ON A THEOREM OF E. H. BROWN

BY

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Introduction

Recently, E. H. Brown proved that the homology of a fibre space can be computed from a chain complex which is the tensor product of the chains of the base and those of the fibre, the differential operator being a relatively simple modification of the one for the corresponding product space; cf. [6].

Brown proves his theorem for fibre spaces in the sense of Hurewicz, and uses the associated techniques.

A recent paper, [5], shows that, for all purposes of homotopy theory, the chain complex of any fibre space in the sense of Serre can be replaced by a "twisted cartesian product". Thus it seemed immediately that the context of semisimplicial complexes and twisted cartesian products was a natural one for Brown's theorem; and in fact it was found that both his theorem and the proof he gave for it could be adapted to this context. The present paper is devoted to this adaptation.

The following remark may be of interest: The existence of *some* differential for the homology of the total space on the tensor product of base chains by fibre chains follows almost immediately from an (unpublished) lemma of H. Cartan, the proof of which is simple. By its very generality, however, this lemma does not seem to enable one to determine the very special form of the differential given by Brown.

The last section of this paper gives a somewhat generalized version of a theorem of Hurewicz and Fadell [7]; this theorem can be proved more directly in the present context by using the map f of Eilenberg and Mac Lane (cf. Section 5 below) for a direct comparison of the spectral sequence of the tensor product (with the "usual" differential) and the *twisted* cartesian product.

The formulation of the algebraic material in Section 2, in particular the use of the differential operator in Hom (A, C), follows a suggestion of J. C. Moore.

I am greatly indebted to E. H. Brown for letting me see a manuscript of his paper.

1. Algebraic preliminaries

We fix, once and for all, a commutative ring Λ with unit; "module" will mean "Λ-module". If A, B are modules, we write $A \otimes B$, Hom (A, B) for $A \otimes_\Lambda B$, $\mathrm{Hom}_\Lambda(A, B)$.

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A sequence $\{A_n\}$ of modules, i.e., the assignment to each integer n of a module A_n, is called a graduated module A; $a \in A$ means $a \in A_n$ for some n, called the dimension of a; $n = \dim a$. If A, B are graduated modules, the graduated module $A \otimes B$ is defined by

$$(A \otimes B)_n = \textstyle\sum_{p+q=n} A_p \otimes B_q ;$$

a homomorphism $f = \{f_k\}:A \to B$ is a sequence of homomorphisms

$$f_k:A_k \to B_{k+n}$$

for a fixed n, called the dimension of f; Hom (A, B) is the graduated module whose n-dimensional component is the module of n-dimensional homomorphisms $f:A \to B$.

Let $f \in \mathrm{Hom}\,(A, B)$, $g \in \mathrm{Hom}\,(C, D)$, and let A, B, C, D be graduated. We define

$$f \otimes g \in \mathrm{Hom}\,(A \otimes C, B \otimes D)$$

by

$$(f \otimes g)(a \otimes c) = (-1)^{tp}(fa \otimes gc)$$

if $\dim g = t$, $\dim a = p$.

A graduated differential module is a graduated module A with a homomorphism $d:A \to A$ such that $dd = 0$ and $dA_p \subset A_{p-1}$.

If A, B are graduated differential modules, we turn $A \otimes B$ into a graduated differential module by the usual definition which, in our notation, can be written

$$d = d \otimes 1 + 1 \otimes d.$$

Similarly we turn Hom (A, B) into a differential module by the definition

$$(df)(a) = d(f(a)) + (-1)^{s+1}f(d(a))$$

if $\dim f = s$. In order to avoid ambiguities, the composition of f and d will be written $d \circ f$. Thus $df = d \circ f \pm fd$; we do not write the composition sign where no ambiguity can arise.

A graduated module C will be called an *algebra* if it has an *associative product* φ, i.e., a homomorphism of dimension zero

$$\varphi:C \otimes C \to C$$

such that

$$\varphi(1 \otimes \varphi) = \varphi(\varphi \otimes 1):C \otimes C \otimes C \to C.$$

The algebra C will be said to operate on a graduated module B if there is an operation τ, i.e., a homomorphism of dimension zero

$$\tau:C \otimes B \to B$$

such that the following diagram commutes:

$$\begin{tikzcd} C \otimes C \otimes B \arrow[r, "\varphi \otimes 1"] \arrow[d, "1 \otimes \tau"] & C \otimes B \arrow[d, "\tau"] \\ C \otimes B \arrow[r, "\tau"] & B, \end{tikzcd}$$

i.e., if

$$\tau(\varphi \otimes 1) = \tau(1 \otimes \varphi).$$

An algebra is a module operating on itself.

A graduated module A will be called a *coalgebra* if it has an associative coproduct ψ, i.e., a homomorphism of dimension zero $\psi: A \to A \otimes A$ such that

$$(\psi \otimes 1)\psi = (1 \otimes \psi)\psi: A \to A \otimes A \otimes A.$$

If C, A are differential modules, and if φ, ψ are admissible, i.e., if $d\varphi = \varphi d$, $\psi d = d\psi$, then we call C, A a *differential* algebra and coalgebra, respectively.

If B is a differential module, we shall also require of the operation τ that $d\tau = \tau d$.

2. Cap and cup products

Let A be a differential coalgebra with coproduct ψ, and let C, B be differential modules. We define

$$\theta: \mathrm{Hom}\,(A, C) \otimes A \otimes B \to A \otimes C \otimes B$$

by

$$\theta(f \otimes a \otimes b) = (1 \otimes f \otimes 1)(\psi \otimes 1)(a \otimes b).$$

2.1 LEMMA. $d\theta = \theta d$.

Proof. Suppose $\dim f = s$, and write $x = a \otimes b$.

$$\begin{aligned} \theta d(f \otimes a \otimes b) &= \theta(df \otimes x) + (-1)^s \theta(f \otimes dx) \\ &= (1 \otimes df \otimes 1)(\psi \otimes 1)(x) + (-1)^s (1 \otimes f \otimes 1)(\psi \otimes 1)(d \otimes 1 + 1 \otimes d)(x) \\ &= (1 \otimes d \otimes 1)(1 \otimes f \otimes 1)(\psi \otimes 1)(x) \\ &\qquad + (-1)^{s+1}(1 \otimes f \otimes 1)(1 \otimes d \otimes 1)(\psi \otimes 1)(x) \\ &\qquad\qquad + (-1)^s (1 \otimes f \otimes 1)(\psi \otimes 1)(d \otimes 1 + 1 \otimes d)(x). \end{aligned}$$

We now use the evident identities

$$(\psi \otimes 1)(1 \otimes d) = (1 \otimes 1 \otimes d)(\psi \otimes 1),$$

$$(\psi \otimes 1)(d \otimes 1) = (1 \otimes d \otimes 1 + d \otimes 1 \otimes 1)(\psi \otimes 1),$$

and our expression becomes

$$
\begin{aligned}
&(1 \otimes d \otimes 1)(1 \otimes f \otimes 1)(\psi \otimes 1)(x)\\
&\qquad + (-1)^s(1 \otimes f \otimes 1)(1 \otimes 1 \otimes d + d \otimes 1 \otimes 1)(\psi \otimes 1)(x)\\
&\quad = (1 \otimes d \otimes 1 + 1 \otimes 1 \otimes d + d \otimes 1 \otimes 1)(1 \otimes f \otimes 1)(\psi \otimes 1)(x)\\
&\quad = d(1 \otimes f \otimes 1)(\psi \otimes 1)(x)\\
&\quad = d\theta(f \otimes a \otimes b).
\end{aligned}
$$

Next, suppose further that C is a differential algebra with product φ, and that it operates on B with operation τ.

