SOME UNIQUENESS THEOREMS ON RIEMANNIAN MANIFOLDS WITH BOUNDARY[1]

BY

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1. Introduction

Let M_n be a differentiable manifold of dimension n, and $X\colon M_n \rightarrow E_{n+m}$ a mapping of M_n into a Euclidean space E_{n+m} of dimension $n + m$ for any $m > 0$. M_n, or rather M_n together with the mapping X, is called an immersed submanifold of E_{n+m} if the functional matrix of X is of rank n everywhere. The submanifold M_n is said to be imbedded, if X is one-one, that is, if $X(P) = X(Q)$, $P, Q \in M_n$, implies that $P = Q$. In particular, when $m = 1$, an immersed (imbedded) submanifold M_n of the space E_{n+m} is called an immersed (imbedded) hypersurface. Throughout this paper all manifolds are supposed to be of class C^3, and the dimension of a manifold M_n is understood to be n.

Now let us consider an oriented immersed manifold M_n. Then to each point $P \in M_n$ there is a unique linear space N of dimension m normal to $X(M_n)$ at the point $X(P)$. For any unit normal vector $e_r(P)$ at the point $X(P)$ in the space N, we put

$$I = dX \cdot dX, \qquad II_r = de_r \cdot dX, \qquad III_r = de_r \cdot de_r, \tag{1.1}$$

where dX and de_r are vector-valued linear differential forms on M_n, and the dot denotes the scalar product of two vectors in the space E_{n+m}. The eigenvalues $k_{r1}, \cdots, k_{rn}$ of II_r relative to I are called the principal curvatures of the manifold M_n associated with the unit normal vector $e_r(P)$. If the Gauss-Kronecker curvature $K_r = k_{r1} \cdots k_{rn}$ associated with the vector $e_r(P)$ is nonzero, the reciprocals $1/k_{r1}, \cdots, 1/k_{rn}$, called the radii of principal curvatures associated with the vector $e_r(P)$, are the eigenvalues of II_r relative to III_r, which is also positive definite due to the assumption $K_r \neq 0$. In this case we introduce the α^{th} elementary symmetric function

$$\binom{n}{\alpha} P_{r\alpha} = \sum 1/k_{r1} \cdots 1/k_{r\alpha} \qquad (1 \leqq \alpha \leqq n). \tag{1.2}$$

If M_n is a hypersurface, then at each point $X(P)$ of M_n there is only one unit normal vector e_r, and for $P_{r\alpha}$ associated with it we shall simply write P_α.

Let M_n be a closed oriented Riemannian manifold immersed in a Euclidean space E_{n+m}. By a normal frame $Xe_{n+1} \cdots e_{n+m}$ on the manifold M_n we mean a point X of the manifold M_n and an ordered set of mutually perpendicular unit vectors $e_{n+1}, \cdots, e_{n+m}$ normal to the manifold M_n at the point

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X. M_n is called a star manifold,[2] if there exist a point O, called a pole, in the manifold M_n and a class C^2 field of normal frames $Xe_{n+1} \cdots e_{n+m}$ over the manifold M_n such that the Gauss-Kronecker curvature K_r of the manifold M_n and the support function $X \cdot e_r$ with respect to the pole O are positive for every vector e_r, $n + 1 \leqq r \leqq n + m$, at every point of the manifold M_n. This normal frame $Xe_{n+1} \cdots e_{n+m}$ is called a fundamental normal frame of the star manifold M_n at the point X. An n-dimensional star manifold with boundary is an n-dimensional compact subset of an n-dimensional star manifold. An n-dimensional convex hypersurface with boundary is an n-dimensional compact subset of the boundary of a convex region in an $(n + 1)$-dimensional Euclidean space E_{n+1}, or is equivalently an n-dimensional compact subset of an n-dimensional imbedded hypersurface with positive Gauss-Kronecker curvature everywhere. An n-dimensional convex hypercap is an n-dimensional convex hypersurface with boundary such that in the space E_{n+1} there is at least one fixed direction, along which every line either is a tangent to the hypersurface or intersects the hypersurface at most at one point. It is obvious that a convex hypercap can never be closed.

Since Christoffel [5] established in 1865 his well-known uniqueness or rigidity theorem on closed convex surfaces in a space E_3, various uniqueness theorems of the same type on closed convex hypersurfaces have been obtained by different authors with different methods. It is natural to ask whether we can extend some of these uniqueness theorems on closed convex hypersurfaces to general immersed manifolds with boundary. In recent years the present and other authors have succeeded in deriving some new integral formulas, by means of which most classical uniqueness theorems can easily be extended to convex hypersurfaces with boundary satisfying a natural boundary condition. For uniqueness theorems on general immersed manifolds with boundary, due to the complication arising from the immersion, the only result we have so far is the generalization [9] of Christoffel's uniqueness theorem to two-dimensional immersed manifolds with boundary. The main purpose of the present paper is to further extend this uniqueness theorem to immersed manifolds of a general dimension $n > 2$ with boundary, and to establish a uniqueness theorem on convex hypercaps by proving the following theorems.

THEOREM I. *Let M_n and M_n^* be two star manifolds, with boundaries B_{n-1} and B_{n-1}^* respectively, in a Euclidean space E_{n+m} for any $m > 0$. Suppose that there exists an orientation-preserving diffeomorphism f of the manifold M_n onto the manifold M_n^* such that, at each pair of corresponding points, the manifolds M_n and M_n^* have a common fundamental normal frame $e_{n+1} \cdots e_{n+m}$ and equal $P_{r,n-1}$ defined by equation* (1.2) *and associated with each common unit normal vector e_r, $r = n + 1, \cdots, n + m$. If the diffeomorphism f restricted to the boundary B_{n-1} is a translation* (*strictly speaking, is induced by a transla-*

[2] The author is indebted to the referee for his comment which leads to the definition of a star manifold in the present form.

lation in the space E_{n+m}) *carrying the boundary* B_{n-1} *onto the boundary* B^*_{n-1}, *then the diffeomorphism* f *is a translation carrying the whole manifold* M_n *onto the whole manifold* M^*_n.

