HOMOMORPHISMS OF MEASURE ALGEBRAS

BY

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1. Introduction

In their recent paper [2] Hewitt and Kakutani prove a truly remarkable theorem: *Let G be a locally compact Abelian group, and let $M(G)$ be the measure algebra on G. Let P be an independent subset of G, and denote by $M(P \cup -P)$ the linear subspace of measures concentrated on $(P \cup -P)$. If L is any linear functional on $M(P \cup -P)$ of norm 1 and satisfying the property $L(\sigma_x)L(\sigma_{-x}) = 1$ for every $x \in P$, then there is a homomorphism h defined on all of $M(G)$ which agrees with L on $M(Q)$.*

Their proof is an existence proof. In this paper we actually *construct* such a homomorphism. This construction, we believe, contributes to a better understanding of the complexities of measure algebras. It is easy to prove, via this construction, that the extension of a linear functional to a homomorphism is *unique* if restricted to the subalgebra M defined below. In a later paper we hope to use this fact to describe the ideal space of M and to give an analysis of this subalgebra.

We outline the procedure for constructing the homomorphism. Let M_0 be the algebra generated by $M(P \cup -P)$ and all the discrete measures. Then let M_1 be the algebra consisting of all those measures which are absolutely continuous to some element of M_0. We let $h = L$ on $M(P \cup -P)$ and extend h to M_1 making use of Šreĭder's "generalized functions" (see [3]). After proving h is well defined and h is a homomorphism on M_1, we extend h to be a homomorphism on the closure M of M_1. Next, we show that the orthogonal complement $M^{\perp}$ of M is an ideal and $M(G)$ is the direct sum of M and $M^{\perp}$. We conclude by defining $h(\mu) = h(\mu_M)$ where $\mu \in M(G)$ and μ_M is the projection of μ on M.

In §3 we prove a "generalized Lebesgue decomposition theorem" which plays a small but important role in our construction. In §4 we construct the homomorphism.

2. Preliminaries

Throughout this paper we assume G is a locally compact Abelian additive group. We let $M(G)$ be the set of all complex-valued regular Borel measures on G. It should be noted that Haar measure m is in $M(G)$ if and only if G is compact. With addition and scalar multiplication defined in the obvious way, $M(G)$ is a Banach space under the norm of total variation, i.e., $\| \mu \| =$

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$\int_G d|\mu|$. (For this and other notation not specifically explained, see Halmos [1].) We can define a multiplication, called convolution, between measures. Let μ and λ be elements of $M(G)$, and let S be any Borel subset of G; define

$$\mu * \lambda(S) = \int_G \mu(S - x)\, d\lambda(x). \tag{2.1}$$

Clearly, $\| \mu * \lambda \| \leqq \| \mu \| \, \| \lambda \|$ so that with this multiplication, $M(G)$ is a Banach algebra. It is commutative since G is Abelian. An equivalent definition (see Stromberg [4]) which we use extensively is as follows: let

$$T = \{(x, y) \in G \times G : x + y \in S\}.$$

Then define

$$\mu * \lambda(S) = \mu \times \lambda(T) \qquad (\mu \times \lambda \text{ is the product measure}). \tag{2.2}$$

Given two measures μ and λ, we say μ is *absolutely continuous with respect to* λ, in symbols $\mu \ll \lambda$, if $\mu(S) = 0$ whenever $|\lambda|(S) = 0$. If $\mu \ll \lambda$ and $\lambda \ll \mu$, then λ and μ are *equivalent* and we write $\mu \equiv \lambda$. A measure μ is *singular* (or *orthogonal*) to another measure λ, in symbols $\mu \perp \lambda$, if there are sets A and B such that $A \cup B = G$ and $|\mu|(S \cap B) = 0 = |\lambda|(S \cap A)$ for every Borel set S. The *Lebesgue decomposition theorem* states that given $\mu, \lambda \in M(G)$ there exist measures μ_1 and μ_2 such that $\mu = \mu_1 + \mu_2$ with $\mu_1 \ll \lambda$ and $\mu_2 \perp \lambda$. We make frequent use of this result. If $M \subset M(G)$, we denote by $M^{\perp}$ the *orthogonal complement* of M, i.e.,

$$M^{\perp} = \{\lambda \in M(G) : \mu \in M \Rightarrow \mu \perp \lambda\}.$$

A measure μ is *concentrated* on a set A if $\mu(B) = 0$ whenever $A \cap B = \emptyset$. If μ is concentrated on a countable set, then μ is called *discrete*. For any $x \in G$, we will always denote by σ_x the measure defined by $\sigma_x(A) = 0$ or 1 depending on whether $x \notin A$ or $x \in A$; thus every discrete measure can be represented by a sum $\sum_{i=1}^{\infty} z_i \sigma_{x_i}$, z_i complex. The measure σ_0 is the identity of $M(G)$. If $\mu(\{x\}) = 0$ for all $x \in G$, then μ is said to be *continuous*. The continuous measures form an ideal of $M(G)$. If $\mu \in M(G)$ and $A \subset G$, the measure $\lambda = \mu \mid A$ is defined by $\lambda(S) = \mu(A \cap S)$.