We define, for $f \in \operatorname{Hom}(A, C)$, $a \in A$, $b \in B$,

$$f \cap (a \otimes b) = (1 \otimes \tau)\theta(f \otimes a \otimes b).$$

The homomorphism $f \cap : A \otimes B \to A \otimes B$ has the dimension of f and, due to 2.1, is easily seen to satisfy

$$d(f \cap x) = df \cap x + (-1)^s f \cap dx \tag{1}$$

if $\dim f = s$ and $x \in A \otimes B$.

Now, let f, $g \in \operatorname{Hom}(A, C)$. We define $f \cup g \in \operatorname{Hom}(A, C)$ by

$$(f \cup g)(a) = \varphi(f \otimes g)\psi(a)$$

and easily deduce

$$d(f \cup g) = df \cup g + (-1)^s f \cup dg. \tag{2}$$

In the case where B, C have zero differential, (1) and (2) are classical.

2.2 Lemma. *Let f, $g \in \operatorname{Hom}(A, C)$, $x \in A \otimes B$. Then*

$$(f \cup g) \cap x = f \cap (g \cap x). \tag{3}$$

Proof.

$$
\begin{aligned}
f \cap (g \cap x) &= (1 \otimes \tau)\theta(f \otimes g \cap x)\\
&= (1 \otimes \tau)(1 \otimes f \otimes 1)(\psi \otimes 1)(g \cap x) \qquad (4)\\
&= (1 \otimes \tau)(1 \otimes f \otimes 1)(\psi \otimes 1)(1 \otimes \tau)\\
&\qquad \cdot (1 \otimes g \otimes 1)(\psi \otimes 1)(x).
\end{aligned}
$$

$$
\begin{aligned}
(f \cup g)(x) &= (1 \otimes \tau)\theta(f \cup g \otimes x)\\
&= (1 \otimes \tau)(1 \otimes f \cup g \otimes 1)(\psi \otimes 1)(x)\\
&= (1 \otimes \tau)(1 \otimes \varphi(f \otimes g)\psi \otimes 1)(\psi \otimes 1)(x) \qquad (5)\\
&= (1 \otimes \tau)(1 \otimes \varphi \otimes 1)(1 \otimes f \otimes g \otimes 1)\\
&\qquad \cdot (1 \otimes \psi \otimes 1)(\psi \otimes 1)(x).
\end{aligned}
$$

In expression (4) we use the evident formula

$$(\psi \otimes 1)(1 \otimes \tau) = (1 \otimes 1 \otimes \tau)(\psi \otimes 1 \otimes 1)$$

and then the formulas

$$(1 \otimes f \otimes 1)(1 \otimes 1 \otimes \tau) = (1 \otimes 1 \otimes \tau)(1 \otimes f \otimes 1 \otimes 1),$$
$$(\psi \otimes 1 \otimes 1)(1 \otimes g \otimes 1) = (1 \otimes 1 \otimes g \otimes 1)(\psi \otimes 1 \otimes 1);$$

thus (4) becomes

$$\begin{aligned}(1 \otimes \tau)&(1 \otimes 1 \otimes \tau)(1 \otimes f \otimes 1 \otimes 1)\\ &\cdot(1 \otimes 1 \otimes g \otimes 1)(\psi \otimes 1 \otimes 1)(\psi \otimes 1)(x)\\ &= (1 \otimes \tau)(1 \otimes 1 \otimes \tau)(1 \otimes f \otimes g \otimes 1)(\psi \otimes 1 \otimes 1)(\psi \otimes 1)(x)'\end{aligned}$$

and this is equal to (5) due to the associativity relation for ψ and the relation $\tau(\varphi \otimes 1) = \tau(1 \otimes \tau)$.

Let $\alpha:A \to A'$ be a map of coalgebras, $\gamma:C \to C'$ a map of algebras, and $\beta:B \to B'$ a homomorphism such that

$$\tau'(\gamma \otimes \beta) = \beta\tau,$$

where $\tau':C' \otimes B' \to B'$ is an operation of C' on B'.

In this context, let $f \in \operatorname{Hom}(A, C)$, $f' \in \operatorname{Hom}(A', C')$ satisfy

$$f'\alpha = \gamma f.$$

Then a straightforward verification shows that

$$(\alpha \otimes \beta)[f \cap (a \otimes b)] = f' \cap (\alpha \otimes \beta)(a \otimes b). \tag{6}$$

3. Twisting cochains

Let the situation be as above, and let $f \in \operatorname{Hom}(A, C)$ be -1-dimensional, i.e., $fA_p \subset C_{p-1}$.

We define the -1-dimensional homomorphism

$$d_f:A \otimes B \to A \otimes B$$

by

$$d_f\, x = dx + f \cap x, \qquad x \in A \otimes B;$$

then

$$\begin{aligned} d_f\, d_f(x) &= d_f(dx + f \cap x)\\ &= ddx + d(f \cap x) + f \cap dx + f \cap (f \cap x)\\ &= df \cap x - f \cap dx + f \cap dx + (f \cup f) \cap x\\ &= (df + f \cup f)(x).\end{aligned}$$

Hence we have

3.1 Lemma. *$d_f\, d_f = 0$ if $df + f \cup f = 0$.*

In this case we call f a *twisting cochain* and d_f the *differential twisted by* f.

If $f = \{f_k\}$ (i.e., f_k is the component $A_k \to C_{k-1}$ of f), then the relation for a twisting cochain can be written

$$d \circ f_n + f_{n-1}\, d + \textstyle\sum_{p+q=n} f_p \cup f_q = 0.$$

In our application of this theory we shall be entirely concerned with the case where A, C have nonnegative graduation, i.e., when $A_i = C_i = 0$ for $i < 0$.

