$. Hence it follows by induction on the degree that $\psi(d \otimes 1) = \gamma^*\psi$.

Now we consider the rational G-module structure on $E(T^*) \otimes A$ that is obtained by forming the tensor product of the G-module structure $x \to s_x$ on $E(T^*)$ with the G-module structure of A, i.e., in which $x \cdot (f \otimes \alpha \otimes a) = (f \cdot x^{-1}) \otimes \alpha \otimes (x \cdot a)$. For $f \in R(G)$, define $f' \in R(G)$ by $f'(x) = f(x^{-1})$. Then we have $f \cdot x^{-1} = (x \cdot f')'$. Hence the $\mathfrak{G}$-module structure of $R(G) \otimes E(\mathfrak{G}^*) \otimes A$ that is induced by our rational G-module structure is such that $\tau \cdot (f \otimes \alpha \otimes a) = (\tau \cdot f')' \otimes \alpha \otimes a + f \otimes \alpha \otimes (\tau \cdot a)$. Now we have $((\tau \cdot f')')^*(u) = (u \cdot (\tau \cdot f')')(1)$. Write $\tau \cdot f = t_\tau(f)$ and $(\tau \cdot f')' = s_\tau(f)$. Then each t_τ commutes with each s_σ, because these operations are derived from the left and the right G-translations, respectively. Hence we have $((\tau \cdot f')')^*(u) = (s_\tau(u \cdot f))(1)$. We claim that, for every $g \in R(G)$ and every $\tau \in \mathfrak{G}$, $(s_\tau(g))(1) = -(t_\tau(g))(1)$. After extension of the base field F to the field of the power series in one variable z with coefficients in F, we have $\exp(zs_\tau)(g) = g \cdot \exp(-z\tau)$ and $\exp(-zt_\tau)(g) = \exp(-z\tau) \cdot g$. Hence $\exp(zs_\tau)(g)(1) = \exp(-zt_\tau)(g)(1)$. This gives our above assertion on comparing the coefficients of z on each side. Hence $(s_\tau(u \cdot f))(1) = -(\tau u \cdot f)(1)$. Thus we have $(s_\tau(f))^*(u) = -f^*(\tau u)$. It follows that ψ transports our $\mathfrak{G}$-module structure on $R(G) \otimes E(\mathfrak{G}^*) \otimes A$ into the $\mathfrak{G}$-module structure of $\mathrm{Hom}_F(U(\mathfrak{G}) \otimes E(\mathfrak{G}), A)$ in which $(\tau \cdot h)(u \otimes e) = \tau \cdot h(u \otimes e) - h(\tau u \otimes e)$.

We claim that, with this $\mathfrak{G}$-module structure, $\mathrm{Hom}_F(U(\mathfrak{G}) \otimes E(\mathfrak{G}), A)$ is injective. Now this $\mathfrak{G}$-module is evidently isomorphic with the direct sum of a finite number of copies of the $\mathfrak{G}$-module $\mathrm{Hom}_F(U(\mathfrak{G}), A)$, where $(\tau \cdot h)(u) = \tau \cdot h(u) - h(\tau u)$, and it suffices to show that this last $\mathfrak{G}$-module is injective. In order to do this, we consider the endomorphism $h \to h^*$, where $h^*(u) = (u \cdot h)(1)$. We claim that $(u \cdot h^*)(v) = (v \cdot h)(u)$. Indeed, this holds for $u = 1$, by the definition of h^*. Suppose it holds for some $u \in U(\mathfrak{G})$ and all $v \in U(\mathfrak{G})$. Let $x \in \mathfrak{G}$. Then

$$(xu \cdot h^*)(v) = x \cdot [(u \cdot h^*)(v)] - (u \cdot h^*)(xv)$$
$$= x \cdot [(v \cdot h)(u)] - (xv \cdot h)(u) = (v \cdot h)(xu).$$

Thus our claim follows by induction. In particular, we have $(u \cdot h^*)(1) = h(u)$, so that $h^{**} = h$. Thus our map $h \rightarrow h^*$ is an F-linear involution of $\mathrm{Hom}_F(U(\mathfrak{G}), A)$. If $x \in \mathfrak{G}$ we have evidently $(x \cdot h)^*(u) = h^*(ux)$. Thus our $\mathfrak{G}$-module structure is transported by the linear automorphism $h \rightarrow h^*$ into the $\mathfrak{G}$-module structure in which $(x \cdot h)(u) = h(ux)$, i.e., into the $U(\mathfrak{G})$-module structure in which $(v \cdot h)(u) = h(uv)$. With this $U(\mathfrak{G})$-module structure, $\mathrm{Hom}_F(U(\mathfrak{G}), A)$ is well known to be injective, from general facts of homological algebra (see [4, Lemma 1]).

If we augment the complex $(\mathrm{Hom}_F(U(\mathfrak{G}) \otimes E(\mathfrak{G}), A), \gamma^*)$ with the $\mathfrak{G}$-monomorphism $A \rightarrow \mathrm{Hom}_F(U(\mathfrak{G}), A)$ defined by $a \rightarrow h_a$, where $h_a(u) = u_0 a$, with u_0 the component of u in F, we obtain an acyclic complex, because this is the dual $\mathrm{Hom}_F(X, A)$ of the usual projective resolution X of F that we described above. Moreover, by what we have just proved, this is an injective resolution of the $\mathfrak{G}$-module A.

Finally, we shall show that the restriction of ψ to the G-fixed part $(E(T^*) \otimes A)^G$ of $E(T^*) \otimes A$ is an isomorphism of $(E(T^*) \otimes A)^G$ onto $(\mathrm{Hom}_F(U(\mathfrak{G}) \otimes E(\mathfrak{G}), A))^{\mathfrak{G}} = \mathrm{Hom}_{U(\mathfrak{G})}(U(\mathfrak{G}) \otimes E(\mathfrak{G}), A)$. We have $(E(T^*) \otimes A)^G = (E(T^*) \otimes A)^{\mathfrak{G}}$, and since ψ is a $\mathfrak{G}$-module homomorphism, it follows that ψ maps $(E(T^*) \otimes A)^G$ into $\mathrm{Hom}_{U(\mathfrak{G})}(U(\mathfrak{G}) \otimes E(\mathfrak{G}), A)$. We may identify $\mathrm{Hom}_{U(\mathfrak{G})}(U(\mathfrak{G}) \otimes E(\mathfrak{G}), A)$ with $\mathrm{Hom}_F(E(\mathfrak{G}), A) = (E(\mathfrak{G}))^* \otimes A$ by the map $h \rightarrow h'$, where $h'(e) = h(1 \otimes e)$. Hence there remains only to prove that the map $v \rightarrow \psi(v)'$ is an isomorphism of $(R(G) \otimes E(\mathfrak{G}^*) \otimes A)^G$ onto $E(\mathfrak{G})^* \otimes A$. We have $(\psi(f \otimes \alpha \otimes a))' = f(1)\alpha^* \otimes a$. Put $V = E(\mathfrak{G})^* \otimes A$, and regard V as a rational G-module such that $x \cdot (\alpha^* \otimes a) = \alpha^* \otimes (x \cdot a)$. Since the map $\alpha \rightarrow \alpha^*$ is an isomorphism of $E(\mathfrak{G}^*)$ onto $E(\mathfrak{G})^*$, what we have to prove amounts simply to the proposition that the map $\sum f \otimes v \rightarrow \sum f(1)v$ is an isomorphism of $(R(G) \otimes V)^G$ onto V.