THEOREM II. *Let* M_n *be a star manifold with a spherical boundary* B_{n-1} *such that at every point* $P^{\mu}_{r,n-2} P^{\nu}_{r,n-1}$ *is constant for* $\mu + \nu > 0$, $\mu \geqq 0$, $\nu > 0$ *and for each vector* e_r *of a fundamental normal frame of the manifold* M_n. *Then the manifold* M_n *is a compact subset of an n-sphere.*

THEOREM III. *Let* M_n *and* M^*_n *be two oriented convex hypercaps with boundaries* B_{n-1} *and* B^*_{n-1} *respectively. Suppose that there exists an orientation-preserving diffeomorphism* f *of the hypercap* M_n *onto the hypercap* M^*_n *such that at each pair of corresponding points the hypercaps* M_n *and* M^*_n *have the same outer normal vector and satisfy either*

$$P_1 \leqq P^*_1, \qquad P_2 \geqq P^*_2, \tag{1.3}$$

or

$$P_1 \geqq P^*_1, \qquad P_2 \leqq P^*_2, \tag{1.4}$$

where P_α *and* P^*_α *are defined by equation* (1.2) *for the hypercaps* M_n *and* M^*_n *respectively. If the diffeomorphism* f *restricted to the boundary* B_{n-1} *is a translation carrying the boundary* B_{n-1} *onto the boundary* B^*_{n-1}, *then the diffeomorphism* f *is a translation carrying the whole hypercap* M_n *onto the whole hypercap* M^*_n.

COROLLARY. *Let* M_n *be a convex hypercap with a spherical boundary* B_{n-1}. *If there is a constant* c *such that, at each point of the hypercap* M_n, *either*

$$P_1 \leqq c \leqq P_2^{1/2} \tag{1.5}$$

or

$$P_1 \geqq c \geqq P_2^{1/2}, \tag{1.6}$$

then the hypercap M_n *is a compact subset of an n-dimensional hypersphere.*

It should be noted that when $n = 2$, the conditions (1.3) and (1.4) together are obviously weaker than the condition of Alexandroff [2], which can be stated as follows: At each pair of corresponding points the hypercaps M_2 and M^*_2 satisfy the condition $F(2P_1, P_2) = F(2P^*_1, P^*_2)$, where $F(U, V)$, for $U > 0$, $U^2 \geqq 4V > 0$, is a continuous function monotonely increasing in both variables U and V. Furthermore, Grotemeyer [8] obtained Theorem III for $n = 2$ in terms of the condition of Alexandroff, but actually only used the conditions (1.3) and (1.4) together in his proof.

2. Immersed submanifolds in Euclidean space

Suppose a Euclidean space E_{n+m} is oriented. By a frame $Xe_1 \cdots e_{n+m}$ in the space E_{n+m} we mean a point X and an ordered set of mutually perpendicular unit vectors $e_1, \cdots, e_{n+m}$ with an orientation coherent with that of the space E_{n+m} so that the determinant $|e_1, \cdots, e_{n+m}|$ is equal to $+1$. To avoid confusion we shall use the following ranges of indices throughout this paper:

$$(2.1)\qquad \begin{gathered} 1 \leqq \alpha, \beta, \gamma \leqq n, \qquad n + 1 \leqq r, s, t \leqq n + m, \\ 1 \leqq i, j, k \leqq n + m. \end{gathered}$$

Then we have

$$(2.2)\qquad e_i \cdot e_j = \delta_{ij},$$

where δ_{ij} are the Kronecker deltas. Let $F(n, m)$ be the space of all frames in the space E_{n+m}, so that $\dim F(n, m) = \frac{1}{2}(n + m)(n + m + 1)$. In $F(n, m)$ we introduce the linear differential forms ω_i', ω_{ij}' by the equations

$$(2.3)\qquad dX = \textstyle\sum_i \omega_i' e_i, \qquad de_i = \textstyle\sum_j \omega_{ij}' e_j,$$

where

$$(2.4)\qquad \omega_{ij}' + \omega_{ji}' = 0.$$

Since $d(dX) = 0$ and $d(de_i) = 0$, from equations (2.3) we have

$$(2.5)\qquad d\omega_i' = \textstyle\sum_j \omega_j' \wedge \omega_{ji}', \qquad d\omega_{ij}' = \textstyle\sum_k \omega_{ik}' \wedge \omega_{kj}',$$

where $\wedge$ denotes the exterior product.

As explained in §1, by an immersed submanifold in the space E_{n+m} we mean an abstract manifold M_n and a mapping $X\colon M_n \to E_{n+m}$ such that the induced mapping X_* on the tangent space is univalent everywhere. Analytically, the mapping can be defined by a vector-valued function $X(P)$, $P \in M_n$. Our assumption implies that the differential $dX(P)$ of $X(P)$, which is a linear differential form on M_n with value in E_{n+m}, has as values a linear combination of n, but not less than n, vectors $t_1, \cdots, t_n$. Since X_* is univalent, we can identify the tangent space of M_n at the point P with the vector space spanned by $t_1, \cdots, t_n$. A linear combination of the vectors $t_1, \cdots, t_n$ is called a tangent vector, and a vector perpendicular to them is called a normal vector. The immersion of M_n in E_{n+m} gives rise to a bundle B, whose bundle space is the subset of $M_n \times F(n, m)$ consisting of

$$(P, X(P)e_1 \cdots e_{n+1} \cdots e_{n+m}) \in M_n \times F(n, m)$$

such that $e_1, \cdots, e_n$ are tangent vectors and $e_{n+1}, \cdots, e_{n+m}$ are normal vectors at the point $X(P)$.