In that case (recall that $\dim f = -1$), $f_i = 0$ for $i \leqq 0$, and the relation for a twisting cochain is

$$(1) \qquad d \circ f_n + f_{n-1}\, d + \textstyle\sum_{i=1}^{n=1} f_i \cup f_{n-1} = 0 \qquad (n \geqq 1).$$

Finally, in our applications A, B, and C will be augmented; i.e., there will be given homomorphisms

$$\eta : A_0\, ,\ B_0\, ,\ C_0 \to \Lambda$$

such that $\eta\, d = 0$, $\eta\psi = \eta$, $\eta\varphi = \eta$, $\eta\tau = \eta$, where the last equations assume that in the tensor product $A \otimes B$ we define an augmentation by

$$\eta(a \otimes b) = (\eta a)(\eta b), \qquad a \in A_0\, , \quad b \in B_0\, .$$

If this, in particular, is to be an augmentation for d_f, we must have $\eta(dx + f \cap x) = 0$ when $\dim x = 1$; this will certainly be the case if f satisfies

$$(2) \qquad \eta f_1 = 0.$$

In the case of augmented complexes we shall always require (2) as well as (1) for a twisting cochain.

We define an increasing filtration on $A \otimes B$ by

$$F_p(A \otimes B) = \textstyle\sum_{i=0}^{p} A_i \otimes B.$$

Now it is easily seen (since $f_0 = 0$) that the operation $f \cap$ decreases filtration by at least one. We shall denote by $E^r_{p,q}(A \otimes B)$ the spectral sequence derived from $A \otimes B$, the differential d_f, and the above filtration. Then the term E^1 will be independent of f, and we get

3.2 LEMMA. *If A is Λ-free, then* $E^1_{p,q} = A_p \otimes H_q(B)$.

Let us now be in the situation of Section 2, formula (6), with $A = A'$, $\alpha =$ identity, and f a twisting cochain. Then it is easily verified that f' is a twisting cochain for $A \otimes B'$.

From a classical spectral sequence argument and 3.2 we now get

3.3 LEMMA. *If A is Λ-free and β is a chain equivalence, then $1 \otimes \beta$ is a chain equivalence* (*the differentials being d_f, $d_{f'}$ respectively*).

4. Acyclic models

We shall use the method of acyclic models. For reasons explained in 7.5 we use the simple theory, due to Eilenberg and Mac Lane [1], in which degeneracy plays no part; in fact, we shall use the method of [2] with trivial degeneracy.

By $d\mathcal{G}$ we denote the category of differential graded (with nonnegative graduation) augmented Λ-modules (DGA modules); cf. Section 2.

Let $\mathfrak{A}$ be any category, $\mathfrak{M}$ a subset of the objects of $\mathfrak{A}$, the "models"; if A is any object, let $S(A)$ denote the set of maps $u:M \to A$ where $M \in \mathfrak{M}$.

Let $K:\mathfrak{A} \to d\mathcal{G}$ be a covariant functor; if $u:M \to A$, $u \in S(A)$, we denote by $K(M, u)$ the ordered pair $(K(M), u)$, regarded as an object of $d\mathcal{G}$.

We define the functor $\hat{K}:\mathfrak{A} \to d\mathcal{G}$ by $\hat{K}(A) = \sum_{u \in S(A)} K(M, u)$ and if $f:A \to B$ is a map of $\mathfrak{A}$, $\hat{K}(f)(x, u) = (x, fu)$ for $x \in K(M)$. Finally, we define the transformation of functors $\Gamma:\hat{K} \to K$ by

$$\Gamma(A)(x, u) = K(u)x, \qquad x \in K(M).$$

4.1 Definition. We say that K is representable (in the category with models $(\mathfrak{A}, \mathfrak{M})$) if there is a transformation of functors $\chi:K \to \hat{K}$ such that $\Gamma\chi$ = identity.

If $K:\mathfrak{A} \to d\mathcal{G}$ is a functor, we denote by $K_i:\mathfrak{A} \to \mathcal{G}$, the category of Λ-modules, its i-dimensional part; similarly if K, $L:\mathfrak{A} \to d\mathcal{G}$ are functors, and $\varphi:K \to L$ a transformation, $\varphi_i:K_i \to L_i$ will denote the evident restriction.

We quote the main theorem on acyclic models (cf. [1]) in exactly the case that we shall use:

4.2 Proposition. *Let K, $L:\mathfrak{A} \to d\mathcal{G}$ be functors. Let K be representable, and $H_q L(M) = 0$ for $q > 0$ and $M \in \mathfrak{M}$.*

(i) *Let $\varphi_0:K_0 \to L_0$, $\varphi_1:K_1 \to L_1$ be given such that $\eta\varphi_0 = \eta$, $d\varphi_1 = \varphi_0 d$; then there is an extension $\varphi:K \to L$ of φ_0, φ_1.*

(ii) *Let φ, ψ be any two extensions of $\varphi_0:K_0 \to L_0$. Then φ, ψ are chain-homotopic.*

We shall also use the following addendum to 4.2.

Suppose K, L are functors with *filtration*, i.e., functors into the category of DGA modules with *increasing filtration*, and suppose the following condition is satisfied:

4.3 Condition. If $x \in K(A)$ has filtration p, then $\chi(A)(x) \in \hat{K}(A)$ has nonzero components only in terms $K(M, u)$ with the property that $L(u)L(M) \subset L(A)$ consists entirely of elements of filtration $\leqq p$.

4.4 *Addendum to* 4.2. If 4.3 is satisfied, then φ and the homotopy between φ and ψ are filtration-preserving.

This is immediate from the proof of 4.2; cf. [1].

Under these conditions φ induces transformations

$$\varphi^r:E^rK \to E^rL \qquad (r \geqq 0)$$

between the spectral sequences of K, L; and

$$\varphi^r = \psi^r \qquad \text{when } r \geqq 1.$$

5. Complexes

The words "complex", "map" will denote "complete semisimplicial complex", "semisimplicial map", respectively; cf. [3] or [4]. We shall denote the face and degeneracy operators by ∂_i, s_i.

Let X be a complex; by $\Lambda(X)$ we denote the *normalized* chain complex with coefficients in Λ; if X, Y are complexes, we denote by $X \times Y$ their cartesian product; cf. [3]. We recall the chain equivalences

$$f: \Lambda(X \times Y) \to \Lambda(X) \otimes \Lambda(Y),$$

$$\nabla: \Lambda(X) \otimes \Lambda(Y) \to \Lambda(X \times Y)$$

introduced by Eilenberg and Mac Lane; cf. [8].

Let X be a complex; we define the diagonal map $\delta: X \to X \times X$ by $\delta x = (x, x)$. It is well known and easily seen that[2]

$$f\delta = \psi: \Lambda(X) \to \Lambda(X) \otimes \Lambda(X)$$

is a coproduct for the DGA module $\Lambda(X)$.

A *group complex* Γ is one for which there is given a map $\bar{\varphi}: \Gamma \times \Gamma \to \Gamma$ which turns Γ_n into a group for each $n \geqq 0$. In this case we define

$$\varphi: \Lambda(\Gamma) \otimes \Lambda(\Gamma) \to \Lambda(\Gamma)$$

by $\varphi = \bar{\varphi}\nabla$.