Suppose that $\sum f(1)v = 0$. Then also $\sum f(1)x \cdot v = 0$, for every $x \in G$. Since $\sum f \otimes v \in (R(G) \otimes V)^G$, this implies that $\sum f(x)v = 0$, whence $\sum f \otimes v = 0$. Thus our map is a monomorphism. Now let $v \in V$, let W be the finite-dimensional G-submodule of V that is generated by v, and let $(w_1, \cdots, w_n)$ be a basis for W. Let $(\varphi_1, \cdots, \varphi_n)$ be the dual basis for $\mathrm{Hom}_F(W, F)$. Define $\varphi_i/w_j \in R(G)$ by $(\varphi_i/w_j)(x) = \varphi_i(x \cdot w_j)$. Then one verifies directly that $\sum_{i=1}^{n} (\varphi_i/w_j) \otimes w_i$ belongs to $(R(G) \otimes V)^G$ and that its image in V is w_j. Hence our map is also an epimorphism. Thus we have proved the following result.

LEMMA 5.1. *Let G be an irreducible algebraic linear group over a field F of characteristic 0, and let A be a rational G-module. Let $E(T^*)$ be the complex of the differential forms on G. Then there is a $U(\mathfrak{G})$-injective resolution Y of A and a homomorphism ψ of the complex $E(T^*) \otimes A$ into Y such that ψ is compatible with the $\mathfrak{G}$-module structures of $E(T^*) \otimes A$ and Y, and maps $(E(T^*) \otimes A)^G$ isomorphically onto $Y^{\mathfrak{G}}$. In particular, ψ induces an isomorphism of the cohomology group $H((E(T^*) \otimes A)^G)$ onto the Lie algebra cohomology group $H(\mathfrak{G}, A)$.*

Let γ be a rational automorphism, with a rational inverse, of the algebraic linear group G. Let A be a rational G-module, and suppose that we are given a linear automorphism γ_A of A such that, for every $a \in A$ and every $x \in G$, $\gamma_A(x \cdot a) = \gamma(x) \cdot \gamma_A(a)$. Let

$$(0) \to A \xrightarrow{\alpha} X_0 \to X_1 \to \cdots$$

be a rationally injective resolution of A. Let $X_i(\gamma)$ denote the rational G-module whose underlying vector space coincides with that of X_i and where the G-operations are "twisted" by γ, i.e., the action of x on $X_i(\gamma)$ is the action of $\gamma(x)$ on X_i, for every $x \in G$. Then, if we replace the first map α of our above resolution by $\alpha \circ \gamma_A$, leave the other maps the same, and replace each X_i by $X_i(\gamma)$, we obtain a new rationally injective resolution of A. The identity map $A \to A$ extends to a map of the first resolution into the second, and the extension is unique up to a homotopy. This means that there is an F-linear endomorphism of the complex X, say γ_X, with the same property we described above for γ_A. Moreover, γ_X is unique up to a homotopy of the complex X, which twists the G-action by γ in the same way as γ_X. Clearly, each $(X_i)^G$ is stable under γ_X, and it follows that the pair (γ, γ_A) defines, via γ_X, an endomorphism of the cohomology group $H(G, A)$. By the uniqueness up to a homotopy of γ_X, this endomorphism does not depend on the particular choice of γ_X. Moreover, this endomorphism of $H(G, A)$ is actually an automorphism, its inverse being induced by a map $(\gamma^{-1})_X$. Finally, our argument can be continued in the usual fashion to show that we thus obtain the *same* automorphism of $H(G, A)$ from *any* rationally injective resolution of A.

In particular, suppose that G is a normal algebraic subgroup of an algebraic linear group M, and that there is a rational representative map $\rho : M \to G$ such that $\rho(xy) = x\rho(y)$, for every $x \in G$ and every $y \in M$. Then, by Proposition 2.2, if A is a rational M-module, a rationally injective resolution X of A is also a rationally injective resolution of the G-module A. Let $y \in M$, and let γ be the automorphism of G defined by $\gamma(x) = yxy^{-1}$. Let γ_A be the automorphism of A that is effected by y. Then we may take for γ_X the automorphism of X that is effected by y. This shows immediately that $H(G, A)$ is thus endowed with the structure of a rational M-module, and also that G acts trivially on $H(G, A)$.

The rational automorphism γ of G induces an automorphism of the Lie algebra $\mathfrak{G}$ of G in the natural fashion. We use the same letter γ to denote this automorphism of $\mathfrak{G}$. If A is regarded as a $\mathfrak{G}$-module, we have $\gamma_A(\tau \cdot a) = \gamma(\tau) \cdot \gamma_A(a)$, for every $\tau \in \mathfrak{G}$. Proceeding in exactly the same way as above with $U(\mathfrak{G})$-injective resolutions of A, we see that (γ, γ_A) induces an automorphism of the Lie algebra cohomology group $H(\mathfrak{G}, A)$. For $G \subset M$, as above, we thus obtain the structure of an M-module on $H(\mathfrak{G}, A)$. If the action of M on $H(\mathfrak{G}, A)$ is made explicit in the standard alternating cochain complex, one sees that $H(\mathfrak{G}, A)$ is a *rational* M-module. If G is irreducible the action of G on $H(\mathfrak{G}, A)$ is trivial.