Consider the inclusion mapping ϕ and the projection p:

$$(2.6)\qquad B \xrightarrow{\ \phi\ } M_n \times F(n, m) \xrightarrow{\ p\ } F(n, m).$$

By putting

$$(2.7)\qquad \omega_i = (p\phi)^*\omega_i', \qquad \omega_{ij} = (p\phi)^*\omega_{ij}',$$

from equations (2.4) and (2.5) we have

$$\omega_{ij} + \omega_{ji} = 0, \tag{2.8}$$

$$d\omega_i = \sum_j \omega_j \wedge \omega_{ji}, \qquad d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}. \tag{2.9}$$

From the definition of the bundle B it follows that $\omega_r = 0$ and that ω_α are linearly independent. Thus the first equation of (2.9) gives

$$\sum_\alpha \omega_\alpha \wedge \omega_{\alpha r} = 0,$$

from which we have

$$\omega_{\alpha r} = \sum_\beta A_{r\alpha\beta}\, \omega_\beta, \qquad A_{r\alpha\beta} = A_{r\beta\alpha}. \tag{2.10}$$

If det $(A_{r\alpha\beta}) \neq 0$ for some r, by introducing the matrix $(\lambda_{r\alpha\beta})$ inverse to the matrix $-(A_{r\alpha\beta})$ we have

$$\omega_\alpha = \sum_\beta \lambda_{r\alpha\beta}\, \omega_{r\beta}. \tag{2.11}$$

By means of equations (2.2), (2.3), (2.7), (2.10), and (2.11), equations (1.1) can be written as

$$\begin{gathered} I = \sum_\alpha \omega_\alpha^2, \qquad III_r = \sum_i \omega_{ri}^2, \\ II_r = +\sum_\alpha \omega_{r\alpha}\, \omega_\alpha = -\sum_{\alpha,\beta} A_{r\alpha\beta}\, \omega_\alpha\, \omega_\beta = +\sum_{\alpha,\beta} \lambda_{r\alpha\beta}\, \omega_{r\alpha}\, \omega_{r\beta}. \end{gathered} \tag{2.12}$$

Suppose

$$\det\,(\delta_{\alpha\beta} + \lambda_{r\alpha\beta}\, y) = \sum_{0 \leqq \gamma \leqq n} \binom{n}{\gamma} P_{r\gamma}(\lambda_r) y^\gamma, \tag{2.13}$$

where y is a parameter. Then $P_{r\gamma}(\lambda_r)$ is a polynomial of degree γ in $\lambda_{r\alpha\beta}$ for a fixed r, and it is easily seen that $P_{r\alpha}(\lambda_r)$ is equal to the invariant $P_{r\alpha}$ defined by equation (1.2).

Through a point in a Euclidean space E_{n+m} let $A_1, \cdots, A_{n+m-1}$ be $n+m-1$ differentiable vector functions of n variables $x^1, \cdots, x^n$, and let J be any vector. Then the scalar product of the vector J and the vector product $A_1 \times \cdots \times A_{n+m-1}$ of the vectors $A_1, \cdots, A_{n+m-1}$ is given by

$$J \cdot (A_1 \times \cdots \times A_{n+m-1}) = (-1)^{n+m-1} | J, A_1, \cdots, A_{n+m-1} |, \tag{2.14}$$

from which it follows that

$$e_1 \times \cdots \times \hat{e}_r \times \cdots \times e_{n+m} = (-1)^{n+m+r} e_r, \tag{2.15}$$

where the circumflex over e_r indicates that the vector e_r is to be deleted. In a previous paper of the author [10] we have combined the vector product of vectors and the exterior product of differentials to define the vector

$$\begin{aligned} &A_1 \otimes \cdots \otimes A_{i-1} \otimes dA_i \otimes \cdots \otimes dA_{n+m-1} \\ &= (A_1 \times \cdots \times A_{i-1} \times A_{i,\alpha_i} \times \cdots \times A_{n+m-1,\alpha_{n+m-1}})\, dx^{\alpha_i} \\ &\qquad \wedge \cdots \wedge dx^{\alpha_{n+m-1}}, \end{aligned} \tag{2.16}$$

where $i = 1, \cdots, n + m - 1$ and $A_{i,\alpha_i} = \partial A_i / \partial x^{\alpha_i}$. It is obvious that the vector (2.16) is independent of the order of the vectors $dA_i, \cdots, dA_{n+m-1}$. Let dA be the area element of an immersed submanifold M_n in the space E_{n+m}. Then by means of the combined operation $\otimes$ we obtain

$$\underbrace{dX \otimes \cdots \otimes dX}_{n} \otimes e_{n+1} \otimes \cdots \otimes \hat{e}_r \otimes \cdots \otimes e_{n+m} = (-1)^{n+m+r} n!\, e_r\, dA, \tag{2.17}$$

$$\underbrace{de_r \otimes \cdots \otimes de_r}_{n} \otimes e_{n+1} \otimes \cdots \otimes \hat{e}_r \otimes \cdots \otimes e_{n+m} = (-1)^{n+m+r} n!\, e_r\, K_r\, dA. \tag{2.18}$$

From equations (2.3), (2.7), (2.15), (2.17), and (2.18) it follows that

$$dA = \omega_1 \wedge \cdots \wedge \omega_n, \tag{2.19}$$

$$K_r\, dA = \omega_{r1} \wedge \cdots \wedge \omega_{rn}. \tag{2.20}$$

3. Integral formulas for a pair of immersed manifolds with boundary

Let M be a compact differentiable manifold of dimension n with boundary, and let M_n and M_n^* be immersed manifolds with boundaries B_{n-1} and B_{n-1}^* given by $X: M \to E_{n+m}$ and $X^*: M \to E_{n+m}$, respectively. Then §2 can be applied to the manifolds M_n, and for the corresponding quantities and equations for the manifold M_n^* we shall use the same symbols and numbers with a star respectively.

