CARTAN INVARIANTS OF ALGEBRAS WITH UNIQUE MINIMAL FAITHFUL REPRESENTATIONS[1]

BY

DRURY W. WALL

Introduction

Let $\mathfrak{A}$ be a finite-dimensional algebra with unit element over a field K. Let $\{e_i\}_{i=1}^{n}$ be a maximal set of nonisomorphic primitive idempotents of $\mathfrak{A}$, and let $\{c_{ij}\}_{i,j=1}^{n}$ be the Cartan invariants of $\mathfrak{A}$. This paper gives relationships among the Cartan invariants of an algebra with a unique minimal faithful representation (a UMFR algebra). Similar relationships are given for an algebra belonging to certain subclasses of the class of UMFR algebras. These results will generalize those obtained by R. M. Thrall [6] for one of the subclasses. The enumeration of these subclasses will be that used in an earlier paper [7] in which some properties of the subclasses were studied.

§1 contains the definitions and notations for the paper including the definitions of certain sets of integers associated with each of the integers $i = 1, \cdots, n$ and certain decomposition numbers associated with the primitive ideals $\mathfrak{A}e_i$ of $\mathfrak{A}$ and their socles. §2 gives relationships among these sets of integers and decomposition numbers when the associated left ideal $\mathfrak{A}e_i$ is weakly subordinate. §3 gives relationships among certain of the Cartan invariants of any algebra $\mathfrak{A}$ in which there are weakly subordinate left ideals. §2′ and §3′ give the corresponding results for right ideals. §4 gives the relationships that hold for the Cartan invariants of a UMFR algebra and gives similar results for the various subclasses. §5 restates the results in terms of the Cartan matrix $C(\mathfrak{A}) = (c_{ij})$.

1. Definitions and notations

Let $\mathfrak{A}$ be a finite-dimensional algebra with unit element over a field K. When referring to ideals of $\mathfrak{A}$ or to $\mathfrak{A}$-modules, the term *isomorphic* will mean isomorphic when considered as $\mathfrak{A}$-modules. If e and f are idempotents, then e and f are *isomorphic* if and only if $\mathfrak{A}e \cong \mathfrak{A}f$ (or equivalently $e\mathfrak{A} \cong f\mathfrak{A}$). Let

$$1 = \sum_{i=1}^{n} \sum_{j=1}^{f_i} e_{ij} \tag{1}$$

be a decomposition of the unit element of $\mathfrak{A}$ into the sum of mutually orthogonal primitive idempotents such that $e_{ij} \cong e_{hk}$ if and only if $i = h$. Let

$$\mathfrak{A} = \sum_{i=1}^{n} \sum_{j=1}^{f_i} \mathfrak{A}e_{ij} \quad \text{and} \quad \mathfrak{A} = \sum_{i=1}^{n} \sum_{j=1}^{f_i} e_{ij}\,\mathfrak{A}$$

be the corresponding decompositions of $\mathfrak{A}$ into the direct sums of primitive left ideals and primitive right ideals, respectively. Let e_i denote e_{i1} for

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$i = 1, \cdots, n$. Then the set $\{e_i\}_{i=1}^n$ is a maximal set of nonisomorphic primitive idempotents, and every primitive left ideal of $\mathfrak{A}$ is isomorphic to one of the $\mathfrak{A}e_i$, and every primitive right ideal of $\mathfrak{A}$ is isomorphic to one of the $e_i\mathfrak{A}$.

Let $\mathfrak{V}_i$ be the indecomposable representation of $\mathfrak{A}$ which has $\mathfrak{A}e_i$ as its representation module, and let $\mathfrak{U}_i$ be the indecomposable representation of $\mathfrak{A}$ with $e_i\mathfrak{A}$ as representation module. Let $\mathfrak{F}_i$ be the irreducible representation of $\mathfrak{A}$ with module $\mathfrak{A}e_i/\mathfrak{N}e_i$, where $\mathfrak{N}$ is the radical of $\mathfrak{A}$. $\mathfrak{F}_i$ is equivalent to the representation with module $e_i\mathfrak{A}/e_i\mathfrak{N}$. The $\mathfrak{V}_i$ are the nonequivalent components of the left regular representation, and the $\mathfrak{U}_i$ are the nonequivalent components of the right regular representation. Every irreducible representation of $\mathfrak{A}$ is equivalent to one of the $\mathfrak{F}_i$. (See [2], [3], and [5].)