THEOREM 5.1. *Let G be a unipotent algebraic linear group over the field F of characteristic* 0, *and let A be a rational G-module. There is an isomorphism φ of the rational cohomology group $H(G, A)$ onto the Lie algebra cohomology group $H(\mathfrak{G}, A)$ with the following property. Given any rationally injective resolution X of the rational G-module A, and any $U(\mathfrak{G})$-injective resolution Y of the $U(\mathfrak{G})$-module A, there is a map of X, regarded as an acyclic $U(\mathfrak{G})$-complex, into Y, and the cohomology map induced by any such map of complexes coincides with the isomorphism φ.*

Proof. By Proposition 2.2, the rational G-module $E(T^*) \otimes A$ is rationally injective. By Theorem 4.1, the complex $E(T^*) \otimes A$, augmented by the G-module monomorphism $a \to 1 \otimes a$ of A into $E^0(T^*) \otimes A$, is therefore a rationally injective resolution of A. Hence the isomorphism of Lemma 5.1 is actually an isomorphism φ of $H(G, A)$ onto $H(\mathfrak{G}, A)$ in this case, i.e., when G is unipotent. There remains only to show the following: Let X and X_1 be any two rationally injective resolutions of A as a rational G-module, and let Y and Y_1 be any two $U(\mathfrak{G})$-injective resolutions of A as a $U(\mathfrak{G})$-module. Let ψ and ψ_1 be $U(\mathfrak{G})$-complex maps $X \to Y$ and $X_1 \to Y_1$, respectively. Then ψ and ψ_1 induce the same cohomology map. In order to see this, let ζ be a G-complex map $X \to X_1$, and let β be a $U(\mathfrak{G})$-complex map $Y_1 \to Y$. Then $\beta \circ \psi_1 \circ \zeta$ and ψ are both $U(\mathfrak{G})$-complex maps $X \to Y$. Since any two such maps are homotopic, they induce the same cohomology maps. The cohomology maps induced by ζ and β are the identity maps on $H(G, A)$ and $H(\mathfrak{G}, A)$, respectively. Hence the cohomology maps induced by ψ and ψ_1 are indeed the same. This completes the proof of Theorem 5.1.

Now consider a rational automorphism γ of G and a compatible automorphism γ_A of A, as discussed above. We claim that the isomorphism $\varphi : H(G, A) \to H(\mathfrak{G}, A)$ transports the automorphism of $H(G, A)$ that is induced by (γ, γ_A) into the automorphism of $H(\mathfrak{G}, A)$ that is induced by (γ, γ_A). To see this, let ψ be a map of a rationally injective resolution X of A into a $U(\mathfrak{G})$-injective resolution Y of A. Let γ_X be a "twisting endomorphism" of X inducing the automorphism of $H(G, A)$, and let $(\gamma^{-1})_Y$ be a "twisting endomorphism" of Y inducing the automorphism of $H(\mathfrak{G}, A)$ that corresponds to $(\gamma^{-1}, (\gamma_A)^{-1})$. Then $(\gamma^{-1})_Y \circ \psi \circ \gamma_X$ is still a $U(\mathfrak{G})$-

complex map $X \to Y$, so that the induced cohomology map $H(G, A) \to H(\mathfrak{G}, A)$ is φ. Hence, denoting the cohomology automorphisms induced by (γ, γ_A) by γ (and similarly for γ^{-1}), we have $\gamma^{-1} \circ \varphi \circ \gamma = \varphi$, which proves our claim.

The isomorphism $H(G, A) \to H(\mathfrak{G}, A)$ of Theorem 5.1 will be referred to as the *canonical* isomorphism. Let G be an arbitrary algebraic linear group, and let N be the maximum unipotent normal subgroup of G. Let $\mathfrak{N}$ be the Lie algebra of N. Then it is clear from our discussion above that *the canonical isomorphism $H(N, A) \to H(\mathfrak{N}, A)$ is a G/N-module isomorphism.*

THEOREM 5.2. *Let G be an algebraic linear group over the field F of characteristic* 0, *let N be the maximum unipotent normal subgroup of G, and let $\mathfrak{N}$ be the Lie algebra of N. Let A be a rational G-module. Then $H(G, A)$ may be identified with $(H(N, A))^{G/N}$, and hence is isomorphic, via the canonical isomorphism, with $H(\mathfrak{N}, A)^{G/N}$.*

Proof. Let X be a rationally injective resolution of A. By Proposition 2.2, X is also a rationally injective resolution of A as an N-module. Hence $H(N, A)$ may be identified with the cohomology group of the complex X^N, and the action of G/N on $H(N, A)$ is induced by the action of G/N (i.e., of G) on X^N. Since G/N is a fully reducible algebraic linear group (for the algebraic structure $R(G/N) = R(G)^N$) and since X^N is a rational G/N-module, X^N is semisimple as a G/N-module, i.e., as a G-module. Since the coboundary map of the complex X^N is a G-module endomorphism, it follows that $H(X^N)^{G/N} = H(X^G)$. Thus we have identified $H(N, A)^{G/N}$ with $H(G, A)$. Together with what we have already seen above, this completes the proof of Theorem 5.2.

Now suppose that G is irreducible. The rational G/N-module $H(\mathfrak{N}, A)$ may be regarded as a $\mathfrak{G}/\mathfrak{N}$-module, and since G/N is irreducible, we have $H(\mathfrak{N}, A)^{G/N} = H(\mathfrak{N}, A)^{\mathfrak{G}/\mathfrak{N}}$. If one examines the proof of [6, Th. 13], observing that all the representations of $\mathfrak{G}/\mathfrak{N}$ that occur in it are semisimple in our present case (being derived from rational representations of G/N), one sees that it carries over exactly to show that $H^n(\mathfrak{G}, A)$ is naturally isomorphic with $\sum_{i+j=n} H^i(\mathfrak{G}/\mathfrak{N}, F) \otimes (H^j(\mathfrak{N}, A))^{G/N}$. Substituting, for the terms on the right, isomorphic terms obtained from Theorems 4.1 and 5.2, we obtain the following result.

THEOREM 5.3. *Let G be an irreducible algebraic linear group over a field F of characteristic* 0, *let A be a rational G-module, and let $E(T^*)$ be the complex of the differential forms on G. Then there is a natural isomorphism*

$$H^n(\mathfrak{G}, A) \approx \textstyle\sum_{i+j=n} H^i(E(T^*)) \otimes H^j(G, A),$$

for each n.