Suppose that there is a diffeomorphism f of the manifold M_n onto the manifold M_n^* such that at each pair of corresponding points the manifolds M_n and M_n^* have parallel tangent spaces. Without loss of generality we may assume that

$$e_i^* = e_i \qquad (i = 1, \cdots, n + m). \tag{3.1}$$

From equations (2.3), (2.7), (2.3)*, (2.7)*, and (3.1) it follows that

$$\omega_{r\alpha}^* = \omega_{r\alpha}. \tag{3.2}$$

Now for the pair of immersed manifolds M_n and M_n^* we introduce the following differential forms:

$$B_{\alpha, n-2-\alpha} = \sum_{r=n+1}^{n+m} (-1)^{r-1} \cdot \left| X, X^*, e_{n+1}, \cdots, \hat{e}_r, \cdots, e_{n+m}, de_r, \underbrace{dX, \cdots, dX}_{\alpha}, \underbrace{dX^*, \cdots, dX^*}_{n-2-\alpha} \right|, \tag{3.3}$$

$$C_{r\beta,n-1-\beta} = (-1)^{m+r} \cdot \left| X, e_{n+1}, \cdots, \hat{e}_r, \cdots, e_{n+m}, de_r, \underbrace{dX, \cdots, dX}_{\beta}, \underbrace{dX^*, \cdots, dX^*}_{n-1-\beta} \right|, \tag{3.4}$$

$$C^*_{r\beta,n-1-\beta} = (-1)^{m+r} \cdot \left| X^*, e_{n+1}, \cdots, \hat{e}_r, \cdots, e_{n+m}, de_r, \underbrace{dX, \cdots, dX}_{\beta}, \underbrace{dX^*, \cdots, dX^*}_{n-1-\beta} \right|, \tag{3.5}$$

$$D_{r\beta,n-1-\beta} = (-1)^{m+r} \cdot \left| e_r, e_{n+1}, \cdots, e_{r-1}, e_{r+1}, \cdots, e_{n+m}, de_r, \underbrace{dX, \cdots, dX}_{\beta}, \underbrace{dX^*, \cdots, dX^*}_{n-1-\beta} \right|, \tag{3.6}$$

where $0 \leqq \alpha \leqq n-2$ and $0 \leqq \beta \leqq n-1$. By means of equation (2.14) and the operation $\otimes$ we obtain

$$\begin{gathered} C_{r\beta,n-1-\beta} = (-1)^{n+r-1} X \cdot E, \qquad C^*_{r\beta,n-1-\beta} = (-1)^{n+r-1} X^* \cdot E, \\ D_{r\beta,n-1-\beta} = (-1)^{n+r-1} e_r \cdot E, \end{gathered} \tag{3.7}$$

where

$$E = e_{n+1} \otimes \cdots \otimes \hat{e}_r \otimes \cdots \otimes e_{n+m} \otimes de_r \otimes \underbrace{dX \otimes \cdots \otimes dX}_{\beta} \otimes \underbrace{dX^* \otimes \cdots \otimes dX^*}_{n-1-\beta}.$$

From the definition of the operation $\otimes$ and the last equation of (3.7) it follows immediately that

$$(-1)^{n+r-1} E = D_{r\beta,n-1-\beta}\, e_r. \tag{3.8}$$

Thus the substitution of equation (3.8) in the first two equations of (3.7) gives

$$C_{r\beta,n-1-\beta} = h_r D_{r\beta,n-1-\beta}, \qquad C^*_{r\beta,n-1-\beta} = h^*_r D_{r\beta,n-1-\beta}, \tag{3.9}$$

where we have placed

$$h_r = X \cdot e_r, \qquad h^*_r = X^* \cdot e_r. \tag{3.10}$$

By using equations (3.3), (3.4), (3.5), and (3.9), applying the ordinary rules for differentiation of determinants, and noticing the pairwise cancellation of terms, we can easily obtain

$$\begin{aligned} dB_{\alpha,n-2-\alpha} &= \textstyle\sum_{r=n+1}^{n+m} (C_{r\alpha,n-1-\alpha} - C^*_{r,\alpha+1,n-2-\alpha}) \\ &= \textstyle\sum_{r=n+1}^{n+m} (h_r D_{r\alpha,n-1-\alpha} - h^*_r D_{r,\alpha+1,n-2-\alpha}) \\ &\qquad (0 \leqq \alpha \leqq n-2). \end{aligned} \tag{3.11}$$

Integrating both sides of equation (3.11) over the manifold M_n and applying Stokes's theorem to the left side, we can arrive at the integral formulas

$$(3.12)\quad \int_{B_{n-1}} B_{\alpha,n-2-\alpha} = \int_{M_n} \sum_{r=n+1}^{n+m} (h_r D_{r\alpha,n-1-\alpha} - h_r^* D_{r,\alpha+1,n-2-\alpha}) \qquad (0 \leqq \alpha \leqq n-2).$$

To apply the formulas (3.12) we introduce the differential forms of order n

$$(3.13)\quad D_{r\alpha\beta} = (-1)^{m+r} \cdot \left| e_r, e_{n+1}, \cdots, e_{r-1}, e_{r+1}, \cdots, e_{n+m}, \underbrace{de_r, \cdots, de_r}_{n-(\alpha+\beta)}, \underbrace{dX, \cdots, dX}_{\alpha}, \underbrace{dX^*, \cdots, dX^*}_{\beta} \right|, \qquad (0 \leqq \alpha, \beta \leqq n).$$

In virtue of equations (2.3), (2.7), (2.8), (2.11), (2.14), (2.15), (2.20), (2.3)*, (2.7)*, (2.11)*, (3.2), and (3.13) we can easily obtain, for two parameters y and y^*,

$$(3.14)\quad \begin{aligned} &\sum_{0 \leqq \alpha+\beta \leqq n} \frac{n!}{\alpha!\,\beta!\,(n-\alpha-\beta)!} y^\alpha y^{*\beta} D_{r\alpha\beta} \\ &= (-1)^{(n+1)(m-1)} \sum_{1 \leqq \alpha_1, \cdots, \alpha_n \leqq n} \varepsilon_{\alpha_1 \cdots \alpha_n} (y\omega_{\alpha_1} + y^*\omega_{\alpha_1}^* + \omega_{r\alpha_1}) \\ &\qquad \wedge \cdots \wedge (y\omega_{\alpha_n} + y^*\omega_{\alpha_n}^* + \omega_{r\alpha_n}) \\ &= (-1)^{(n+1)(m-1)} n!\,(y\omega_1 + y^*\omega_1^* + \omega_{r1}) \wedge \cdots \wedge (y\omega_n + y^*\omega_n^* + \omega_{rn}) \\ &= (-1)^{(n+1)(m-1)} n! \det (y\lambda_{r\alpha\beta} + y^*\lambda_{r\alpha\beta}^* + \delta_{\alpha\beta}) K_r\, dA, \end{aligned}$$