Let c_{ij} be the number of irreducible constituents of $\mathfrak{V}_i$ which are equivalent to $\mathfrak{F}_j$, and let $\tilde{c}_{ij}$ be the number of irreducible constituents of $\mathfrak{U}_i$ which are equivalent to $\mathfrak{F}_j$. It is known [1, p. 106] that $c_{ij} = \tilde{c}_{ji}$. The integers c_{ij} are known as the *Cartan invariants* of $\mathfrak{A}$, and the matrix $C(\mathfrak{A}) = (c_{ij})$ is the *Cartan matrix* of $\mathfrak{A}$. The Cartan invariant c_{ij} can be characterized in a number of additional ways: the number of constituents of any composition series of $\mathfrak{A}e_i$ which are isomorphic to $\mathfrak{A}e_j/\mathfrak{N}e_j$; the number of constituents of any composition series of $e_j\mathfrak{A}$ which are isomorphic to $e_i\mathfrak{A}/e_i\mathfrak{N}$; the composition length of $e_j\mathfrak{A}e_i$ as an $e_j\mathfrak{A}e_j$-module (see [1, p. 106]).

For any primitive ideal $\mathfrak{J}$, the *socle* $\mathfrak{S}(\mathfrak{J})$ of $\mathfrak{J}$ is the sum of all minimal subideals of $\mathfrak{J}$ (see [4, p. 63], [7, §5]). For each $\mathfrak{A}e_i$ let

$$\mathfrak{S}(\mathfrak{A}e_i) = \sum_{k=1}^{n} \sum_{j=1}^{g_{ik}} \mathfrak{L}_{i,kj}$$

be a decomposition of $\mathfrak{S}(\mathfrak{A}e_i)$ into the direct sum of minimal subideals of $\mathfrak{A}e_i$ such that $\mathfrak{L}_{i,kj} \cong \mathfrak{A}e_k/\mathfrak{N}e_k$ for all j and k. For each $e_i\mathfrak{A}$ let

$$\mathfrak{S}(e_i\mathfrak{A}) = \sum_{k=1}^{n} \sum_{j=1}^{\tilde{g}_{ik}} \mathfrak{R}_{i,kj}$$

be a decomposition of $\mathfrak{S}(e_i\mathfrak{A})$ into the direct sum of minimal subideals of $e_i\mathfrak{A}$ such that $\mathfrak{R}_{i,kj} \cong e_k\mathfrak{A}/e_k\mathfrak{N}$ for all j and k.

DEFINITION. Let $\Sigma_i = \{k \mid g_{ik} \neq 0\}$ and $\Pi_i = \{k \mid \tilde{g}_{ik} \neq 0\}$. If Σ_i is a set with only one element, denote it by $\sigma(i)$, and if Π_i has one only element, denote it by $\pi(i)$.

If a primitive left ideal $\mathfrak{A}e$ is dual to a primitive right ideal $f\mathfrak{A}$, then $\mathfrak{A}e$ and $f\mathfrak{A}$ are *dominant* ideals. An algebra in which every primitive ideal is dominant is a *quasi-Frobenius* algebra. Assume $\mathfrak{A}e_i$ is dual to $e_j\mathfrak{A}$. Then $\mathfrak{A}e_i$ has a unique minimal subideal which is $\mathfrak{S}(\mathfrak{A}e_i)$ and whose representation of $\mathfrak{A}$ is equivalent to $\mathfrak{F}_j$. Dually, $e_j\mathfrak{A}$ has a unique subideal whose representation is equivalent to $\mathfrak{F}_i$. Thus, $\Sigma_i = \{j\}$ and $\Pi_j = \{i\}$, and so $\sigma(i) = j$ and $\pi(j) = i$. If $\mathfrak{A}e_i$ and $e_j\mathfrak{A}$ are dual, then the representations of $\mathfrak{A}$ that they generate are equivalent. In terms of the Cartan invariants this implies that for all k

$$c_{ik} = \tilde{c}_{\sigma(i)k} = c_{k\sigma(i)} \quad \text{and} \quad c_{kj} = \tilde{c}_{jk} = c_{\pi(j)k} \, . \tag{2}$$

DEFINITION. Let $\Sigma = \{i \mid e_i \mathfrak{A} \text{ is dominant}\}$ and $\Pi = \{i \mid \mathfrak{A}e_i \text{ is dominant}\}$. (Since an algebra $\mathfrak{A}$ need not have dominant ideals, these sets may be empty.)