6. Group extensions

Let N be a unipotent algebraic linear group over the field F of characteristic 0. A rational automorphism α of N induces a Lie algebra automorphism

α^* of the Lie algebra $\mathfrak{N}$ of N such that $\alpha(\exp(\zeta)) = \exp(\alpha^*(\zeta))$, for every $\zeta \in \mathfrak{N}$. Conversely, a Lie algebra automorphism β of $\mathfrak{N}$ induces a rational automorphism β' of N, which is given by $\beta'(x) = \exp(\beta(\log(x)))$, for every $x \in N$, so that $(\beta')^* = \beta$. Thus the map $\beta \to \beta^*$ is a group isomorphism of the group $A(N)$ of all rational automorphisms of N onto the group $A(\mathfrak{N})$ of all Lie algebra automorphisms of $\mathfrak{N}$. Evidently, $A(\mathfrak{N})$ is an algebraic linear group. Hence $A(N)$ may be regarded as an algebraic linear group by identifying it with $A(\mathfrak{N})$.

We observe that the algebra of the rational representative functions on $A(N)$ can be defined without reference to the Lie algebra $\mathfrak{N}$. In fact, for every pair (f, x), where $f \in R(N)$ and $x \in N$, let us define the function $p_{(f,x)}$ on $A(N)$ by setting $p_{(f,x)}(\alpha) = f(\alpha(x))$. The composite $f \circ \exp$ is a polynomial function on $\mathfrak{N}$, and we have $f(\alpha(x)) = (f \circ \exp)(\alpha^*(\log(x)))$. Hence it is clear that $p_{(f,x)}$ is a polynomial function on $A(N)$, for the algebraic structure transported from $A(\mathfrak{N})$. Conversely, the polynomial functions on $A(\mathfrak{N})$ are the polynomials in the functions $\alpha^* \to g(\alpha^*(\zeta))$, where g ranges over the space of the linear functions on $\mathfrak{N}$, and ζ ranges over $\mathfrak{N}$. We have

$$g(\alpha^*(\zeta)) = (g \circ \log)(\alpha(\exp(\zeta))) = p_{(g \circ \log, \exp(\zeta))}(\alpha).$$

Since $g \circ \log$ is a polynomial function on N, we may therefore conclude that the algebra of the polynomial functions on $A(N)$ is precisely the algebra generated by the functions $p_{(f,x)}$. The algebra $R(A(N))$ of the rational representative functions on $A(N)$ is therefore the algebra generated by the functions $p_{(f,x)}$ and their "duals" $(p_{(f,x)})'$, where $(p_{(f,x)})'(\alpha) = p_{(f,x)}(\alpha^{-1})$.

Now let K be another algebraic linear group over F, and let ρ be a rational homomorphism of K into $A(N)$. As an abstract group, the semidirect product $N \times_\rho K$ (or simply $N \cdot K$) is the group of all pairs (x, y) with $x \in N$ and $y \in K$, where $(x_1, y_1)(x_2, y_2) = (x_1 \rho(y_1)(x_2), y_1 y_2)$. We wish to define the structure of an algebraic linear group on $N \cdot K$, such that N and K become identifiable with algebraic subgroups of $N \cdot K$, and such that the projection $N \cdot K \to K$ induces a rational isomorphism, with rational inverse, of $(N \cdot K)/N$ onto K, where $(N \cdot K)/N$ is regarded as an algebraic linear group as in Theorem 3.1.

We identify the elements of $R(K) \otimes R(N)$ with functions on $N \cdot K$ such that $(f \otimes g)(x, y) = f(y)g(\rho(y)^{-1}(x))$. The algebraic structure on $N \cdot K$ will be such that $R(N \cdot K) = R(K) \otimes R(N)$. It is verified directly that, as an algebra of functions on $N \cdot K$, $R(K) \otimes R(N)$ is stable under the left translations. In fact, we have

$$(x, y) \cdot (f \otimes g) = (y \cdot f) \otimes (x \cdot (g \circ \rho(y)^{-1})).$$

Next we show that our algebra of functions on $N \cdot K$ is stable also under the involution $h \to h'$, where $h'(u) = h(u^{-1})$. We have $(x, y)^{-1} = (\rho(y)^{-1}(x^{-1}), y^{-1})$. Hence

$$(f \otimes g)'(x, y) = (f \otimes g)(\rho(y)^{-1}(x^{-1}), y^{-1}) = f(y^{-1})g(x^{-1}) = f'(y)g'(x).$$

Hence it suffices to show that, if $h \in R(N)$, the function h^*, defined by $h^*(x, y) = h(x)$, belongs to $R(K) \otimes R(N)$. We may write $h^*(x, y) = (h \circ \rho(y))(\rho(y)^{-1}(x))$. As y ranges over K, the functions $h \circ \rho(y)$ remain in a certain *finite-dimensional* subspace of $R(N)$, because $h \circ \rho(y) = h \circ \exp \circ \log \circ \rho(y) = h \circ \exp \circ \rho(y)^* \circ \log$. We can find a basis $h_1, \cdots, h_n$ for this space, and elements $x_1, \cdots, x_n$ in N, such that $h_i(x_j) = \delta_{ij}$. Now we write $h \circ \rho(y) = \sum_{i=1}^{n} u_i(y)h_i$, and we get $u_i(y) = h(\rho(y)(x_i))$. Thus, in the notation introduced at the beginning of this section, $u_i = p_{(h,x_i)} \circ \rho \in R(K)$. Now we have $h^* = \sum_{i=1}^{n} u_i \otimes h_i \in R(K) \otimes R(N)$. Thus $R(K) \otimes R(N)$ is stable under the involution $h \to h'$. Since it is stable under the left translations, it follows that it is stable also under the right translations.

Now we show that every proper automorphism of $R(K) \otimes R(N)$ is a left translation. Let α be any proper automorphism of $R(K) \otimes R(N)$. Then the restriction of α to $R(K)$ is a proper automorphism of $R(K)$, regarded as the algebra of representative functions on K. By Lemma 3.1, there is an element $y \in K$ such that α coincides on $R(K)$ with the left translation by y. Hence we may now assume that α leaves the elements of $R(K)$ fixed. Now α induces a unitary homomorphism $\alpha' : R(N) \to F$, where $\alpha'(g) = \alpha(g)(1)$. By Lemma 3.1 and [5, Prop. 2.5], there is an element $x \in N$ such that $\alpha'(g) = g(x)$, for every $g \in R(N)$. Let t_x denote the left translation by $(1, x)$ on $R(K) \otimes R(N)$, and put $\beta = t_x^{-1} \alpha$. Then β leaves the elements of $R(K)$ fixed, and $\beta'(g) = g(1)$, for every $g \in R(N)$. Hence $\beta'(h) = h(1)$, for every $h \in R(K) \otimes R(N)$. Since β commutes with the right translations, this implies that β is the identity map, i.e., that $\alpha = t_x$. Thus we have proved that every proper automorphism of $R(K) \otimes R(N)$ is a left translation.