It is easily seen that φ is a product for the DGA algebra $\Lambda(\Gamma)$.

Let Γ be as before, and Y a complex. Γ is said to *operate on* Y if there is given an *operation* $\bar{\tau}$ of Γ on Y, i.e., a map $\bar{\tau}: \Gamma \times Y \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
\Gamma \times \Gamma \times Y \arrow[r, "\bar{\varphi} \times 1"] \arrow[d, "1 \times \bar{\tau}"'] & \Gamma \times Y \arrow[d, "\bar{\tau}"] \\
\Gamma \times Y \arrow[r, "\bar{\tau}"] & Y.
\end{tikzcd}$$

We shall write $\tau(\gamma, y) = \gamma y$.

In this case the definition

$$\tau = \tau\nabla : \Lambda(\Gamma) \otimes \Lambda(Y) \to \Lambda(Y)$$

defines an operation in the sense of Section 2; indeed we obtain a case of the

[2] We use the convention of denoting a map $X \to Y$ and the induced homomorphism $\Lambda(X) \to \Lambda(Y)$ by the same symbol.

entire situation of that section if we write $A = \Lambda(X)$, $B = \Lambda(Y)$, $C = \Lambda(\Gamma)$ and adopt the definitions of the present section.

6. The category $\mathcal{P}$

6.1 DEFINITION. Let X be a complex, and Γ a group complex. By an (X, Γ)-twisting function ξ we mean a set of functions $\xi_n : X_n \to \Gamma_{n-1}$ $(n > 0)$ such that

$$\begin{aligned}
\partial_0 \xi(x) &= [\xi(\partial_0 x)]^{-1}\xi(\partial_1 x), & \\
\partial_i \xi(x) &= \xi(\partial_{i+1} x), & i > 0, \\
s_i \xi(x) &= \xi(s_{i+1} x), & i \geqq 0, \\
\xi(s_0 x) &= 1. &
\end{aligned}$$

6.2 DEFINITION. The category $\mathcal{P}$ has as objects triples (X, Γ, ξ) where X is a complex, Γ a group complex, and ξ an (X, Γ)-twisting function. A map of $\mathcal{P}$

$$(a, b, \theta) : (X, \Gamma, \xi) \to (X', \Gamma', \xi')$$

consists of

(i) a map $a : X \to X'$,
(ii) a homomorphic map $b : \Gamma \to \Gamma'$,
(iii) a dimension-preserving function

$$\theta : X \to \Gamma',$$

these data to satisfy the following axioms:

(1) $b\xi = \xi' a$,

(2) $\partial_i \theta = \theta \partial_i$ if $i > 0$,

$\theta(\partial_0 x) \cdot b\xi(x) = \xi'(ax) \cdot \partial_0 \theta(x)$.

The identity map on (X, Γ, ξ) is $(1_x, 1_y, e)$, where e assigns the unit of Γ'_n to every element of X_n.

Composition is defined as follows: Let (a, b, θ) be as before, and $(a', b', \theta') : (X', \Gamma', \xi') \to (X'', \Gamma'', \xi'')$. We define

$$(a', b', \theta')(a, b, \theta) = (a'a, b'b, \theta^*),$$

where $\theta^* : X \to \Gamma''$ is defined by

$$\theta^*(x) = \theta'(ax) \cdot b'\theta(x).$$

The associative law is easily verified.

6.3 DEFINITION. $\mathcal{P}_0 \subset \mathcal{P}$ is the subcategory whose objects (X, Γ, ξ) are such that X has only one zero-dimensional element. $\mathcal{Q}_0$ is the category of complexes with only one zero-dimensional element.

We now describe a covariant functor

$$\mathcal{G}:\mathcal{A}_0 \to \mathcal{P}_0$$

due to Kan; cf. [4].

$$\mathcal{G}(X) = (X, GX, \xi^K),$$

where GX is the group complex defined as follows:

(i) $(GX)_n$ is a group with a generator $\{x\}$ for each element $x \in X_{n+1}$.
(ii) For every $x \in X_n$, $(GX)_n$ has a relation $\{s_0 x\} = 1$.
(iii) The face and degeneracy operators are defined by

$$\partial_i\{x\} = \{\partial_{i+1} x\}, \qquad i > 0,$$

$$s_i\{x\} = \{s_{i+1} x\}, \qquad i > 0,$$

$$\partial_0\{x\} = \{\partial_0 x\}^{-1}\{\partial_1 x\}.$$

Finally, ξ^K is defined by

$$\xi^K x = \{x\}.$$

Now, let $a:X \to X'$ be a map. We define

$$\mathcal{G}(a) = (a, G(a), e):(X, GX, \xi^K) \to (X', GX', \xi^K)$$

as follows:

$$G(a)\{x\} = \{ax\}, \qquad ex = 1.$$

The main property of this construction will be found in Proposition 7.2.

6.4. Let ξ be an (X, Γ)-twisting function. Then a homomorphic map

$$G_\xi:G(X) \to \Gamma$$

is defined by

$$G_\xi\{x\} = \xi(x).$$

Suppose $f:Z \to X$ is a map. We define

$$f_\xi:GZ \to \Gamma$$

by

$$f_\xi = G_\xi \circ G(f).$$

7. The category $\mathcal{B}$

This is the category of "regular twisted cartesian products" (RTCP) (cf. [5]).

7.1 Definition. An object of $\mathcal{B}$ is a quadruple (X, Γ, ξ, Y) where (X, Γ, ξ) is an object of $\mathcal{P}$ and Y is a complex on which Γ operates.

A map $(a, b, \theta, c):(X, \Gamma, \xi, Y) \to (X', \Gamma', \xi', Y')$ consists of a map $(a, b, \theta):(X, \Gamma, \xi) \to (X', \Gamma', \xi')$ of $\mathcal{P}$ and a map $c:Y \to Y'$ such that

$$c(\gamma y) = (b\gamma)(cy) \qquad (\gamma \in \Gamma_n, \quad y \in Y_n).$$

In this definition the RTCP of [5] makes no explicit appearance; we introduce it through a functor $[\quad]:\mathfrak{B} \to \mathfrak{C}$, the category of complexes and maps, as follows:

$$[X, \Gamma, \xi, Y] = E$$

is a complex defined by

$$\begin{aligned} E_n &= X_n \times Y_n, \\ \partial_i(x, y) &= (\partial_i x, \partial_i y), && i > 0, \\ \partial_0(x, y) &= (\partial_0 x, \xi(x)\cdot\partial_0 y), \\ s_i(x, y) &= (s_i x, s_i y), && i \geqq 0, \end{aligned}$$

while the map

$$[a, b, \theta, c]:[X, \Gamma, \xi, Y] \to [X', \Gamma', \xi', Y']$$

is defined by

$$(x, y) \to (ax, \theta(x)\cdot by).$$

The various identities assure that this is a map. We also introduce the "fibre map" $p:E \to X$ by $p(x, y) = x$. Then clearly

$$p'[a, b, \theta, c] = ap.$$

Let (X, Γ, ξ, Y) be an object of $\mathfrak{B}$, and Z a complex. We define the operation of Γ on $Y \times Z$ by $\gamma(y, z) = (\gamma y, z)$ and define

$$(X, \Gamma, \xi, Y) \times Z = (X, \Gamma, \xi, Y \times Z);$$

and if $g:Z \to Z'$ is a map,

$$(a, b, \theta, c) \times g = (a, b, \theta, c \times g).$$

It is easily seen that

$$\begin{aligned} [(X, \Gamma, \xi, Y) \times Z] &= [X, \Gamma, \xi, Y] \times Z, \\ [(a, b, \theta, c) \times g] &= [a, b, \theta, c] \times g. \end{aligned}$$