When we want to say a relation $p(x) = q(x)$ holds almost everywhere with respect to a measure μ, we write $p(x) = q(x)[\mu]$. By this we mean $|\mu|(\{x : p(x) \neq q(x)\}) = 0$.

For any set $A \subset G$, we set $A^0 = \{0\}$, $A^1 = A$, and $A^n = A + A^{n-1}$ for $n = 2, 3, \cdots$. Again set $A^{(1)} = A$ and $A^{(n)} = A \times A^{(n-1)}$ for $n = 2, 3, \cdots$.

A subset $P \subset G$ is said to be *independent* (over the integers) if whenever $x_1, \cdots, x_n$ are distinct elements of P and $q_1, \cdots, q_n$ are integers not all zero, we have $q_1 x_1 + \cdots + q_n x_n \neq 0$.

A *regular family* of sets in G is a collection F of subsets of G satisfying: (1) if $A \in F$, then every Borel subset of A is again in F, (2) F is closed under countable unions, (3) F is closed under arithmetic sums, i.e., $A, B \in F$ implies

$(A + B) \in F$, and (4) all countable sets are in F. It is not hard to see that for a given set $A \subset G$, the *regular family generated by* A is the collection $\{\{\bigcup_{n=1}^{\infty}(A_n + x_n) : A_n \subset A^n, x_n \in G\}\}$, where if $A_n = \emptyset$, then $(A_n + x_n) = \{x_n\}$. A measure μ is *concentrated in* F if μ is concentrated on some element of F. We say μ is *concentrated outside* F if $\mu(A) = 0$ for all $A \in F$. D. A. Raĭkov (see Šreĭder [3]) has proved (1) *the set* H *of measures concentrated in* F *is an algebra*; (2) *the set* I *of measures concentrated outside* F *is an ideal*, and (3) $M(G)$ *is the direct sum of* H *and* I.

One final preliminary remark.

We make use of Šreĭder's "generalized functions." A *generalized function* L is a function $L: M(G) \times G \to C$ (C = complex plane) such that (1) for fixed $\mu \in M(G)$, $L(\mu, x)$ is μ-measurable, and (2) if $\mu \ll \lambda$, then $L(\mu, x) = L(\lambda, x)[\mu]$. Šreĭder [3] proved these generalized functions characterize the dual space of $M(G)$ in the following way: If L is any bounded linear functional on $M(G)$, then there is a generalized function $L(\mu, x)$ such that

$$L(\mu) = \int_G L(\mu, x)\, d\mu(x) \quad \text{and} \quad \| L \| = \sup_{\mu \in M(G)}\{\text{ess. sup}_{x \in G} | L(\mu, x) |\}.$$

3. Generalized Lebesgue decomposition theorem

Let M *be a closed linear subspace of* $M(G)$ *with the property that* $\mu \in M$ *and* $\lambda \ll \mu$ *implies* $\lambda \in M$. *Then* $M(G)$ *can be decomposed into the direct sum* $M(G) = M + M^{\perp}$.

Remark. By the statement $M(G)$ is the direct sum of M and $M^{\perp}$ we mean that each element $\mu \in M(G)$ has a *unique* representation $\mu = \mu' + \mu''$ with $\mu' \in M$ and $\mu'' \in M^{\perp}$.

Proof. The proof is by contradiction; we suppose that $\mu \in M(G)$ and μ cannot be written as $\mu = \mu' + \mu''$ with $\mu' \in M$ and $\mu'' \in M^{\perp}$. Since $\mu \notin M^{\perp}$, there exists a $\lambda \in M$ such that $\mu = \nu_1 + \tau_1$ with $\nu_1 \ll \lambda$ and $\tau_1 \perp \lambda$. Thus (1) $\nu_1 \neq 0$, (2) $\nu_1 \in M$, and (3) there is a subset $K_1 \subset G$ with $\mu \mid K_1 = \nu_1$. We proceed by transfinite induction. Let Φ be the first uncountable ordinal, and suppose for each ordinal ψ, $\psi < \psi_0 < \Phi$, we have defined a ν_ψ such that (1), (2), and (3) hold and, in addition, (4) $K_{\psi_1} \cap K_{\psi_2} = \emptyset$ if $\psi_1 \neq \psi_2$. It follows from (3) and (4) and the fact ψ_0 is a countable ordinal that the sum $\nu_0 = \sum_{\psi < \psi_0} \nu_\psi$ makes sense; for

$$\nu_0(S) = \textstyle\sum_{\psi<\psi_0} \nu_\psi(S) = \sum_{\psi<\psi_0} \mu(S \cap K_\psi) = \mu(\bigcup_{\psi<\psi_0}(S \cap K_\psi)).$$

Furthermore, $\nu_0 \in M$ since M is closed. Now write $\mu = \mu_1 + \mu_2$ with $\mu_1 \ll \nu_0$ and $\mu_2 \perp \nu_0$. By our assumption $\mu_2 \neq 0$ and $\mu_2 \notin M^{\perp}$; by decomposing μ_2 in the right way we can produce ν_{ψ_0} in such a way that properties (1)–(4) hold for $\psi \leqq \psi_0$. Thus, for each countable ordinal ψ we have a nonzero ν_ψ. This is obviously not possible since there are uncountably many countable ordinals and μ is a *finite* measure.