Now $R(K) \otimes R(N)$ is finitely generated, and the translates of every element of $R(K) \otimes R(N)$ lie in a finite-dimensional subspace of $R(K) \otimes R(N)$. Hence there is a finite-dimensional subspace T of $R(K) \otimes R(N)$ such that T is stable under the left and the right translations by the elements of $N \cdot K$ and $F[T] = R(K) \otimes R(N)$. The representation of $N \cdot K$ by left translations on T is evidently faithful. The image of $N \cdot K$ in the group of linear automorphisms of T coincides with the restriction image of the group of all proper automorphisms of $R(K) \otimes R(N)$, by what we have proved above. The proof of [5, Prop. 2.6] applies to the present case and shows that this last image is an algebraic subgroup of the group of all linear automorphisms of T. Thus we have a faithful representation of $N \cdot K$ as an algebraic linear group. It is clear from the construction that the algebra of representative functions for this algebraic structure of $N \cdot K$ is precisely $R(K) \otimes R(N)$. The restrictions to N and to K of the elements of $R(K) \otimes R(N)$ make up precisely $R(N)$ and $R(K)$, respectively. Moreover, the subgroup $N = (N, 1)$ of $N \cdot K$ is the subgroup consisting of all elements $x \in N \cdot K$ such that $(f - f(1))(x) = 0$, for every $f \in R(K)$, and the subgroup $K = (1, K)$ of $N \cdot K$ is the subgroup of all elements $x \in N \cdot K$ such that $(g - g(1))(x) = 0$, for all $g \in R(N)$. Hence N and K are identified with algebraic subgroups of

$N \cdot K$. Finally, $(R(K) \otimes R(N))^{(1,N)} = R(K)$, whence K is rationally isomorphic with $(N \cdot K)/N$, when $(N \cdot K)/N$ has the algebraic structure of Theorem 3.1.

Let P be an algebraic linear group, and let φ be a rational group epimorphism of P onto an algebraic linear group G. Let Q be the kernel of φ. We say that the pair (P, φ) is a *rational group extension* of Q by G if the map $f \rightarrow f \circ \varphi$ is an isomorphism of $R(G)$ *onto* $R(P)^Q$. Suppose that we are given a vector group Q (regarded as a unipotent linear algebraic group), an arbitrary algebraic linear group G, and a rational homomorphism ρ of G into the group $A(Q)$ of the rational automorphisms of Q (actually, $A(Q)$ is simply the group of all linear automorphisms of the vector group Q). We wish to examine the rational group extensions (P, φ) of Q by G that are compatible with ρ, in the sense that $\rho(\varphi(p))(q) = pqp^{-1}$, for all $p \in P$ and all $q \in Q$. We assume that the base field F is of characteristic 0. Then it is clear from Theorem 3.1 that, by means of the Baer composition of group extensions, one obtains the structure of an abelian group on the set of the rational equivalence classes of these extensions. Moreover, a well known construction that is quite similar to the Baer composition yields scalar multiplications by elements of F, so that *the set of the rational equivalence classes of the extensions of Q by G that are compatible with ρ is equipped with the structure of a vector space over F.* We wish to prove that this vector space is naturally isomorphic with the rational cohomology space $H^2(G, \mathfrak{Q})$. Here $\mathfrak{Q}$ is the Lie algebra of Q. Since Q is a commutative unipotent algebraic group, we could identify $\mathfrak{Q}$ with Q. We have already seen above that $A(Q)$ may be compatibly identified with $A(\mathfrak{Q})$. The G-module structure on $\mathfrak{Q}$ is obtained by transporting ρ through the identification of $A(Q)$ with $A(\mathfrak{Q})$.

We shall use the following rationally injective resolution X of the G-module $\mathfrak{Q}$. For each $n \geqq 0$, X_n is the tensor product $R \otimes \cdots \otimes R \otimes \mathfrak{Q}$, with $n+1$ factors $R = R(G)$. The G-module structure on X_n is such that

$$x \cdot (f_0 \otimes f_1 \otimes \cdots \otimes f_n \otimes q) = (f_0 \cdot x^{-1}) \otimes f_1 \otimes \cdots \otimes f_n \otimes (x \cdot q).$$

The coboundary operator $d : X_n \rightarrow X_{n+1}$ is given, in the functional notation, by

$$(dh)(x_0, \cdots, x_{n+1}) = \textstyle\sum_{i=0}^{n} (-1)^i h(x_0, \cdots, x_i x_{i+1}, \cdots, x_{n+1}) + (-1)^{n+1} h(x_0, \cdots, x_n).$$

The augmentation $\mathfrak{Q} \rightarrow X_0$ is given by $q \rightarrow 1 \otimes q$. It is easily checked that this complex is acyclic; a homotopy ζ is given by $(\zeta h)(x_1, \cdots, x_n) = h(1, x_1, \cdots, x_n)$. By Proposition 2.2, each X_n is rationally injective.

Now let (P, φ) be a rational group extension of Q by G that is compatible with our G-module structure on $\mathfrak{Q}$. By Theorem 3.1, there exists a rational representative map $\psi : G \rightarrow P$ such that $\varphi \circ \psi$ is the identity map on G. We define $h \in R \otimes R \otimes \mathfrak{Q}$ by $h(x, y) = \log(\psi(x)\psi(y)\psi(xy)^{-1})$. There is one and only one element $k \in (X_2)^G$ such that $k(1, x, y) = h(x, y)$. One checks immediately that $dk = 0$, so that k defines an element $u \in H^2(G, \mathfrak{Q})$. Now one shows in the usual manner, and without encountering any special difficulties, that the correspondence $(P, \varphi) \to u$ induces a linear monomorphism of the space of the equivalence classes of the rational group extensions of Q by G into $H^2(G, \mathfrak{Q})$. Our above results on the rational cohomology of G will enable us to show that this monomorphism is actually an isomorphism, i.e., that every element of $H^2(G, \mathfrak{Q})$ arises, in the way described above, from a rational group extension of Q by G.