By $\mathfrak{B}_0$ we denote the subcategory of $\mathfrak{B}$ with objects (X, Γ, ξ, Y) such that X has only one zero-dimensional element.

We now return to the construction $\mathcal{G}$ of Section 5. Let the group complex $G(X)$ operate on itself by multiplication; thus we can form the object $(X, G(X), \xi^K, G(X))$ of $\mathfrak{B}_0$. Then we have the fundamental result of Kan.

7.2 Proposition. $[X, G(X), \xi^K, G(X)]$ *is contractible.*

We use this proposition to introduce into our category the *models* we are going to use. Let Δ^n be the usual n-dimensional model of semisimplicial theory. By $\bar{\Delta}^n$ denote the complex obtained by identifying all zero-dimensional elements of Δ^n. We now define

$$\begin{aligned} M_{n,m} &= (\bar{\Delta}^n, G\bar{\Delta}^n, \xi^K, G\bar{\Delta}^n) \times \Delta^m \\ &= (\bar{\Delta}^n, G\bar{\Delta}^n, \xi^K, G\bar{\Delta}^n \times \Delta^m). \end{aligned}$$

Since $[\bar{\Delta}^n, G\bar{\Delta}^n, \xi^K, G\bar{\Delta}^n \times \Delta^m] = [\bar{\Delta}^n, G\bar{\Delta}^n, \xi^K, G\bar{\Delta}^n] \times \Delta^m$, and since Δ^m is contractible, 7.2 gives at once

7.3 LEMMA. *$[M_{n,m}]$ is contractible.*

We now introduce two functors K, $L_F : \mathcal{B}_0 \to d\mathcal{G}$ (cf. Section 4).

$$K(X, \Gamma, \xi, Y) = \Lambda[X, \Gamma, \xi, Y],$$

the normalized chain complex with the usual differential, and

$$K(a, b, \theta, c) = \Lambda[a, b, \theta, c].$$

From 7.3 we have immediately

7.4 PROPOSITION. *$H_q K(M_{n,m}) = 0$ for $q > 0$.*

7.5 PROPOSITION. *K is representable.*

Proof. Here we meet a technical difficulty; we can easily prove the representability of $\bar{K}$, the functor defined like K but using the *unnormalized* chain complex; we then deduce the representability of K by the method of Lemma 6.3 in [1].

Several alternatives suggest themselves: We could use the unnormalized complex; but then the theory of filtration of L_F (see below) is not so simple; or we could use the more elaborate theory of [2], introducing a category with models and degeneracy. This is complicated, but possible. The trouble however *then* is that acyclicity in our present sense becomes insufficient; we need contracting chain homotopies that are natural in a certain sense. In the case of K this is not difficult, but in the case of L_F the necessary verifications would be very formidable, due to the indirect way in which its acyclicity on models is proved. The present method seems to be the simplest.

We prove, then, the representability of $\bar{K}$: Let $(x, y) \in X_n \times Y_n$, $A = (X, \Gamma, \xi, Y)$. We define $\chi(A)(x, y) = ((\bar{\delta}^n, 1^n \times \delta^n), u) \in \bar{K}(M_{n,n}, u)$, where δ^n denotes the n-dimensional generator of Δ^n, $\bar{\delta}^n$ its image in $\bar{\Delta}^n$, and

$$u = (\tilde{x}, \tilde{x}_\xi, e, c) : M_{n,n} \to A$$

is the map defined as follows: $\tilde{x} : \bar{\Delta}^n \to X$ is the map defined by $\tilde{x}\bar{\delta}^n = x$; $\tilde{x}_\xi : G\bar{\Delta}^n \to \Gamma$ is the map defined in terms of $\tilde{x}$ and ξ as in 6.4; $e : \bar{\Delta}^n \to \Gamma$ assigns to every element the unit of Γ; and

$$c : G\bar{\Delta}^n \times \Delta^n \to Y$$

is defined by

$$c(g, \sigma) = (\tilde{x}_\xi\, g) \cdot \tilde{y}\sigma \qquad (g \in G\bar{\Delta}^n, \quad \sigma \in \Delta^n),$$

where $\tilde{y} : \Delta^n \to Y$ is defined by $\tilde{y}\delta^n = y$.

The necessary verifications are easy.

7.6 DEFINITION. A *twisting cochain F on the category $\mathcal{P}$* is a function F which assigns to every object (X, Γ, ξ) a twisting cochain

$$F(X, \Gamma, \xi) : \Lambda(X) \to \Lambda(\Gamma)$$

(cf. Section 3) in such a way that if $(a, b, \theta):(X, \Gamma, \xi) \to (X', \Gamma', \xi')$ is a map, then the diagram

$$\begin{tikzcd} \Lambda(X) \arrow[r, "F(X,\Gamma,\xi)"] \arrow[d, "a"] & \Lambda(\Gamma) \arrow[d, "b"] \\ \Lambda(X') \arrow[r, "F(X',\Gamma',\xi')"] & \Lambda(\Gamma') \end{tikzcd}$$

is commutative.

In defining a twisting cochain we shall usually define (with an abuse of notation!) a function $F:X \to \Lambda(\Gamma)$ which is zero on degenerate simplexes.

Now let F be a twisting cochain on the category $\mathcal{P}_0$. We define the functor

$$L = L_F:\mathcal{B}_0 \to d\mathcal{G}$$

as follows: $L_F(X, \Gamma, \xi, Y) = \Lambda(X) \otimes \Lambda(Y)$, with the differential d_f where $f = F(X, \Gamma, \xi)$;

$$L_F(a, b, \theta, c) = \Lambda(a) \otimes \Lambda(c).$$

The fact that this map commutes with d_f follows from the condition of 7.6 and formula (3) in Section 2.

7.7 Proposition. *$L = L_F$ is representable.*

Proof. Again (cf. 7.5) we prove the representability of $\bar{L}_F$, obtained by using the tensor product of the *unnormalized* chain complexes, and then apply the method of Lemma 6.3 in [1].