2. Weakly subordinate left ideals

An ideal $\mathfrak{J}$ (left or right) is *subordinate* to an ideal $\mathfrak{J}'$ if there exists a subideal $\mathfrak{J}^*$ of $\mathfrak{J}'$ such that $\mathfrak{J} \cong \mathfrak{J}^*$. An ideal $\mathfrak{J}$ is *weakly subordinate* to a set of ideals $\{\mathfrak{J}_i\}_{i=1}^{s}$ if there exists a set of ideals $\{\mathfrak{J}'\}_{j=1}^{m}$ with each $\mathfrak{J}'_j$ a subideal of some $\mathfrak{J}_i$, such that $\mathfrak{J}$ is isomorphic to some submodule of the direct sum $\sum_{j=1}^{m} \mathfrak{J}'_j$. If an ideal is weakly subordinate to a set of ideals, then it is weakly subordinate to a set of mutually nonisomorphic ideals [7, Theorem 1]. An ideal $\mathfrak{J}$ is *subordinate* to a set of ideals if it is subordinate to an ideal of $\mathfrak{A}$ which is their direct sum.

Consider the case of a primitive left ideal $\mathfrak{J}$ weakly subordinate to a set of dominant ideals. Then $\mathfrak{J}$ is isomorphic to one of the $\mathfrak{A}e_i$, $i = 1, \cdots, n$, and both $\mathfrak{J}$ and the set of dominant ideals may be chosen from among the $\mathfrak{A}e_i$, $i = 1, \cdots, n$. If $\mathfrak{A}e_i$ is weakly subordinate to a set of dominant ideals, let Φ_i be the subset of the integers $1, \cdots, n$ such that $\mathfrak{A}e_i$ is weakly subordinate to $\{\mathfrak{A}e_k \mid k \in \Phi_i\}$ and is not weakly subordinate to $\{\mathfrak{A}e_k \mid k \in X\}$ where X is any proper subset of Φ_i. In this notation $\mathfrak{A}e_i$ is weakly subordinate to $\{\mathfrak{A}e_k \mid k \in \Phi_i\}$ if and only if $\mathfrak{A}e_i$ is isomorphic to a submodule $\mathfrak{M}_i^*$ of the left $\mathfrak{A}$-module

$$\mathfrak{M}_i = \sum_{k \in \Phi_i} \sum_{j=1}^{h_{ik}} \mathfrak{A}e_k \, , \tag{3}$$

where for each $k \in \Phi_i$, h_{ik} is the smallest possible integer. By setting $h_{jk} = 0$ for all $k \notin \Phi_i$, the summation in (3) can be extended so that

$$\mathfrak{M}_i = \sum_{k=i}^{n} \sum_{j=1}^{h_{ik}} \mathfrak{A}e_k \, . \tag{4}$$

THEOREM 1. *$\mathfrak{A}e_i$ is weakly subordinate to a set of dominant ideals if and only if $\Sigma_i \subset \Sigma$. If $\mathfrak{A}e_i$ is weakly subordinate to a set of dominant ideals $\{\mathfrak{A}e_i \mid k \in \Phi_i\}$, then*

$$\Sigma_i = \{\sigma(k) \mid k \in \Phi_i\}, \tag{5}$$

$$\Phi_i = \{\pi(k) \mid k \in \Sigma_i\}. \tag{6}$$

Proof. The first statement is merely a rephrasing of an earlier result [7, Theorem 4].