Note. The above theorem can be obtained from a general result on ordered linear spaces due to F. Riesz (see N. BOURBAKI, *Integration*). The proof is included here for completeness.

4. Construction of the homomorphism

Let P be an independent subset of G, and set $Q = P \cup -P$. We denote by $M(Q)$ the linear subspace of all measures in $M(G)$ which are concentrated on Q.

Suppose L is any linear functional on $M(Q)$ of norm 1 with the property $L(\sigma_x)L(\sigma_{-x}) = 1$ for any $x \in P$. We wish to construct a homomorphism h defined on all of $M(G)$ which agrees with L when restricted to $M(Q)$.

First, observe that by the Hahn-Banach theorem and Šreĭder's work, we know there is a generalized function $L(\mu, x)$ such that $L(\mu) = \int L(\mu, x)\, d\mu(x)$ for all $\mu \in M(Q)$.

Next, we note that $\| L \| = 1$ and $L(\sigma_x)L(\sigma_{-x}) = 1$ together imply $L(\sigma_x) = \overline{L(\sigma_{-x})}$. Since P is independent, it now follows that the function $\chi(x) = L(\sigma_x, x)$ for $x \in Q$ can be extended to a homomorphism of the entire group G into the circle group. We denote this extension also by χ. For any discrete measure δ, define $h(\delta) = \int \chi(x)\, d\delta(x)$. Then $h(\delta) = L(\delta)$ if $\delta \in M(Q)$, and furthermore, *h is a multiplicative linear functional on the algebra of all discrete measures in $M(G)$.*

Third, we let M_0 be the algebra generated by all discrete measures and $M(Q)$. Then a general element μ of M_0 may be represented as

$$u = \delta + \textstyle\sum_{j=1}^{n} \delta_j * \mu_{1,j} * \cdots * \mu_{m,j} \tag{4.1}$$

$$(\delta, \delta_j \text{ discrete}; m = 1, 2, \cdots; \mu_{m,j} \text{ continuous members of } M(Q)).$$

Abbreviating $\mu_j = \delta_j * \mu_{1,j} * \cdots * \mu_{m,j}$, we use the notation above to define

$$\begin{aligned} h(\mu) &= \int \chi(x)\, d\delta(x) \\ &\quad + \sum_{j=1}^{n} \int \chi(s) L(\mu_{1,j}, t) \cdots L(\mu_{m,j}, v)\, d\mu_j(s + t + \cdots + v). \end{aligned} \tag{4.2}$$

Let us suppose for the moment that h is well defined on M_0 by (4.2).

Clearly, h agrees with L on $M(Q)$. Applying the Fubini theorem and the generalized version of (2.2) to the second term of (4.2) yields

$$\begin{aligned} h(\mu) &= h(\delta) + \textstyle\sum_{j=1}^{n} h(\delta_j) L(\mu_{1,j}) \cdots L(\mu_{m,j}) \\ &= h(\delta) + \textstyle\sum_{j=1}^{n} h(\delta_j) h(\mu_{1,j}) \cdots h(\mu_{m,j}). \end{aligned}$$

This, together with the fact h is already a homomorphism on the discrete measures implies that *h is a homomorphism on M_0*.

Now let M_1 be the set of all measures in $M(G)$ which are absolutely continuous with respect to some measure in M_0. Observe that if

$$\mu = \delta + \textstyle\sum_{j=1}^{n} \delta_j * \mu_{1,j} * \cdots * \mu_{m,j}$$

is in M_0, then so is $\tilde{\mu} = |\delta| + \sum_{j=1}^{n} |\delta_j| * |\mu_{1,j}| * \cdots * |\mu_{m,j}|$. It follows that M_1 is an algebra. (Addition is trivial; for convolution see, for example, Šreĭder [3], p. 9.) We now extend h to M_1 by the following device. If $\mu = \delta + \sum_{j=1}^{n} \mu_j$ (μ_j is as in (4.2)) is in M_0 and $\lambda \ll \mu$, we write λ as the sum $\lambda = \sum_{j=0}^{n} \lambda_j$ of mutually singular components with $\lambda_0 \ll \delta$ and $\lambda_j \ll \mu_j$, $1 \leqq j \leqq n$. Then we define h by

$$h(\lambda) = \int \chi(x)\, d\lambda_0(x) \tag{4.3}$$
$$+ \sum_{j=1}^{n} \int \chi(s) L(\mu_{1,j}, t) \cdots L(\mu_{m,j}, v)\, d\lambda_j(s + t + \cdots + v).$$

We interrupt our construction at this point to prove h is well defined by (4.2) and (4.3) and these definitions are consistent. We need the following lemmas.