Let $x \in X_p$, $y \in Y_q$. We define

$$\bar{\chi}(A)(x \otimes y) = (\bar{\delta}^p \otimes 1^q \times \delta^q, u) \in \bar{L}(M_{p,q}, u),$$

where

$$u = (\tilde{x}, \tilde{x}_\xi, e, c):M_{p,q} \to A,$$

and

$$c:G\bar{\Delta}^p x \Delta^q \to Y$$

is defined by

$$c(g \times \sigma) = \tilde{x}_\xi(g)\cdot\tilde{y}\sigma \qquad (g \in G\bar{\Delta}^p, \quad \sigma \in \Delta^q);$$

cf. 7.5 for the notations.

We now give filtrations to L_F and K. L_F is filtered by using the filtration on the tensor product explained in Section 3, i.e.,

$$F_p(\Lambda(X) \otimes \Lambda(Y)) = \textstyle\sum_{i \leqq p} \Lambda_i(X) \otimes \Lambda(Y).$$

K is filtered, as usual, by letting $(x, y) \in [X, \Gamma, \xi, Y]$ have filtration $\leqq p$ if x is the degeneration of an at most p-dimensional element.

7.8 PROPOSITION. *The representations of* 7.5 *and* 7.7 *are mutually related by the property* 4.3.

This follows immediately from the occurrence of the map $\tilde{x}$ in both representations.

8. The main theorem

8.1 PROPOSITION. *There exists a twisting cochain on the category* $\mathcal{P}_0$ *with the following properties*:

(i) $F_1(X, \Gamma, \xi)(x) = 1 - \xi^{-1}(x) \quad (x \in X_1)$.
(ii) *If* $\xi(x) = 1 \in \Gamma_{q-1}$ *if* $\dim x = q < n$, *then* $F_q(X, \Gamma, \xi) = 0$ *for* $q < n$.

Proof. Direct verification shows that the given F_1 and

$$F_2(x) = -\xi^{-1}(x) \cdot s_0 \, \xi^{-1}(\partial_0 \, x) \qquad (x \in X_2)$$

will satisfy both conditions.

We now suppose that F_i has been defined for $i < q > 2$. Then F_q is to satisfy

$$\begin{aligned} d \circ F_q &= -F_{q-1} \circ d - \textstyle\sum_{i=1}^{q-1} F_i \cup F_{q-1} \\ &= \nu_q, \quad \text{say.} \end{aligned}$$

It is easily verified that $d \circ \nu_q = 0$.

Consider the object $(\bar{\Delta}^q, G\bar{\Delta}^q, \xi)$ (cf. 6.3). It is easily seen (e.g. by considering its realization) that the homotopy groups of $\bar{\Delta}^q$ above π_1 are all zero; hence the homotopy groups of $G\bar{\Delta}^q$ (its "loop space", cf. [4]) above π_0 are all zero. Hence, if $\bar{\delta}^q$ denotes again the generating element of $\bar{\Delta}^q$, $\nu_q(\bar{\delta}^q) \in \Lambda_{q-2}(G\bar{\Delta}^q)$, being a cycle, is a boundary; i.e., there is $c \in \Lambda_{q-1}(G\bar{\Delta}^q)$ such that $dc = \nu_q(\bar{\delta}^q)$.

We now prove the existence of F_q on objects of type (X, GX, ξ^K). Let $x \in X_q$. This defines a map $\langle x \rangle : \bar{\Delta}^q \to X$, and hence $G\langle x \rangle : G(\bar{\Delta}^q) \to G(X)$. If $\bar{\Lambda}(X)$ denotes the *unnormalized* chain complex of X, any chain $g \in \bar{\Lambda}_q(X)$ thus induces, by linear extension, a chain map

$$G\langle g \rangle : \bar{\Lambda}G(\bar{\Delta}^q) \to \bar{\Lambda}G(X).$$

Now we define (cf. p. 195 of [1]) a homomorphism

$$\eta : \bar{\Lambda}^q(X) \to \bar{\Lambda}_q(x)$$

by

$$\eta(x) = (1 - s_0 \, \partial_1)(1 - s_1 \, \partial_2) \cdots (1 - s_{q-1} \, \partial_q)x, \qquad x \in X_q,$$

so that $\eta(x) = 0$ if x is degenerate. Now, for $x \in X_q$, we define

$$F_q(x) = G\langle \eta(x) \rangle c.$$

Clearly $F_q(x) = 0$ if x is degenerate. Now

$$\begin{aligned} dF_q(x) &= dG\langle \eta x\rangle c = G\langle \eta x\rangle\, dc \\ &= G\langle \eta x\rangle \nu_q(\bar{\delta}^q) = \nu_q(\langle \eta x\rangle \bar{\delta}^q) \\ &= \nu_q(x) \end{aligned}$$

since, by the inductive hypothesis, ν_q is zero on degenerate elements.

Finally, consider (X, Γ, ξ). F_q is defined, by the above, on $(X, G(X), \xi^K)$. We now define

$$F_q(X, \Gamma, \xi) = G_\xi\, F_q(X, GX, \xi^K) \qquad \text{(cf. 6.4).}$$

Notation. From now on, F will denote a twisting cochain satisfying the properties of 8.1.

8.2 LEMMA. $H_q\, L_F(\bar{\Delta}^n, G\bar{\Delta}^n, \xi^K) = 0$ *if* $q > 0$.

As remarked before, $H_i(G\bar{\Delta}^n) = 0$ if $i > 0$; also by the same argument, $H_0(G\bar{\Delta}^n) = \Lambda(\pi)$, the group ring of $\pi = \pi_1(\bar{\Delta}^n)$. Now (cf. Section 9) π is generated by elements $[x]$, one for each $x \in \bar{\Delta}_1^n$. Hence, regarding $\Lambda(\pi)$ as a DGA algebra in which all elements have graduation 0, we define

$$h\colon \Lambda(G\bar{\Delta}^n) \to \Delta(\pi)$$

by letting h be zero on elements of dim > 0, and putting $h\bar{x} = [x]$ if $x \in \bar{\Delta}_1^n$. $H_0(G\bar{\Delta}^n)$ has generators in 1-1 correspondence with the elements of π, and hence h is a chain equivalence. Hence, by using 3.3, the tensor product $\Lambda(\bar{\Delta}^n) \otimes \Lambda G(\bar{\Delta}^n)$ with twisting function F is chain-equivalent to $\Lambda(\bar{\Delta}^n) \otimes \Lambda(\pi)$ with twisting function hF.

A direct computation[3] immediately shows that

$$d_{hF}(x \otimes \gamma) = \textstyle\sum_{i=0}^{q-1} \partial_i\, x \otimes \gamma + (-1)^q \partial_q\, x \otimes [\partial_0^{q-1} x]\gamma,$$

and this is exactly the boundary formula for the universal *covering complex* of $\bar{\Delta}^n$ (cf. Section 9), which is contractible since all higher homotopy groups of $\bar{\Delta}^n$ vanish.