where $\varepsilon_{\alpha_1 \cdots \alpha_n}$ is $+1$ or -1 according as $\alpha_1, \cdots, \alpha_n$ form an even or odd permutation of $1, \cdots, n$, and is zero otherwise. Now suppose

$$(3.15)\quad \det (y\lambda_{r\alpha\beta} + y^*\lambda_{r\alpha\beta}^* + \delta_{\alpha\beta}) = \sum_{0 \leqq \alpha+\beta \leqq n} \frac{n!}{\alpha!\,\beta!\,(n-\alpha-\beta)!} y^\alpha y^{*\beta} P_{r\alpha\beta},$$

so that $P_{r\alpha\beta}$ is a homogeneous polynomial of degrees α and β in $\lambda_{r\rho\sigma}$ and $\lambda_{r\rho\sigma}^*$ $(\rho, \sigma = 1, \cdots, n)$ respectively. In particular, from equation (2.13) it follows that $P_{r\alpha0} = P_{r\alpha}(\lambda_r)$. Comparison of equation (3.14) with equation (3.15) yields immediately

$$(3.16)\quad D_{r\alpha\beta} = (-1)^{(n+1)(m-1)} n!\, P_{r\alpha\beta} K_r\, dA \qquad (0 \leqq \alpha, \beta \leqq n).$$

Substituting equation (3.16) in equation (3.12) we therefore obtain

$$(3.17)\quad \begin{aligned} &\int_{B_{n-1}} B_{\alpha,n-2-\alpha} \\ &= (-1)^{(n+1)(m-1)} n! \int_{M_n} \sum_{r=n+1}^{n+m} (h_r P_{r\alpha,n-1-\alpha} - h_r^* P_{r,\alpha+1,n-2-\alpha}) K_r\, dA, \\ &\qquad (0 \leqq \alpha \leqq n-2). \end{aligned}$$

If the diffeomorphism f of the manifold M_n onto the manifold M_n^* restricted to the boundary B_{n-1} is a translation carrying the boundary B_{n-1} onto the boundary B_{n-1}^*, then on the boundary B_{n-1}, $dX^* = dX$, and therefore

$B_{0,n-2} = B_{n-2,0}$. From the two equations of (3.17), for which $\alpha = 0$, $n - 2$ respectively, we can easily obtain the integral formula

$$(3.18)\quad \begin{aligned} &2\int_{M_n} \sum_{r=n+1}^{n+m} h_r(P_{r0,n-1} - P_{r,n-2,1})K_r\, dA \\ &= \int_{M_n} \sum_{r=n+1}^{n+m} [h_r^*(P_{r1,n-2} - P_{r,n-1,0}) - h_r(P_{r,n-2,1} - P_{r0,n-1})]K_r\, dA. \end{aligned}$$

Addition of equation (3.18) to the one obtained by interchanging the roles of the two manifolds M_n and M_n^* in equation (3.18) thus gives

$$(3.19)\quad \int_{M_n} \sum_{r=n+1}^{n+m} [h_r(P_{r0,n-1} - P_{r,n-2,1}) + h_r^*(P_{r,n-1,0} - P_{r1,n-2})]K_r\, dA = 0.$$

Let $F_r(u_1, \cdots, u_n)$, $r = n + 1, \cdots, n + m$, be m functions in n positive variables $u_1, \cdots, u_n$. We shall say that each function F_r is of type $n - 1$, if the following two conditions for each r are satisfied:

(i) $F_r(P_{r10}, \cdots, P_{rn0}) = F_r(P_{r01}, \cdots, P_{r0n})$ implies that $P_{r,n-2,1} \geqq P_{r0,n-1}$, and therefore, by interchanging the two manifolds M_n and M_n^*, that $P_{r1,n-2} \geqq P_{r,n-1,0}$,

(ii) $F_r(P_{r10}, \cdots, P_{rn0}) = F_r(P_{r01}, \cdots, P_{r0n})$ and $P_{r,n-2,1} = P_{r0,n-1}$ (or $P_{r1,n-2} = P_{r,n-1,0}$) if and only if $\lambda^*_{r\alpha\beta} = \lambda_{r\alpha\beta}$ for $\alpha, \beta = 1, \cdots, n$.

THEOREM 3.1. *Let M_n and M_n^* be two star manifolds, with boundaries B_{n-1} and B^*_{n-1} respectively, in a Euclidean space E_{n+m} for any $m > 0$, and let $F_r(u_1, \cdots, u_n)$, $r = n + 1, \cdots, n + m$, be m functions of type $n - 1$ in n positive variables $u_1, \cdots, u_n$. Suppose that there exists an orientation-preserving diffeomorphism f of the manifold M_n onto the manifold M_n^* such that, at each pair of corresponding points, the manifolds M_n and M_n^* have a common fundamental normal frame $e_{n+1} \cdots e_{n+m}$, and the functions $F_r(P_{r10}, \cdots, P_{rn0})$ and $F_r(P_{r01}, \cdots, P_{r0n})$ have the same value for each r. If the diffeomorphism f restricted to the boundary B_{n-1} is a translation carrying the boundary B_{n-1} onto the boundary B^*_{n-1}, then the diffeomorphism f is a translation carrying the whole manifold M_n onto the whole manifold M_n^*.*