Lemma 1. *Let x and y be arbitrary elements of G, and let $\mu_1, \cdots \mu_n$ be continuous measures in $M(Q)$. If $n > m$, then*

$$|\sigma_x * \mu_1 * \cdots * \mu_n|(y + Q^m) = 0.$$

Proof. Let $\mu = \sigma_x * \mu_1 * \cdots * \mu_n$; then μ is concentrated on $x + Q^n$. We will show $|\mu|((x + Q^n) \cap (y + Q^m)) = 0$. To that end, let S be any Borel subset of that intersection, and let

$$S_n = \{(x, s_1, \cdots, s_n) \in \{x\} \times Q^{(n)} : x + s_1 + \cdots + s_n \in S\}.$$

By definition (2.2),

$$\mu(S) = \sigma_x \times \mu_1 \times \cdots \times \mu_n(S_n).$$

Now if $(x, s_1, \cdots, s_n) \in S_n$, then there is a set $\{t_1, \cdots, t_m\} \subset Q$ such that

$$x + s_1 + \cdots + s_n = y + t_1 + \cdots + t_m.$$

Write $s_i = \varepsilon u_i$ and $t_j = \varepsilon v_j$ where $\varepsilon = \pm 1$ and u_i and v_j are in P. Thus

$$\varepsilon u_1 + \cdots + \varepsilon u_n - \varepsilon v_1 - \cdots - \varepsilon v_m = y - x.$$

If $s_i \pm s_j \neq 0$ for $1 \leqq i < j \leqq n$, then the independence of P and the hypothesis $n > m$ insures the existence of u_k, $1 \leqq k \leqq n$, such that every such representation of $y - x$ contains the term $u_k = \pm s_k$. Clearly, the subset of S_n consisting of those elements for which some coordinate (larger than 1) is $\pm s_k$ has measure zero w.r.t. $\sigma_x \times \mu_1 \times \cdots \times \mu_n$. Now consider those elements of S_n for which $s_i \pm s_j \neq 0$ and $\pm s_k$ does not appear. Then $\pm s_k$ must appear as some v_k, and we will be left, as before, with a $u_r = \pm s_r$. This can only proceed a finite number of times, and at each step we have a set of measure zero. We conclude that the subset of S for which $s_i \pm s_j \neq 0$ has measure zero w.r.t. $\sigma_x * \mu_1 * \cdots * \mu_n$. If $s_i \pm s_j = 0$ for some $i \neq j$, the situation is somewhat more complicated. For $1 \leqq i < j \leqq n$, let

$$T_{i,j} = \{(x, s_1, \cdots, s_n) \in S_n : s_i \pm s_j = 0\},$$

$$T_{i,j,s_j} = \{(x, y_1, \cdots, y_{n-1}) \in \{x\} \times Q^{(n-1)} : (x, y_1, \cdots, y_{j-1}, s_j, y_j, \cdots, y_{n-1}) \in T_{i,j}\}.$$

Now $T_{i,j,s_j} \subset \{x\} \times Q \times \cdots \times \{\pm s_j\} \times \cdots \times Q$ where $\{\pm s_j\}$ appears as the i^{th} factor. Since μ_i is continuous,

$$\sigma_x \times \mu_1 \times \cdots \times \mu_{j-1} \times \mu_{j+1} \times \cdots \times \mu_n(T_{i,j,s_j}) = 0.$$

It follows from the definition of product measures (see Halmos [1]) that $\sigma_x \times \mu_1 \times \cdots \times \mu_n(T_{i,j}) = 0$. Thus we divide S into a finite number of sets each of which has μ-measure zero. Our lemma is proved.

COROLLARY. *Let δ_1 and δ_2 be any discrete measures, and let $\mu_1, \cdots, \mu_n, \lambda_1, \cdots, \lambda_m$ be continuous elements of $M(Q)$. If $n > m$, then $\delta_1 * \mu_1 * \cdots * \mu_n$ is singular to $\delta_2 * \lambda_1 * \cdots * \lambda_m$.*

The corollary is an immediate consequence of Lemma 1.

The next lemma plays an important role in our construction and is quite interesting in its own right.

LEMMA 2. *Let δ_1 and δ_2 be any two discrete measures, and let $\mu_1, \cdots, \mu_n, \lambda_1, \cdots, \lambda_n$ be continuous elements of $M(Q)$. Suppose that $\{\mu_i\}_{i=1}^{k}$ and $\{\lambda_j\}_{j=1}^{m}$ are orthogonal collections, i.e., $\mu_i \perp \lambda_j$ for $1 \leqq i \leqq k \leqq n$ and $1 \leqq j \leqq m \leqq n$. If $k + m > n$, then $\mu = \delta_1 * \mu_1 * \cdots * \mu_n$ is singular to $\lambda = \delta_2 * \lambda_1 * \cdots * \lambda_n$.*

Proof. It is sufficient to prove the statement for the case $\delta_1 = \sigma_x$ and $\delta_2 = \sigma_y$. It follows from the orthogonality condition there exist sets A and B contained in Q such that $A \cap B = \emptyset$, each μ_i, $1 \leqq i \leqq k$, is concentrated on A, and each λ_j, $1 \leqq j \leqq m$, is concentrated on B. Hence μ is concentrated on $(x + A^k + Q^{n-k})$, while λ is concentrated on $(y + B^m + Q^{n-m})$. Again let S be any Borel subset of $((x + A^k + Q^{n-k}) \cap (y + B^m + Q^{n-m}))$. If $x + a_1 + \cdots + a_k + q_{k+1} + \cdots + q_n \in S$, then there are elements $b_1, \cdots, b_m$ and $r_{m+1}, \cdots, r_n$ of B and Q, respectively, with

$$x_1 + a_1 + \cdots + q_n = y + b_1 + \cdots + r_n.$$

In other words

$$(a_1 + \cdots + a_k - b_1 - \cdots - b_m) + (q_{k+1} + \cdots + q_n - r_{m+1} - \cdots - r_n) = y - x.$$

Since $k + m > (n - k) + (n - m)$ and $A \cap B = \emptyset$ we may split S into two sets: S_1, the set where at least one a_i is not cancelled by any q or r, and S_2, the complement. Using precisely the same argument as in Lemma 1, $\mu(S_1) = \lambda(S_2) = 0$ holds. It follows that μ and λ are singular.