Let N be the maximum unipotent normal subgroup of G, and write G as a semidirect product $N \cdot K$, where K is a fully reducible algebraic subgroup of G. Let $u \in H^2(G, \mathfrak{Q})$, and let u_N be the image of u in $H^2(N, \mathfrak{Q})$ by the canonical "restriction" homomorphism $H^2(G, \mathfrak{Q}) \to H^2(N, \mathfrak{Q})$. The precise meaning of the first part of Theorem 5.2 is that the restriction homomorphism $H(G, \mathfrak{Q}) \to H(N, \mathfrak{Q})$ sends $H(G, \mathfrak{Q})$ isomorphically onto $H(N, \mathfrak{Q})^G = H(N, \mathfrak{Q})^K$. In particular, we have $u_N \in (H^2(N, \mathfrak{Q}))^K$. The complex X^G may be identified with the complex of the nonhomogeneous rational representative cochains for G in $\mathfrak{Q}$, as described at the end of Section 2. The identification is given by the map $h \to h'$, where

$$h'(x_1, \cdots, x_n) = h(1, x_1, \cdots, x_n).$$

The restriction map $H(G, \mathfrak{Q}) \to H(N, \mathfrak{Q})$ is then induced by the restriction of these cochains from G^n to N^n. The action of K on $H(N, \mathfrak{Q})$ is induced by the cochain action $h \to y \cdot h$, where

$$(y \cdot h)(x_1, \cdots, x_n) = y \cdot h(y^{-1} x_1 y, \cdots, y^{-1} x_n y).$$

The cochain complex for N in $\mathfrak{Q}$ thereby becomes evidently a rational, and hence semisimple K-module. It follows that u_N has a cochain representative h that is K-fixed, i.e., is such that, for all $y \in K$ and all x_1, x_2 in N, $y \cdot h(y^{-1} x_1 y, y^{-1} x_2 y) = h(x_1, x_2)$.

Now suppose that we have already constructed a rational group extension (P_N, φ_N) of Q by N whose corresponding cohomology class is u_N. Since the above cochain h is a representative of u_N, it follows that there is a rational representative map ψ of N into P_N such that $\varphi_N \circ \psi$ is the identity map on N and $\log(\psi(x_1)\psi(x_2)\psi(x_1 x_2)^{-1}) = h(x_1, x_2)$, for all x_1, x_2 in N. This enables us to define a rational homomorphism ρ of K into the group of the rational automorphisms of P_N (note that P_N is a *unipotent* algebraic linear group) that is compatible with the action of K on Q and with the action of K on N by conjugation. In fact, for $z \in P_N$ and $y \in K$, we put

$$\rho(y)(z) = (y \cdot (z\psi\varphi_N(z)^{-1}))\psi(y\varphi_N(z)y^{-1}).$$

The fact that $\rho(y)$ is an automorphism of P_N is a formal consequence of the fact that h is K-fixed, and it is then clear that $\rho(y)$ is a *rational* automorphism of P_N, and that ρ is a rational homomorphism of K into the group of the rational automorphisms of P_N.

Now we can form the semidirect product $P_N \times_\rho K$, with the structure of an algebraic linear group as defined at the beginning of this section. The rational epimorphism $\varphi_N : P_N \to N$ induces a rational epimorphism $\varphi : P_N \times_\rho K \to N \cdot K = G$, and it is clear that the pair $(P_N \times_\rho K, \varphi)$ is a rational group extension of Q by G. Let v be the element of $H^2(G, \mathfrak{Q})$ that corresponds to this group extension. Then v_N is evidently the element of $H^2(N, \mathfrak{Q})$ that corresponds to (P_N, φ_N). Thus $v_N = u_N$. Since, by Theorem 5.2, the restriction map $H^2(G, \mathfrak{Q}) \to H^2(N, \mathfrak{Q})$ is a monomorphism, it follows that $v = u$. Hence there remains only to construct the rational group extension (P_N, φ_N).

We shall construct this from a suitable Lie algebra extension of $\mathfrak{Q}$ by $\mathfrak{N}$, by exponentiation. The canonical isomorphism of $H^2(N, \mathfrak{Q})$ onto $H^2(\mathfrak{N}, \mathfrak{Q})$ sends u_N onto an element $(u_N)^*$ of $H^2(\mathfrak{N}, \mathfrak{Q})$. Let (A, α) be a Lie algebra extension of $\mathfrak{Q}$ by $\mathfrak{N}$ whose corresponding element in $H^2(\mathfrak{N}, \mathfrak{Q})$ is $(u_N)^*$. Since A is a nilpotent Lie algebra, there is a faithful representation of A by nilpotent linear endomorphisms of a finite-dimensional F-space V. We let P_N be the algebraic subgroup of the full linear group on V that consists of the exponentials of the elements of A. We identify Q with the subgroup of P_N whose Lie algebra is $\mathfrak{Q}$, i.e., with the subgroup consisting of the exponentials of the elements of $\mathfrak{Q}$. Then, since P_N is unipotent, α induces a rational epimorphism φ_N of P_N onto N, and Q is the kernel of φ_N. In fact, φ_N is given by $\varphi_N = \exp_{\mathfrak{N}} \circ \alpha \circ \log_{P_N}$.

In order to see that the cohomology class in $H^2(N, \mathfrak{Q})$ that is associated with (P_N, φ_N) is actually u_N, we must make the canonical isomorphism $H(N, \mathfrak{Q}) \to H(\mathfrak{N}, \mathfrak{Q})$ explicit in a suitable way. This will be accomplished by defining an explicit map of the rationally injective resolution X (constructed for N in the way explained above for G) of the rational N-module $\mathfrak{Q}$ into the $U(\mathfrak{N})$-injective resolution $\mathrm{Hom}_F(U(\mathfrak{N}) \otimes E(\mathfrak{N}), \mathfrak{Q})$ of $\mathfrak{Q}$, which we used in the proof of Lemma 5.1.