Proof. Applying a translation in the space E_{n+m} if necessary, without loss of generality we may assume the poles in the star manifolds M_n and M_n^* to be coincident, so that $h_r > 0$ and $h_r^* > 0$ for $r = n + 1, \cdots, n + m$ over the whole manifolds M_n and M_n^*. Thus, due to the first property of the functions F_r and the assumption that $K_r > 0$, each term of the integrand of equation (3.19) is nonpositive, and equation (3.19) holds when and only when

$$(3.20)\quad P_{r0,n-1} = P_{r,n-2,1}, \qquad P_{r,n-1,0} = P_{r1,n-2} \qquad (r = n + 1, \cdots, n + m).$$

By the second property of the functions F_r we therefore obtain

$$(3.21)\quad \lambda^*_{r\alpha\beta} = \lambda_{r\alpha\beta} \qquad (\alpha, \beta = 1, \cdots, n).$$

Substitution of equation (3.21) in equations (2.11) and (2.11)* gives immediately

$$\omega_\alpha^* = \omega_\alpha \qquad (\alpha = 1, \cdots, n). \tag{3.22}$$

From equations (3.22), (2.3), (2.7), (2.3)*, and (2.7)* it follows that $dX^* = dX$ over the whole manifold M_n, and hence the proof of the theorem is complete.

In particular, if the second manifold M_n^* in Theorem 3.1 is a compact subset of an n-sphere, then Theorem 3.1 becomes

THEOREM 3.2. *Let M_n be a star manifold with a spherical boundary B_{n-1} in a Euclidean space E_{n+m} for any $m > 0$. If there are m functions*

$$F_r(u_1, \cdots, u_n), \qquad r = n+1, \cdots, n+m,$$

in n positive variables $u_1, \cdots, u_n$ with the following two properties for each vector e_r of a fundamental normal frame at every point of the manifold M_n:

(i) *$F_r(P_{r1}, \cdots, P_{rn}) = F_r(a, \cdots, a^n) =$ constant implies that $P_{r,n-2} \geqq a^{n-2}$,*

(ii) *$F_r(P_{r1}, \cdots, P_{rn}) = F_r(a, \cdots, a^n)$ and $P_{r,n-2} = a^{n-2}$ imply that $\lambda_{r\alpha\beta} = a\delta_{\alpha\beta}$ for $\alpha, \beta = 1, \cdots, n$,*

then the manifold M_n is a compact subset of an n-sphere of radius a.

For $m = 1$, the integral formulas (3.12), (3.17), (3.18) and Theorems I, II, 3.1, 3.2 were obtained by Chern [3].

4. Proofs of Theorems I and II

Proof of Theorem I. Theorem I follows from Theorem 3.1 immediately if we can show that the m functions $F_r = P_{r,n-1}$, $r = n+1, \cdots, n+m$, are of type $n-1$. To this end we need the following inequality of Gårding [7]:

Let $P_{r,n-1}(\lambda_r^{(1)}, \cdots, \lambda_r^{(n-1)})$ be the completely polarized form of the polynomial $P_{r,n-1}(\lambda_r)$ defined by equation (2.13), so that

$$P_{r,n-1}(\underbrace{\lambda_r, \cdots, \lambda_r}_{n-1}) = P_{r,n-1}(\lambda_r), \qquad P_{r,n-1}(\underbrace{\lambda_r, \cdots, \lambda_r}_{n-2}, \lambda_r^*) = P_{r,n-2,1}.$$

Then for positive definite symmetric matrices $(\lambda_{r\alpha\beta}^{(1)}), \cdots, (\lambda_{r\alpha\beta}^{(n-1)})$ the following inequality is valid:

$$P_{r,n-1}(\lambda_r^{(1)}, \cdots, \lambda_r^{(n-1)}) \geqq P_{r,n-1}(\lambda_r^{(1)})^{1/(n-1)} \cdots P_{r,n-1}(\lambda_r^{(n-1)})^{1/(n-1)}, \tag{4.1}$$

where the equality holds when and only when the $n-1$ matrices are pairwise proportional.

Suppose now $P_{r,n-1,0} = P_{r0,n-1}$, which can be written as $P_{r,n-1}(\lambda_r) =$

$P_{r,n-1}(\lambda_r^*)$. Since $(\lambda_{r\alpha\beta})$ and $(\lambda_{r\alpha\beta}^*)$ are positive definite, from the inequality (4.1) it follows that

$$P_{r,n-2,1} = P_{r,n-1}(\underbrace{\lambda_r, \cdots, \lambda_r}_{n-2}, \lambda_r^*) \tag{4.2}$$
$$\geqq P_{r,n-1}(\lambda_r)^{(n-2)/(n-1)} P_{r,n-1}(\lambda_r^*)^{1/(n-1)} = P_{r,n-1}(\lambda_r^*),$$

which is the first condition for the functions $P_{r,n-1}$ to be of type $n - 1$. By interchanging the two manifolds M_n and M_n^* we have

$$P_{r1,n-2} \geqq P_{r,n-1}(\lambda_r). \tag{4.3}$$

The equality holds in (4.2) and (4.3) when and only when $\lambda_{r\alpha\beta}^* = \rho\lambda_{r\alpha\beta}$ for $\alpha, \beta = 1, \cdots, n$. On the other hand, as in the proof of Theorem 3.1, by using equations (4.2), (4.3), and (3.19) it is easily seen that the equality holds in (4.2) and (4.3). Since $P_{r,n-1}(\lambda_r) = P_{r,n-1}(\lambda_r^*)$, $\rho = 1$, and therefore the second condition for the functions $P_{r,n-1}$ to be of type $n - 1$ is satisfied.