LEMMA 3. *Let* $\mu = \delta_1 * \mu_1 * \cdots * \mu_n$, *and let* $\lambda = \delta_2 * \lambda_1 * \cdots * \lambda_m$, *where* δ_1, δ_2 *are discrete and the* μ_i*'s and* λ_j*'s are continuous elements of* $M(Q)$. *Suppose* $\mu = \gamma_1 + \gamma_2$ *where* $\gamma_1 \perp \lambda$ and $\gamma_2 \ll \lambda$. *Then*

$$\chi(s)L(\mu_1, t) \cdots L(\mu_n, v) = \chi(s)L(\lambda_1, t) \cdots L(\lambda_m, v)[\gamma_2],$$

where $t, \cdots, v$ *are in* Q *and* s *is arbitrary.*

Remark. It makes sense to talk about these products being equal almost everywhere w.r.t. γ_2 since, if we disregard the variable s, γ_2 is concentrated on Q^n. Hence an element in the "domain" of γ_2 looks like $t + \cdots + v$.

Proof. First observe if $\gamma_2 = 0$ the statement is trivial; if $\gamma_2 \neq 0$, then $n = m$ holds by the corollary to Lemma 1.

Write λ_i as the sum of 2^{n-1} mutually orthogonal components

$$\lambda_i = \sum \alpha_{i, j(1), \cdots, j(n)},$$

where $j(r) = 0$ or 1 according to whether this component is singular or absolutely continuous to λ_r. Since $j(i)$ is always 1, there are 2^{n-1} different components. This decomposition is accomplished as follows: write

$$\lambda_1 = \alpha_{1,0} + \alpha_{1,1}$$

with $\alpha_{1,0} \perp \lambda_2$ and $\alpha_{1,1} \ll \lambda_2$. Then $\alpha_{1,0} = \alpha_{1,0,0} + \alpha_{1,0,1}$ with $\alpha_{1,0,0} \perp \lambda_3$ and $\alpha_{1,0,1} \ll \lambda_3$, etc. It is important to note that *given any component* α_i *of* λ_i *and any* λ_r, $1 \leqq r \leqq n$, *then* $\alpha_i \perp \lambda_r$ *or* $\alpha_i \ll \lambda_r$.

We list these components in the form of an $n \times 2^{n-1}$ matrix where the i^{th} row is the decomposition of λ_i. For each k, $1 \leqq k \leqq n$, we write $\mu_k = \beta_{k,1,1} + \gamma$ where $\beta_{k,1,1}$ is absolutely continuous to the (1,1) entry in the matrix and γ is singular to it. Next write $\gamma = \beta_{k,1,2} + \gamma'$ with $\beta_{k,1,2}$ absolutely continuous to the (1,2) entry and γ' singular to it. Continuing in this way we can write μ_k as the sum of $n2^{n-1} + 1$ measures: $n2^{n-1}$ measures $\beta_{k,i,j}$, $1 \leqq i \leqq n$ and $1 \leqq j \leqq 2^{n-1}$ plus one measure $\beta_{k,0,0}$ which is singular to each entry and, hence, singular to each λ_r. Again, it is important to notice that *for a given* $\beta_{k,i,j}$ *and any* λ_r, *either* $\beta_{k,i,j} \ll \lambda_r$ *or* $\beta_{k,i,j} \perp \lambda_r$.