8.3 LEMMA. *For any twisting cochain* F *on* $\mathcal{P}_0$, $L_F(M_{n,m})$ *and* $L_F(\bar{\Delta}^n, G\bar{\Delta}^n, \xi^K)$ *are chain-equivalent.*

Proof. Since Δ^m is contractible, it is easy to write down a chain equivalence

$$G\bar{\Delta}^n \to G\bar{\Delta}^n \times \Delta^m.$$

From this and 3.3 the result is immediate. 8.2 and 8.3 give

8.4 PROPOSITION. $H_q\, L_F(M_{n,m}) = 0$ *if* $q > 0$.

We now come to the main theorem of this paper.

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[3] The sign $(-1)^q$ originates in our definition of the cap product.

8.5 THEOREM. *There are filtration-preserving chain maps* $\varphi : L_F \to K$, $\psi : K \to L_F$; *and* $\varphi\psi$, $\psi\varphi$ *are chain-homotopic to the identity by filtration-preserving chain homotopies.*

Proof. Letting $x_i \in X_i$, $y_i \in Y_i$, we define $\varphi_i = \varphi_i(X, \Gamma, \xi, Y)$, $\psi_i = \psi_i(X, \Gamma, \xi, Y)$ for $i = 0, 1$ as follows:

$$\begin{aligned} \varphi_0(x_0 \otimes y_0) &= (x_0 , y_0), \\ \varphi_1(x_0 \otimes y_1) &= (s_0 x_0 , y_1), \\ \varphi_1(x_1 \otimes y_0) &= (x_1 , s_0(\xi^{-1}(x_1) \cdot y_0)), \\ \psi_0(x_0 , y_0) &= (x_0 \otimes y_0), \\ \psi_1(x_1 , y_1) &= x_1 \otimes \xi(x_1) \cdot \partial_0 y_1 + \partial_1 x_1 \otimes y_1 . \end{aligned}$$

The necessary verifications are straightforward. The theorem now follows from 7.7, 7.4, 7.5, 8.2, 7.8, 4.2, and 4.4.

8.6 COROLLARY. φ *induces isomorphisms*

$$\varphi^r : E^r L_F \to E^r K,$$

$$\varphi_r : E_r L_F \to E_r K, \qquad \textit{for } r \geqq 1,$$

where E^r, E_r *denote the homology or cohomology spectral sequences.*

9. The universal covering space

We give brief indications how a theory of the universal covering space can be given in the semisimplicial context, purely to complete the proof of 8.2. A complete theory of covering spaces could easily be developed by the same method.

In the theory of twisted cartesian products there are two alternatives: One can give preferred treatment to ∂_0, as we have done, in accordance with [5]; or one can give preferred treatment to the *last* face operator, as is done in [4]; in the present section we adopt the latter view; this seeming inconsistency is required by the form of the formula that one is led to in 8.2; in some sense neither of the two methods seems to be quite self-contained.

Let X be a complex with a single vertex o. In [4], it is proved that $\pi_1(X, o)$ is the group with a generator $\{x\}$ for every $x \in X_1$ and a relation $\{\partial_0 \sigma\}\{\partial_2 \sigma\} = \{\partial_1 \sigma\}$ for every $\sigma \in X_2$. We denote by $[x]$ the element of $\pi = \pi_1(x, o)$ represented by $\{x\}$.

We now define the *universal covering complex* $\tilde{X}$ of X as follows:

$$\begin{aligned} \tilde{X}_n &= X_n \times \pi, \\ \partial_i(X, \gamma) &= (\partial_i x, \gamma), \qquad i < n, \quad x \in X_n, \quad \gamma \in \pi. \\ \partial_n(x, \gamma) &= (\partial_n x, [\partial_0^{n-1} x]\gamma), \\ s_i(x, \gamma) &= (s_i x, \gamma). \end{aligned}$$

The c.s.s. relations are easily verified, as is the fact that the boundary in X is given exactly by the formula of 8.2.

We can regard $\tilde{X}$ as a RTCP as follows: Let $S(\pi)$ be the complex obtained by regarding the elements of π as zero-dimensional elements. Then $\tilde{X}$ is isomorphic to $[X, S(\pi), \xi, S(\pi)]$, where ξ is given by

$$\xi(x) = s_{n-2} \cdots s_0[\partial_0^{n-1} x].$$

We now examine the homotopy sequence of $[X, S(\pi), \xi, S(\pi)]$ and obtain: Since $\pi_i(S(\pi)) = 0$ when $i > 0$, $\pi_i(\tilde{X}) \approx \pi_i(X)$ for $i > 1$; also $\pi_1(x) \to \pi_0(S(\pi))$, as can be verified, is exactly given by $[x] \to [x]$, i.e., is an isomorphism; whence $\pi_1(\tilde{X}) = 0$.

Thus $\tilde{X}$ has exactly the properties of the universal covering space.

10. Cohomology and the theorem of Hurewicz-Fadell

Let us return to the algebraic situation of Section 2, and consider the case of cohomology. Let G be any graded Λ-module. We define a pairing

$$\operatorname{Hom}(B, G) \otimes C \xrightarrow{\tau} \operatorname{Hom}(B, G)$$

by

$$\tau(g \otimes c)(b) = g\tau(c \otimes b), \qquad c \in C,\ b \in B,\ g \in \operatorname{Hom}(B, G),$$

where τ on the right denotes the given pairing $C \otimes B \to B$, as in Section 2.

Using this pairing, and given $h \in \operatorname{Hom}(A, \operatorname{Hom}(B, G))$, $f \in \operatorname{Hom}(A, C)$, we can define

$$h \cup f \in \operatorname{Hom}(A, \operatorname{Hom}(B, G))$$

by $h \cup f = \tau(h \otimes f)\psi$, as in Section 2.

By applying the natural isomorphism

$$\operatorname{Hom}(A, \operatorname{Hom}(B, G)) = \operatorname{Hom}(A \otimes B, G)$$

it is then easy to verify that

$$(1) \qquad (h \cup f)(x) = h(f \cap x), \qquad x \in A \otimes B.$$

A straightforward verification proves that

$$(2) \qquad d(h \cup f) = dh \cup f + (-1)^p h \cup df,$$

and that if $g \in \operatorname{Hom}(A, C)$, then

$$(3) \qquad h \cup (f \cup g) = (h \cup f) \cup g.$$

In these formulas we have used the definition $(dh)x = (-1)^{p+1}h(dx)$ for the "coboundary" as in Section 1, i.e., we have regarded G as a differential group with zero differential.