Proof of Theorem II. By putting

$$\lambda_{r\alpha\beta}^{(1)} = \cdots = \lambda_{r\alpha\beta}^{(n-2)} = \lambda_{r\alpha\beta}, \qquad \lambda_{r\alpha\beta}^{(n-1)} = a\delta_{\alpha\beta} \qquad (\alpha, \beta = 1, \cdots, n),$$

from inequality (4.1) we obtain

$$P_{r,n-2}^{1/(n-2)} \geqq P_{r,n-1}^{1/(n-1)}, \tag{4.4}$$

where the equality holds when and only when $\lambda_{r\alpha\beta} = b\delta_{\alpha\beta}$ for $\alpha, \beta = 1, \cdots, n$. Let $a > 0$ be defined by

$$P_{r,n-2}^{\mu} P_{r,n-1}^{\nu} = a^{\mu(n-2)+\nu(n-1)}, \tag{4.5}$$

so that

$$P_{r,n-1} = a^{(\mu/\nu)((n-2)+n-1)} P_{r,n-2}^{-\mu/\nu}. \tag{4.6}$$

From inequality (4.4) and equation (4.6) it follows that

$$P_{r,n-2} \geqq P_{r,n-1}^{(n-2)/(n-1)} = a^{(n-2)((\mu/\nu)(n-2)/(n-1)+1)} P_{r,n-2}^{-(\mu/\nu)(n-2)/(n-1)}, \tag{4.7}$$

which implies $P_{r,n-2} \geqq a^{n-2}$, where the equality holds when and only when it holds in (4.4). Thus, if $P_{r,n-2} = a^{n-2}$, then $\lambda_{r\alpha\beta} = b\delta_{\alpha\beta}$ for $\alpha, \beta = 1, \cdots, n$, and therefore $b = a$ in consequence of equation (4.5). Hence Theorem II follows from Theorem 3.2 by taking $F_r(P_{r1}, \cdots, P_{rn}) = P_{r,n-2}^{\mu} P_{r,n-1}^{\nu}$.

5. Integral formulas for convex hypercaps

Let M be a compact differentiable manifold of dimension n with boundary; and let M_n be a convex hypercap with boundary B_{n-1}, so that M_n is an imbedded manifold given by $X:M \to E_{n+1}$ with positive Gauss-Kronecker curvature K_{n+1} everywhere. Then §2 with $m = 1$ can be applied.

In the space E_{n+1}, let ξ be a fixed direction along which every line either is a tangent to the hypercap M_n or intersects the hypercap M_n at most at one point, so that by the definition of a convex hypercap we have

$$\tau \equiv \xi \cdot e_{n+1} \geqq 0. \tag{5.1}$$

For the hypercap M_n we introduce the following differential forms:

$$\begin{aligned} A_\alpha &= |\xi, X, \underbrace{de_{n+1}, \cdots, de_{n+1}}_{n-1-\alpha}, \underbrace{dX, \cdots, dX}_{\alpha}| \qquad (0 \leqq \alpha \leqq n-1), \\ D_\beta &= |\xi, \underbrace{de_{n+1}, \cdots, de_{n+1}}_{n-\beta}, \underbrace{dX, \cdots, dX}_{\beta}| \qquad (0 \leqq \beta \leqq n). \end{aligned} \tag{5.2}$$

As in §3, by using exterior differentiation and Stokes's theorem we obtain

$$\int_{M_n} D_{\alpha+1} = \int_{B_{n-1}} A_\alpha \qquad (0 \leqq \alpha \leqq n-1). \tag{5.3}$$

Now let M_n^* be another convex hypercap given by the imbedding $X^*: M \to E_{n+1}$, and suppose that there is a diffeomorphism f of the hypercap M_n onto the hypercap M_n^* such that at each pair of corresponding points the hypercaps M_n and M_n^* have the same unit normal vector e_{n+1}. For this pair of hypercaps M_n and M_n^* we introduce the following differential forms:

$$\begin{aligned} A_{\alpha\beta} &= |\xi, X, \underbrace{de_{n+1}, \cdots, de_{n+1}}_{n-1-(\alpha+\beta)}, \underbrace{dX, \cdots, dX}_{\alpha}, \underbrace{dX^*, \cdots, dX^*}_{\beta}|, \\ A^*_{\alpha\beta} &= |\xi, X^*, \underbrace{de_{n+1}, \cdots, de_{n+1}}_{n-1-(\alpha+\beta)}, \underbrace{dX, \cdots, dX}_{\alpha}, \underbrace{dX^*, \cdots, dX^*}_{\beta}|, \\ B_{\alpha\beta} &= |X, X^*, \underbrace{de_{n+1}, \cdots, de_{n+1}}_{n-1-(\alpha+\beta)}, \underbrace{dX, \cdots, dX}_{\alpha}, \underbrace{dX^*, \cdots, dX^*}_{\beta}|, \\ C_{\rho\sigma} &= |X, \underbrace{de_{n+1}, \cdots, de_{n+1}}_{n-(\rho+\sigma)}, \underbrace{dX, \cdots, dX}_{\rho}, \underbrace{dX^*, \cdots, dX^*}_{\sigma}|, \\ C^*_{\rho\sigma} &= |X^*, \underbrace{de_{n+1}, \cdots, de_{n+1}}_{n-(\rho+\sigma)}, \underbrace{dX, \cdots, dX}_{\rho}, \underbrace{dX^*, \cdots, dX^*}_{\sigma}|, \\ D_{\rho\sigma} &= |\xi, \underbrace{de_{n+1}, \cdots, de_{n+1}}_{n-(\rho+\sigma)}, \underbrace{dX, \cdots, dX}_{\rho}, \underbrace{dX^*, \cdots, dX^*}_{\sigma}|, \end{aligned} \tag{5.4}$$

where $0 \leqq \alpha, \beta \leqq n-1$ and $0 \leqq \rho, \sigma \leqq n$. As in §3, it is easily seen that

$$\tau C_{\rho\sigma} = h D_{\rho\sigma}, \qquad \tau C^*_{\rho\sigma} = h^* D_{\rho\sigma} \qquad (0 \leqq \rho, \sigma \leqq n), \tag{5.5}$$

where we have placed

$$h = X \cdot e_{n+1}, \qquad h^* = X^* \cdot e_{n+1}. \tag{5.6}$$

Exterior differentiation gives

$$
\begin{aligned}
(5.7)\qquad & dA_{\alpha\beta} = D_{\alpha+1,\beta}\,, \qquad dA^*_{\alpha\beta} = D_{\alpha,\beta+1}\,,\\
& dB_{\alpha\beta} = C_{\alpha,\beta+1} - C^*_{\alpha+1,\beta} = (1/\tau)(hD_{\alpha,\beta+1} - h^*D_{\alpha+1,\beta}), \qquad \text{if } \tau \neq 0.
\end{aligned}
$$