Now a general term in the product $\delta_1 * \mu_1 * \cdots * \mu_n$ looks like

$$\delta_1 * \beta_{1,i_1,j_1} * \cdots * \beta_{n,i_n,j_n},$$

where the i's and j's run over the proper ranges. Now if this term is singular to λ, we are not interested in it; therefore we assume this term is not singular to λ. If this is the case, Lemma 2 assures us there is some factor $\beta_{k,i_k,j_k} \ll \lambda_1$; for the sake of economy in notation call it β_1. Using this notation, we proceed by induction. Suppose we have arranged the factors so that $\beta_k \ll \lambda_k$ for $k = 1, 2, \cdots, r < n$. If, in the remaining factors, one is absolutely continuous to λ_{r+1}, we call it β_{r+1}, and our induction is complete; thus we must assume the remaining $(n - r)$ factors are all singular to λ_{r+1}. Let $\{\beta_{k_1}, \cdots, \beta_{k_p}\}$ be the subset of $\{\beta_1, \cdots, \beta_r\}$ each of whose elements is absolutely continuous to λ_{r+1} and let $\{\beta_{m_1}, \cdots, \beta_{m_s}\}$ be that subset each of whose elements is singular to λ_{r+1}. Since each β-factor is either singular or absolutely continuous to λ_{r+1}, we have $p + s = r$. We know $p > 0$ because if it were 0 we could invoke Lemma 2 to produce a contradiction to our assumption of nonsingularity. If any one of the remaining β-factors is absolutely continuous to some λ_{k_i}, $1 \leqq i \leqq p$, we can rearrange to let this new β-factor become β_{k_i} and let the original β_{k_i} stand for β_{r+1}. If this is not the case, then the remaining $(n - r)$ factors are all singular to each λ_{k_i}, and there are p of these; hence $s > 0$. Now let $\{\beta_{q_1}, \cdots, \beta_{q_t}\}$ be that subset of $\{\beta_{m_1}, \cdots, \beta_{m_s}\}$ each of whose elements is absolutely continuous to some λ_{k_i}, $1 \leqq i \leqq p$, and let $\{\beta_{\mu_1}, \cdots, \beta_{\mu_v}\}$ be that subset each of whose elements is singular to every λ_{k_i}. As before, $t + v = s$; if $t = 0$, then each β_{m_j}, $1 \leqq j \leqq s$, is singular to $\lambda_{k_1}, \cdots, \lambda_{k_p}$ and λ_{r+1}. Thus the set $\{\beta_{m_1}, \cdots, \beta_{m_s}\}$ together with the remaining $(n - r)$ factors are each singular to $(p + 1)$ λ-factors; but $(n - r + s) + (p + 1) = n + 1$, so we know $t > 0$. If any one of the remaining $(n - r)$ factors is absolutely continuous to some λ_{q_i}, $1 \leqq i \leqq t$, we make two rearrangements similar to the one above and end our proof. If not, then the $(n - r)$ β-factors are singular to each λ_{q_i}. So far then, they are singular to $(p + t + 1)$ λ-factors. Lemma 2 and our assumption will call an early halt to such proceedings, and we conclude an arrangement may be made so that $\beta_i \ll \lambda_i$ for $i = 1, \cdots, n$. This being so, we know, by a property of generalized functions, that

$$L(\beta_i, x) = L(\lambda_i, x)[\beta_i].$$

We further conclude the lemma is proved.

In view of Lemma 3, h is certainly well defined by (4.2); for if

$$\delta + \textstyle\sum_{j=1}^{n} \mu_j = \delta' + \sum_{k=1}^{m} \mu_k',$$

then $\delta = \delta'$, and we can write

$$\mu_j = \gamma_{1,j} + \cdots + \gamma_{m,j} \quad \text{and} \quad \mu_k' = \gamma_{1,k}' + \cdots + \gamma_{n,k}'$$

with $\gamma_{i,j} \equiv \gamma_{j,i}'$. It follows that (4.2) yields the same value for each representation. That (4.3) is well defined and is consistent with (4.2) is now immediate.

Recall that M_0 is the smallest algebra containing $M(Q)$ and all discrete measures, and M_1 is the algebra of all measures absolutely continuous w.r.t. some element of M_0. We wish to show that *h defined on M_1 by* (4.3) *is a homomorphism.* First h is additive, for suppose

$$\lambda \ll \delta + \textstyle\sum_{j=1}^{n} \mu_j \quad \text{and} \quad \nu \ll \delta' + \sum_{k=1}^{m} \mu_k'.$$

There is no loss of generality in assuming each of these measures is positive. Then λ, ν, and $(\lambda + \nu)$ are absolutely continuous to the sum of these two measures. As in the definition (4.3) we write λ, μ, and $(\lambda + \nu)$ as the sum of $1 + n + m$ components since there are that many terms in the sum, where $\lambda_0 \ll \delta + \delta'$, $\lambda_i \ll \mu_i$, $1 \leqq i \leqq n$, and $\lambda_i \ll \mu_i'$, $1 \leqq i - n \leqq n + m$;

and similarly for ν and $(\lambda + \nu)$. Now given any Borel set S, there is a set $K_i \subset G$ (μ_i is concentrated on K_i), and $\lambda_i(S) = \lambda(S \cap K_i)$; the equality remains true if we replace λ_i and λ by ν_i and ν, or by $(\lambda + \nu)_i$ and $(\lambda + \nu)$. Thus

$$(\lambda + \nu)_i(S) = (\lambda + \nu)(S \cap K_i) = \lambda(S \cap K_i) + \nu(S \cap K_i)$$
$$= \lambda_i(S) + \nu_i(S);$$

i.e., $(\lambda + \nu)_i = \lambda_i + \nu_i$. It follows that h is additive on M_1. Clearly h is homogeneous, and to prove multiplicity we let

$$\lambda = \sum_{j=0}^{n} \lambda_j \quad \text{and} \quad \nu = \sum_{k=0}^{m} \nu_k$$

as in (4.3). Since h is additive, we have $h(\lambda * \nu) = \sum_{j,k} h(\lambda_j * \nu_k)$. Now $\lambda_j * \nu_k \ll \mu_j * \mu_k'$. Referring to the definition of μ_j and μ_k' and using the Fubini theorem we see that $h(\lambda_j * \nu_k) = h(\lambda_j)h(\nu_k)$. Thus

$$h(\lambda * \nu) = \sum_{j,k} h(\lambda_j * \nu_k) = \sum_{j,k} h(\lambda_j)h(\nu_k)$$
$$= \left(\sum_j h(\lambda_j)\right)\left(\sum_k h(\nu_k)\right) = h(\lambda)h(\nu).$$