For $\zeta \in \mathfrak{N}$ and $n > 0$, we define a linear map $X_n \to X_{n-1}$, denoted $h \to \zeta \cdot h$, such that $\zeta \cdot (f_0 \otimes f_1 \otimes \cdots \otimes f_n \otimes q) = f_0 \otimes f_1 \otimes \cdots \otimes f_{n-1} \otimes \zeta(f_n)(1)q$. Let $f \to f^*$ denote the map $R(N) \to \mathrm{Hom}_F(U(\mathfrak{N}), F)$ that we defined in Section 5, i.e., $f^*(u) = u(f)(1)$, for every $u \in U(\mathfrak{N})$. We generalize this map to a map $h \to h^*$ of X_0 into $\mathrm{Hom}_F(U(\mathfrak{N}), \mathfrak{Q})$ such that $(f \otimes q)^* = f^* \otimes q$. Now we define a linear map $\tau : X_n \to \mathrm{Hom}_F(U(\mathfrak{N}) \otimes E^n(\mathfrak{N}), \mathfrak{Q})$ by setting

$$\tau(h)(u \otimes \zeta_1 \cdots \zeta_n) = \textstyle\sum_p e(p)(\zeta_{p(1)} \cdots \zeta_{p(n)} \cdot h)^*(u),$$

where the summation is over all permutations p of the set $(1, \cdots, n)$, and $e(p)$ is the signature of p. It is clear from what we have seen in Section 5 that τ is an $\mathfrak{N}$-module homomorphism of X_n into $\mathrm{Hom}_F(U(\mathfrak{N}) \otimes E^n(\mathfrak{N}), \mathfrak{Q})$.

Let d and d^* denote the coboundary operators on X and on $\mathrm{Hom}_F(U(\mathfrak{N}) \otimes E(\mathfrak{N}), \mathfrak{Q})$, respectively. We claim that $d^* \circ \tau = \tau \circ d$. We recall the well known explicit formula for d^*:

$$(d^*h)(u \otimes \zeta_1 \cdots \zeta_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i-1} h(u\zeta_i \otimes \zeta_1 \cdots \hat{\zeta}_i \cdots \zeta_{n+1}) + \sum_{r<s} (-1)^{r+s} h(u \otimes [\zeta_r, \zeta_s]\zeta_1 \cdots \hat{\zeta}_r \cdots \hat{\zeta}_s \cdots \zeta_{n+1}).$$

It is clearly sufficient to prove that $d^* \circ \tau = \tau \circ d$ in the case where $\mathfrak{Q} = F$. Furthermore, it suffices to verify that $d^*(\tau(h)) = \tau(d(h))$ in the case where $h = f_0 \otimes f_1 \otimes \cdots \otimes f_n$. Then we have

$$(dh)(x_0, \cdots, x_{n+1}) = \sum_{i=0}^{n} (-1)^i f_0(x_0) \cdots f_i(x_i x_{i+1}) \cdots f_n(x_{n+1}) + (-1)^{n+1} f_0(x_0) \cdots f_n(x_n).$$

Applying successively the maps $k \to \zeta_i \cdot k$, observing that the $\mathfrak{N}$-operations on $R(N)$ commute with the right translations, we obtain

$$(\zeta_1 \cdots \zeta_{n+1} \cdot (dh))(x_0) = \sum_{i=0}^{n} (-1)^i u_i,$$

where, for $i > 0$,

$$u_i = f_0(x_0)\zeta_1(f_1)(1) \cdots \zeta_{i-1}(f_{i-1})(1)\zeta_i \zeta_{i+1}(f_i)(1)\zeta_{i+2}(f_{i+1})(1) \cdots \zeta_{n+1}(f_n)(1),$$

and $u_0 = \zeta_1(f_0)(x_0)\zeta_2(f_1)(1) \cdots \zeta_{n+1}(f_n)(1)$.

This gives, on collecting the terms arising from u_0,

$$\tau(dh)(u \otimes \zeta_1 \cdots \zeta_{n+1}) = \sum_{j=1}^{n+1} (-1)^{j-1} \tau(h)(u\zeta_j \otimes \zeta_1 \cdots \hat{\zeta}_j \cdots \zeta_{n+1}) + S,$$

where

$$S = f_0^*(u) \sum_{i=1}^{n} (-1)^i \sum_p e(p) v_p,$$

with

$$v_p = \zeta_{p(1)}(f_1)(1) \cdots \zeta_{p(i)}\zeta_{p(i+1)}(f_i)(1) \cdots \zeta_{p(n+1)}(f_n)(1).$$

Now let P_i denote the set of all permutations p for which $p(i) < p(i+1)$. By suitably combining the terms $e(p)v_p$, we may write

$$S = f_0^*(u) \sum_{i=1}^{n} (-1)^i \sum_{p \in P_i} e(p) w_p,$$

where

$$w_p = \zeta_{p(1)}(f_1)(1) \cdots [\zeta_{p(i)}, \zeta_{p(i+1)}](f_i))(1) \cdots \zeta_{p(n+1)}(f_n)(1).$$

For each fixed pair of indices (r, s) with $r < s$, let $S_{(r,s)}$ be the sum of all those terms in the above expression for S in which $p(i) = r$ and $p(i+1) = s$. For a given triple (i, r, s), and for p such that $p(i) = r$ and $p(i+1) = s$, let p' be the map of the set $(1, \cdots, i-1, i+2, \cdots, n+1)$ onto the set $(1, \cdots, r-1, r+1, \cdots, s-1, s+1, \cdots, n+1)$ that is given by the restriction of p. If we define the signature $e(p')$ of p' from the total number of inversions, we have $e(p') = (-1)^{r+i+s+i+1} e(p) = (-1)^{r+s+1} e(p)$. Hence the term

$$f_0^*(u)\zeta_{p'(1)}(f_1)(1) \cdots [\zeta_r, \zeta_s](f_i)(1) \cdots \zeta_{p'(n+1)}(f_n)(1)$$

of $S_{(r,s)}$ occurs with the sign $(-1)^{r+s+i-1} e(p')$. Hence we see that

$$S_{(r,s)} = (-1)^{r+s} \tau(h)(u \otimes [\zeta_r, \zeta_s]\zeta_1 \cdots \hat{\zeta}_r \cdots \hat{\zeta}_s \cdots \zeta_{n+1}).$$

Now it is clear that $\tau(dh) = d^*(\tau(h))$, so that τ is a map of $\mathfrak{N}$-complexes and therefore induces the canonical isomorphism of $H(N, \mathfrak{Q})$ onto $H(\mathfrak{N}, \mathfrak{Q})$, by Theorem 5.1.