Now, let f be a twisting cochain, as in Section 3. The "coboundary" d_f dual to d_f is given by

$$(4) \qquad (d_f h)x = (-1)^{p+1}h(d_f x).$$

A simple calculation then shows

$$d_f h = dh + (-1)^{p+1} h \cup f. \tag{5}$$

Applying this to Theorem 8.5 we thus get a formula for the differential that "computes" the *cohomology* of a RTCP. Usually G has no graduation; then we put $G = G_0$, $G_i = 0$, $i > 0$. What is usually called an n-cochain is $-n$-dimensional in our notation.

We now come to the theorem of Hurewicz-Fadell (cf. [7]), which is an application of condition (ii) in 8.1. From now on, we suppose that A, B, C are Λ-free, as they are in the application, and that Λ is a principal ideal ring.

10.1 Definitions. A twisting cochain f such that $f_i = 0$ for $i \leqq n \geqq 1$ will be called *n-trivial.*

A (X, Γ)-twisting function ξ such that $\xi(x) = 1 \in \Gamma_{i-1}$ for $x \in X_i$, $i \leqq n \geqq 1$, will be called n-trivial.

From 8.1 (ii) we have at once

10.2 Proposition. *If ξ is n-trivial, so is $F(X, \Gamma, \xi)$.*

Examples. (1) If $\pi_i(X) = 0$ for $i < n$, then we can, without changing the homotopy type of (X, Γ, ξ, Y), replace X by a complex for which X_i is the degeneration of a single vertex for $i < n$; cf [5]. This is the case of the theorem of Hurewicz-Fadell. ξ becomes n-trivial.

(2) More generally, suppose (cf. [5], in particular Lemma IV. 2.6) that in (X, Γ, ξ, Y), Γ can be reduced so that $\Gamma_i = 1_i$ for $i < n$. Then clearly ξ becomes n-trivial.

Let f be n-trivial, $n \geqq 1$, and let $x \in A \otimes B$ have filtration p. Then $f \cap x$ has filtration at most $p - n$; thus in this case

$$(d - d_f)F_p(A \otimes B) \subset F_{p-n-1}(A \otimes B).$$

Hence by a simple lemma on spectral sequences, the spectral sequences induced by d and d_f are isomorphic up to E^{n+1}; the spectral sequence for d, however, as is well known, has the property $d_r = 0$ if $r \geqq 2$, $E^r = E^2$ if $r \geqq 2$, and similarly for cohomology.

Hence we have

10.3 Theorem. *If f is n-trivial, $n \geqq 1$, then*

$$E^r(A \otimes B) = E^2(A \otimes B) = H_*(A, H_*(B)),$$

$$E^r(A \otimes B, G) = E_2(A \otimes B, G) = H^*(A, H^*(B, G)), \quad 2 \leqq r \leqq n + 1.$$

If we continue in the same situation, it is easily seen that d^{n+1} and d_{n+1} depend on f_{n+1} only and are, indeed, induced by the homomorphisms

$$x \to f_{n+1} \cap x, \qquad x \in A \otimes B,$$

$$h \to (-1)^{p+1} h \cup f_{n+1}, \qquad h \in \operatorname{Hom}(A \otimes B, G), \dim h = p.$$

Now formula (1) in Section 3 shows that $f_{n+1} \in \operatorname{Hom}(A, C)$ satisfies "$f_{n+1}(da)$ is a boundary in C"; hence f_n induces a cocycle

$$k \in \operatorname{Hom}(A, H(C)).$$

Let $\mathbf{k} \in H^{n+1}(A, H_n(C))$ be its cohomology class. Then 10.3 gives

10.4 THEOREM. *If f is n-trivial, $n \geqq 1$, then d^{n+1}, d_{n+1} are given by*

$$d^{n+1} x = \mathbf{k} \cap x, \qquad x \in H_*(A, H_*(B)),$$
$$d_{n+1} h = (-1)^{p+1} h \cup \mathbf{k}, \qquad h \in H^p(A, H^*(B, G)).$$

We finish by giving a new interpretation to $\mathbf{k}$. If f is n-trivial, we can (cf. example (2) above) replace C by $\sum_{i=n}^{\infty} C_i$; every element $c \in C_n$ is a cycle and represents $[c] \in H_n(C)$; we can thus define the *fundamental cocycle* $\gamma \in \Lambda_n(C, H_n(C))$ by $\gamma c = [c]$.

Next, consider the "associated principal bundle", i.e., in our algebraic situation the tensor product $A \otimes C$ with twisting cochain F. We regard $\gamma \in \Lambda_n(A \otimes C, H_n(C))$ by the usual method, i.e., $\gamma(1 \otimes c) = [c]$ if 1 denotes the 0-dimensional generator of A, and $\gamma(a \otimes c) = 0$ otherwise. Similarly, we identify $a \in A$ with $a \otimes 1 \in A \otimes C$.

Now let $a \in A_{n+1}$.

$$\begin{aligned}(\gamma \cup f)(a \otimes 1) &= \gamma(f \cap (a \otimes 1)) \\ &= \gamma(1 \otimes f(a)) \\ &= [f(a)] = k(a \otimes 1),\end{aligned}$$

where we have used the fact that the coproduct satisfies

$$\psi a = 1 \otimes a + a \otimes 1 + \textstyle\sum_{i=1}^{n} a_i' \otimes a_{n+1-i}''$$

where $\dim a_i' = i$, $\dim a_j'' = j$.

Hence $\gamma \cup f = k$, or $\boldsymbol{\gamma} \cup \mathbf{k} = \mathbf{k}$. Hence by 10.4

$$d_{n+1} \boldsymbol{\gamma} = (-1)^{n+1} \mathbf{k}.$$

We translate this back into geometric language:

10.5 THEOREM. *Let (X, Γ, ξ, Y) be a bundle in which Γ has been reduced to a group for which $\Gamma_i = 1_i$, $i < n \geqq 0$; and let $\boldsymbol{\gamma} \in H^n(\Gamma, H_n(\Gamma))$ be the cohomology class of the fundamental cocycle. Then we have for the spectral sequences associated with this bundle:*

$$E_{p,q}^r = H_p(X, H_q(Y)),$$
$$E_r^{p,q} = H^p(X, H^q(Y)), \qquad 2 \leqq r \leqq n+1,$$

while

$$d^{n+1}: H_p(X, H_q(Y)) \to H_{p-n-1}(A, H_{q+n}(Y)),$$
$$d_{n+1}: H^p(X, H^q(Y)) \to H^{p+n+1}(A, H^{q-n}(Y))$$

are given by

$$d^{n+1}x = \mathbf{k} \cap x,$$

$$d_{n+1}\, h = (-1)^{n+1} h \cup \mathbf{k},$$

where $\mathbf{k} \in H^{n+1}(X, H_n(\Gamma))$ *is the transgression of* $(-1)^{n+1}\gamma$ *in the associated principal bundle, the products* $\cap$ *and* $\cup$ *being defined by using the operation of* Γ *on* Y.

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