So h is a bounded homomorphism on M_1; *extend h uniquely to a homomorphism on the closure M of M_1.*

Now M satisfies the hypothesis of the generalized Lebesgue decomposition theorem. To see this, let $\mu \in M$ and let $\lambda \ll \mu$. There is a sequence $\{\mu_n\} \subset M_1$ with $\mu_n \to \mu$. Write $\lambda = \lambda_{1,n} + \lambda_{2,n}$, where $\lambda_{1,n} \perp \mu_n$ and $\lambda_{2,n} \ll \mu_n$. It follows that $\lambda_{1,n} \to 0$ and $\lambda_{2,n} \to \lambda$. But each $\lambda_{2,n} \in M_1$, so $\lambda \in M$. Therefore we may decompose $M(G)$ into the direct sum $M(G) = M + M^{\perp}$.

We now extend h to the entire algebra by the usual device: if $\mu \in M(G)$, define

(4.4) $$h(\mu) = h(\mu_M) \qquad (\mu_M \text{ is the projection of } \mu \text{ on } M).$$

A simple calculation shows that h is linear on $M(G)$. *If we can prove that $M^{\perp}$ is an ideal, it will follow h is also multiplicative.*

Consider the regular family of sets F generated by Q (see §2). Let H be the algebra of all measures concentrated in F, and let I be the ideal of all measures concentrated outside F. We know $M(G) = H + I$ and, clearly, $M \subset H$ and $I \subset M^{\perp}$. To prove our assertion above, let $\nu \in M^{\perp}$ and $\lambda \in M(G)$. Write $\nu = \nu_H + \nu_I$ and $\lambda = \lambda_H + \lambda_I$, where ν_H, etc. are the projections on H and I. So $\nu * \lambda = \nu_H * \lambda_H + \gamma$ where $\gamma \in I \subset M^{\perp}$; hence we may as well assume that ν and λ are concentrated in F. Because of our earlier remarks on regular families, and because H and $M^{\perp}$ are "translation invariant" (this means $\mu \in H \Leftrightarrow \mu_x \in H$ for all $x \in G$; μ_x is a measure defined by $\mu_x(A) = \mu(A - x)$), we may, and do, assume that ν and λ are concentrated on Q^s and Q^t, respectively. We make one further observation; it is sufficient to prove $\nu * \lambda \perp \mu_1 * \cdots * \mu_m$ where $\mu_i \in M(Q)$, $1 \leqq i \leqq m$. For, if this is true for all $\lambda \in H$, then $\nu * (\lambda * \sigma_{-x}) \perp \mu_1 * \cdots * \mu_m$ which implies

$$\nu * \lambda \perp \sigma_x * \mu_1 * \cdots * \mu_m .$$

It would follow that $\nu * \lambda \perp M_0$, and consequently, $\nu * \lambda \in M^{\perp}$.

Therefore, *we assume* $\nu \in (M^{\perp} \cap H)$, $\lambda \in H$, ν *is concentrated on* Q^s, λ *is concentrated on* Q^t, *and* $\mu = \mu_1 * \cdots * \mu_m$ *where* $\mu_i \in M(Q)$. *We will prove* $\nu * \lambda \perp \mu$.

Let $Q_0 = \{0\}$, $Q_1 = Q$, and for each $n = 2, 3, \cdots$, let $Q_n = Q^n - \bigcup_{i=1}^{n-1} Q^i$. Then $Q^n = \bigcup_{i=1}^{n} (Q_i \cap Q^n)$, and the sets $(Q_i \cap Q^n)$ are mutually disjoint. Let $\nu_i = \nu \mid (Q_i \cap Q^s)$ and $\lambda_j = \lambda \mid (Q_j \cap Q^t)$ for $1 \leqq i \leqq s$ and $1 \leqq j \leqq t$.

LEMMA 4. *For each* $k = 2, 3, \cdots, i < s$, *there are* only *a countable number of elements* $\{x_{k,j}\}_{j=1}^{\infty} \subset Q_{i-k}$ *such that* $(Q_k + x_{k,j}) \subset Q_i$ *and* $\nu_i(Q_k + x_{k,j}) \neq 0$.

Proof. Clearly $\nu_i(Q_1 + x) = 0$ for every $x \in G$ since $\nu_i \perp M$. Let $k = 2$. For $x_1 \neq x_2 \in Q_{i-2}$, let $x_1 = q_1 + \cdots + q_{i-2}$ and $x_2 = r_1 + \cdots + r_{i-2}$. Then $(Q_2 + x_1) \cap (Q_2 + x_2)$ is empty, one point, or a translation of Q_1 depending on whether x_1 and x_2 have $(i - 5)$ or less common terms, $(i - 4)$ common terms, or $(i - 3)$ common terms. In any case

$$\nu_i((Q_2 + x_1) \cap (Q_2 + x_2)) = 0;$$

this surely implies the lemma is true for $k = 2$. Using induction, suppose the statement is true for $k < n \leqq i$. Now $x_1 \neq x_2$ are in Q_{i-n}, and