Now let us return to our Lie algebra extension (A, α) of $\mathfrak{Q}$ by $\mathfrak{N}$. We choose a linear map γ of $\mathfrak{N}$ into A such that $\alpha \circ \gamma$ is the identity map on $\mathfrak{N}$. Define $g \in \mathrm{Hom}_F(E^2(\mathfrak{N}), \mathfrak{Q})$ by $g(\zeta_1 \zeta_2) = [\gamma(\zeta_1), \gamma(\zeta_2)] - \gamma([\zeta_1, \zeta_2])$. Then, by construction of (A, α), g is a representative cochain for the cohomology class $(u_N)^* \in H^2(\mathfrak{N}, \mathfrak{Q})$. Define $\psi : N \rightarrow P_N$ by $\psi = \exp_A \circ \gamma \circ \log_N$. Then ψ is a rational representative map, and $\varphi_N \circ \psi$ is the identity map on N. Hence, if f is defined by $f(x, y) = \log(\psi(x)\psi(y)\psi(xy)^{-1})$, then f is a representative cochain for the cohomology class, v say, in $H^2(N, \mathfrak{Q})$ that is associated with the rational group extension (P_N, φ_N). We determine the canonical image of v in $H^2(\mathfrak{N}, \mathfrak{Q})$ by using the map τ constructed above. This shows that a representative cochain for the image of v in $H^2(\mathfrak{N}, \mathfrak{Q})$ is h, where $h(\zeta_1 \zeta_2) = \zeta_1 \zeta_2 \cdot f - \zeta_2 \zeta_1 \cdot f$. Here, the right-hand side is to be interpreted as follows: Define $f_x(y) = f(x, y)$. Then $\zeta \cdot f$ is the map of N into $\mathfrak{Q}$ defined by $(\zeta \cdot f)(x) = \zeta(f_x)(1)$, and $\zeta_1 \zeta_2 \cdot f = \zeta_1(\zeta_2 \cdot f)(1)$. Here, if $k = \sum_i k_i \otimes q_i$, with $k_i \in R(N)$ and $q_i \in \mathfrak{Q}$, $\zeta(k)$ stands for $\sum_i \zeta(k_i) \otimes q_i$. We shall show that $h = g$.

In order to do this, we enlarge the base field F to the field of the power series in one variable t with coefficients in F. If p is any power series in t, we shall denote by p_1 the coefficient of t in p. Then we have (computing in the enveloping algebraf o linear endomorphisms)

$$\begin{aligned}
\zeta(f_x)(1) &= (f_x(\exp(t\zeta)))_1 = (\log(\psi(x)\psi(\exp(t\zeta))\psi(x\exp(t\zeta))^{-1}))_1 \\
&= (\psi(x)\psi(\exp(t\zeta))\psi(x\exp(t\zeta))^{-1})_1 \\
&= \psi(x)(\psi(\exp(t\zeta)))_1\psi(x)^{-1} + (\psi(x)\psi(x\exp(t\zeta))^{-1})_1 \\
&= \psi(x)(\psi(\exp(t\zeta)))_1\psi(x)^{-1} - (\psi(x\exp(t\zeta)))_1\psi(x)^{-1}.
\end{aligned}$$

If β is any rational representative map, let us write $\beta^*(\zeta) = (\beta(\exp(t\zeta)))_1$. The above result may be written

$$\begin{aligned}
\zeta(f_x)(1) &= \psi(x)\psi^*(\zeta)\psi(x)^{-1} - (\psi \cdot x)^*(\zeta)\psi(x)^{-1} \\
&= \psi(x)\psi^*(\zeta)\psi(x)^{-1} - \zeta(\psi \cdot x)(1)\psi(x)^{-1} \\
&= \psi(x)\psi^*(\zeta)\psi(x)^{-1} - \zeta(\psi)(x)\psi(x)^{-1}.
\end{aligned}$$

Hence we find

$$\begin{aligned}
\zeta_1 \zeta_2 \cdot f &= \psi^*(\zeta_1)\psi^*(\zeta_2) - \psi^*(\zeta_2)\psi^*(\zeta_1) - \zeta_1 \zeta_2(\psi)(1) + \psi^*(\zeta_2)\psi^*(\zeta_1) \\
&= \psi^*(\zeta_1)\psi^*(\zeta_2) - \zeta_1 \zeta_2(\psi)(1).
\end{aligned}$$

Thus we have

$$h(\zeta_1 \zeta_2) = [\psi^*(\zeta_1), \psi^*(\zeta_2)] - \psi^*([\zeta_1, \zeta_2]).$$

Since $\psi = \exp_A \circ \gamma \circ \log_N$, we have $\psi^* = \gamma$, whence $h = g$.

We conclude that the canonical image of v in $H^2(\mathfrak{N}, \mathfrak{Q})$ coincides with $(u_N)^*$, whence $v = u_N$. This means that the rational group extension (P_N, φ_N) has u_N for its associated element in $H^2(N, \mathfrak{Q})$.

Actually, our above discussion of τ and the proof that $h = g$ did not make use of the unipotency of N, so that we have the following general result. Let (P, φ) be a rational group extension of Q by G, and let ψ be a rational representative map of G into P such that $\psi(1) = 1$ and $\varphi \circ \psi$ is the identity map on G. Let f and h be defined from ψ as above. Then $h(\zeta_1 \zeta_2) = [\psi^*(\zeta_1), \psi^*(\zeta_2)] - \psi^*([\zeta_1, \zeta_2])$. Now ψ^* is simply the differential at 1 of ψ, and thus is a linear map of $\mathfrak{G}$ into $\mathfrak{P}$ whose composite with the differential of φ is the identity map on $\mathfrak{G}$. Hence our result shows that h is a representative cocycle for the element of $H^2(\mathfrak{G}, \mathfrak{Q})$ that is associated with the Lie algebra extension induced by (P, φ). This means that the natural passage from group extensions to Lie algebra extensions induces the *canonical* map $H^2(G, Q) \to H^2(\mathfrak{G}, \mathfrak{Q})$ obtained from maps of resolutions. We have established the following result.

THEOREM 6.1. *Let Q be a vector group, G an arbitrary algebraic linear group over a field F of characteristic* 0. *Let ρ be a rational homomorphism of G into the group of all linear automorphisms of Q. Then the usual "factor set" correspondence between group extensions and 2-dimensional cohomology classes induces an isomorphism of the space of the equivalence classes of the rational group extensions of Q by G that are compatible with ρ onto the rational cohomology space $H^2(G, Q)$. Moreover, this isomorphism transports the natural map from group extensions to Lie algebra extensions into the canonical monomorphism of $H^2(G, Q)$ into $H^2(\mathfrak{G}, Q)$, induced by any $\mathfrak{G}$-complex map of a rationally injective resolution of Q into a $U(\mathfrak{G})$-injective resolution of Q.*

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