Integrating both sides of each of equations (5.7) over the hypercap M_n and applying Stokes's theorem to the left side we have

$$
\begin{aligned}
(5.8)\qquad & \int_{B_{n-1}} A_{\alpha\beta} = \int_{M_n} D_{\alpha+1,\beta}\,, \qquad \int_{B_{n-1}} A^*_{\alpha\beta} = \int_{M_n} D_{\alpha,\beta+1}\,,\\
& \int_{B_{n-1}} B_{\alpha\beta} = \int_{M_n} (1/\tau)(hD_{\alpha,\beta+1} - h^*D_{\alpha+1,\beta}), \qquad \text{if } \tau \neq 0.
\end{aligned}
$$

Essentially the same argument as that used in deriving equation (3.16) shows that

$$
(5.9)\qquad D_{\alpha\beta} = (-1)^n n!\,\tau P_{\alpha\beta} K_{n+1}\, dA \qquad (0 \leqq \alpha, \beta \leqq n),
$$

where $P_{\alpha\beta}$ are defined, in terms of two parameters y and y^*, by

$$
\begin{aligned}
(5.10)\qquad & \det (y\lambda_{n+1,\alpha\beta} + y^*\lambda^*_{n+1,\alpha\beta} + \delta_{\alpha\beta})\\
& \qquad = \sum_{0\leqq \alpha+\beta\leqq n} \frac{n!}{\alpha!\,\beta!\,(n-\alpha-\beta)!}\, y^\alpha y^{*\beta} P_{\alpha\beta}\,.
\end{aligned}
$$

We shall also write $P_{\alpha 0} = P_\alpha(\lambda_{n+1})$ and $P_{0\alpha} = P_\alpha(\lambda^*_{n+1})$. Substituting equations (5.9) in equations (5.8) we thus arrive at the integral formulas

$$
\begin{aligned}
& \int_{B_{n-1}} A_{\alpha\beta} = (-1)^n n! \int_{M_n} \tau P_{\alpha+1,\beta} K_{n+1}\, dA,\\
(5.11)\qquad & \int_{B_{n-1}} A^*_{\alpha\beta} = (-1)^n n! \int_{M_n} \tau P_{\alpha,\beta+1} K_{n+1}\, dA,\\
& \int_{B_{n-1}} B_{\alpha\beta} = (-1)^n n! \int_{M_n} (1/\tau)(hP_{\alpha,\beta+1} - h^*P_{\alpha+1,\beta}) K_{n+1}\, dA,\\
& \qquad\qquad \text{if } \tau \neq 0,
\end{aligned}
$$

where $0 \leqq \alpha, \beta \leqq n-1$.

6. Proof of Theorem III

From equations (5.11) we can easily deduce different integral formulas, but for proving Theorem III we need only the following one:

$$
\begin{aligned}
(6.1)\qquad & \int_{B_{n-1}} (A_{10} - A_{01} + A^*_{01} - A^*_{10})\\
& \qquad = (-1)^n n! \int_{M_n} \tau (P_{20} + P_{02} - 2P_{11}) K_{n+1}\, dA.
\end{aligned}
$$

From the assumption of Theorem III that the given diffeomorphism f restricted to the boundary B_{n-1} is a translation carrying the boundary B_{n-1} onto the boundary B^*_{n-1}, it follows that over the boundary B_{n-1}, $dX^* = dX$, and therefore $A_{10} - A_{01} + A^*_{01} - A^*_{10} = 0$. Thus the integral formula (6.1) is reduced to

$$\int_{M_n} \tau(P_{20} + P_{02} - 2P_{11})K_{n+1}\, dA = 0. \tag{6.2}$$

On the other hand, in a recent paper [4] we have established the

LEMMA. *If $\mu = (\mu_{\alpha\beta})$ and $\mu^* = (\mu^*_{\alpha\beta})$ are two positive definite symmetric matrices of order n such that for a fixed γ, $2 \leqq \gamma \leqq n$,*

$$P_{\gamma-1}(\mu) \leqq P_{\gamma-1}(\mu^*), \qquad P_\gamma(\mu) \geqq P_\gamma(\mu^*), \tag{6.3}$$

then

$$Q_\gamma(\mu, \mu^*) \equiv P_\gamma(\mu) + P_\gamma(\mu^*) - 2P_{\gamma-1,1}(\mu, \mu^*) \leqq 0, \tag{6.4}$$

where the equality implies that $P_\gamma(\mu) = P_\gamma(\mu^*)$.

At each pair of corresponding points of the hypercaps M_n and M^*_n under the diffeomorphism f, we take the common unit outer normal vector to be the vector e_{n+1}; so that the matrices $(\lambda_{n+1,\alpha\beta})$ and $(\lambda^*_{n+1,\alpha\beta})$ are positive definite everywhere, and the conditions (1.3) or (1.4) are equivalent to the conditions (6.3) with $\gamma = 2$. Since $Q_2(\mu, \mu^*)$ is symmetric with respect to the matrices μ and μ^*, the above lemma gives

$$P_2(\lambda_{n+1}) + P_2(\lambda^*_{n+1}) - 2P_{11}(\lambda_{n+1}, \lambda^*_{n+1}) \leqq 0, \tag{6.5}$$

where the equality implies that $P_2(\lambda_{n+1}) = P_2(\lambda^*_{n+1})$. From the inequalities (5.1), (6.5), and the assumption that $K_{n+1} > 0$, it follows immediately that the integrand of equation (6.2) is nonpositive, and equation (6.2) holds when and only when the equality holds in (6.5). Thus

$$P_2(\lambda_{n+1}) = P_2(\lambda^*_{n+1}),$$

and hence Theorem III follows from the uniqueness theorem of Alexandroff-Fenchel-Jessen for convex hypersurfaces with boundary.[3]

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