$$(Q_n + x_j) \subset Q_i , \qquad j = 1, 2.$$

By using the above argument, if they have $i - 2n + 1$ or less elements in common, $\nu_i((Q_n + x_1) \cap (Q_n + x_2)) = 0$. On the other hand, for each $j = 1, 2, \cdots, n - 2$, if x_1 and x_2 have $i - 2n + 1 + j$ common terms, then $(Q_n + x_1) \cap (Q_n + x_2) \subset (Q_{j+1} + y)$ where $y \in Q_{i-j-1}$. Observe, since all of these sets are in Q_i, a term appears in y if and only if it appears either in x_1 or x_2. (It is assumed, of course, if a term appears more than once, it is counted as a separate term each time.) Thus, there are at most $\binom{i-j-1}{i-n}$ sets $(Q_n + x)$ whose pairwise intersections are contained in $(Q_{j+1} + y)$ for each $y \in Q_{i-j-1}$. By the induction hypothesis only a countable number of sets $(Q_n + x)$ can have pairwise intersections of nonzero ν_i-measure; so the rest must have pairwise intersections of zero ν_i-measure. The desired conclusion is now immediate.

Let $\{x_{i,k,j}\}$, $i = 2, \cdots, s - 1$; $k = 2, \cdots, i$; and $j = 1, 2, \cdots$ be *the* sequence of elements such that (1) $x_{i,k,j} \in Q_{i-k}$, (2) $(Q_k + x_{i,k,j}) \subset Q_i$, and (3) $\nu_i(Q_k + x_{i,k,j}) \neq 0$. For convenience, we also allow 0 to be in this sequence.

We assert the existence of subsets $A, B_1, \cdots, B_m$ such that

(1) ν is concentrated on A and μ_p is concentrated on B_p, $1 \leqq p \leqq m$, and

(2) $\mu_{p_1} * \cdots * \mu_{p_r}((B_{p_1} + \cdots + B_{p_r}) \cap (A \pm x_{i,k,j})) = 0$

(here we want 0 to be one of the $x_{i,k,j}$) for any combination of μ_p's and all $x_{i,k,j}$'s. The construction of such sets is not hard; we consider $\mu_1, \cdots, \mu_m$ and all products of these. There are only countably many finite sums $\sum \sigma_{\pm x_{i,k,j}}$, and $\nu * \sum \sigma_{\pm x_{i,k,j}} = \sum \nu_{\pm x_{i,k,j}}$ is singular to M. The rest is straightforward.

We are now ready to prove $\nu * \lambda \perp \mu$. Recall λ is concentrated on Q^t, and now ν is concentrated on $Q^s \cap A$, and μ is concentrated on

$$\textstyle\sum_{p=1}^m B_p = B_1 + \cdots + B_p .$$

Also $\nu_i = \nu \mid (Q_i \cap Q^s \cap A)$ and $\lambda_j = \lambda \mid (Q_j \cap Q^t)$. It is sufficient to prove $\nu_i * \lambda_j \perp \mu$ for all i and j, $1 \leqq i \leqq s$ and $1 \leqq j \leqq t$. *We shall show that any Borel set* $S \subset \left(\sum_{p=1}^m B_p \cap ((Q_i \cap A) + Q_j)\right)$ *can be written as the union of sets each of which is either of* $\nu_i * \lambda_j$*-measure zero or of* μ*-measure zero.*

First, if $m > i + j$, then Lemma 1 provides $\mu(S) = 0$; we therefore assume $m \leqq i + j$ and $\sum_{p=1}^m B_p \subset Q_m$. Next, if $m = i + j$, then some finite sum of b_p's is in A, and by condition (2) above it would follow that $\mu(S) = 0$. So it reduces to the case $m < i + j$. If $s \in S$, there is an $x = q_1 + \cdots + q_i \in Q_i \cap A$, a $y = r_1 + \cdots + r_j \in Q_j$, and a $b = b_1 + \cdots + b_m \in \sum_{p=1}^m B_p$ such that $s = x + y = b$. Since $m < i + j$, we must have $q_{u_1} = -r_{v_1}, \cdots, q_{u_w} = -r_{v_w}$ where $w = \frac{1}{2}(i + j - m)$.

Thus $x \in (Q_{i-w} + z)$ where $z = -(q_{u_1} + \cdots + q_{u_w}) \in Q_w$. Divide S into sets

$$S_1 = \{s \in S : s = x + y;\, x \in (Q_{i-w} + z) \cap A;\, \nu_i(Q_{i-w} + z) \neq 0\}$$

and its complement S_2. If $s \in S_1$, then z is some $x_{i,k,j}$. But $x = q + z$, $q \in Q_{i-w}$; this makes $q = x - z \in (A - z)$. This compels a finite sum of b_p's to be in $(A - z)$, and, as before, $\mu(S_1) = 0$. Now

$$S_2 = \{s \in S : s = x + y;\, x \in (Q_{i-w} + z) \cap A;\, \nu_i(Q_{i-w} + z) = 0\}.$$

So for each fixed y, $\nu_i(Q_{i-w} + z) = 0$, and we infer that $\nu_i * \lambda_j(S_2) = 0$. This completes the proof and the construction.

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