Assume that $\mathfrak{A}e_i$ is weakly subordinate to $\{\mathfrak{A}e_k \mid k \in \Phi_i\}$, a set of dominant ideals. Since $\mathfrak{A}e_i \cong \mathfrak{M}_i^*$, where $\mathfrak{M}_i^*$ is a submodule of $\mathfrak{M}_i$ given by (3), every minimal subideal of $\mathfrak{A}e_i$ is isomorphic to a minimal subideal of one of the $\mathfrak{A}e_k$, $k \in \Phi_i$ (see proof of [7, Theorem 4]). For $k \in \Phi_i$, $\mathfrak{A}e_k$ is dual to $e_{\sigma(k)} \mathfrak{A}$, and thus the minimal subideal of $\mathfrak{A}e_k$ is isomorphic to $\mathfrak{A}e_{\sigma(k)}/\mathfrak{N}e_{\sigma(k)}$. Thus, $\Sigma_i \subset \{\sigma(k) \mid k \in \Phi_i\}$. From the minimality of Φ_i it follows that

$$\{\sigma(k) \mid k \in \Phi_i\} \subset \Sigma_i \, ,$$

and, therefore, (5) is proved. (6) follows from (5) and from the fact that $\pi(\sigma(k)) = k$ for $k \in \Phi_i$.

COROLLARY. *Let* $\mathfrak{A}e_i$ *be weakly subordinate to a set of dominant ideals.* Σ_i *has a single element* j, *i.e.*, $\Sigma_i = \{j\}$, *if and only if* $\mathfrak{A}e_i$ *is weakly subordinate to the single dominant ideal* $\mathfrak{A}e_{\pi(j)}$.

Note that the g_{ij} are defined for any i and j, but the h_{ij} are defined only for those i such that $\mathfrak{A}e_i$ is weakly subordinate to a set of dominant ideals. If i is such that $\mathfrak{A}e_i$ is weakly subordinate, then for every j

$$g_{ij} = h_{i\pi(j)} \quad \text{and} \quad h_{ij} = g_{i\sigma(j)}. \tag{7}$$

Hence, the number h_{ik} of times $\mathfrak{A}e_k$ appears as a component of $\mathfrak{M}_i$ is exactly the number $g_{i\sigma(k)}$ of components $\mathfrak{L}_{i,\sigma(k)j}$ in $\mathfrak{S}(\mathfrak{A}e_i)$.

2′. Weakly subordinate right ideals

The case in which $\mathfrak{J}$ is a primitive right ideal weakly subordinate to a set of dominant ideals is exactly dual to the left ideal case. The definitions and results will merely be stated.

If $e_i\,\mathfrak{A}$ is weakly subordinate to a set of dominant ideals, let Ψ_i be the set of integers such that $e_i\,\mathfrak{A}$ is weakly subordinate to $\{e_k\,\mathfrak{A} \mid k \in \Psi_i\}$ but not to $\{e_k\,\mathfrak{A} \mid k \in \mathrm{X}\}$, where X is any proper subset of Ψ_i. Thus, $e_i\,\mathfrak{A}$ is weakly subordinate to $\{e_k\,\mathfrak{A} \mid k \in \Psi_i\}$ if and only if $e_i\,\mathfrak{A}$ is isomorphic to a submodule $\tilde{\mathfrak{M}}_i^*$ of the right $\mathfrak{A}$-module

$$\tilde{\mathfrak{M}}_i = \sum_{k\in\Psi_i} \sum_{j=1}^{\tilde{h}_{ik}} e_k\,\mathfrak{A}. \tag{3′}$$

By setting $\tilde{h}_{ik} = 0$ for all $k \notin \Psi_i$, the summation in (3′) can be extended so that

$$\tilde{\mathfrak{M}}_i = \sum_{k=1}^{n} \sum_{j=1}^{\tilde{h}_{ik}} e_k\,\mathfrak{A}. \tag{4′}$$

THEOREM 1′. $e_i\,\mathfrak{A}$ *is weakly subordinate to a set of dominant ideals if and only if* $\Pi_i \subset \Pi$. *If* $e_i\,\mathfrak{A}$ *is weakly subordinate to a set of dominant ideals* $\{e_k\,\mathfrak{A} \mid k \in \Psi_i\}$, *then*

$$\Pi_i = \{\pi(k) \mid k \in \Psi_i\}, \tag{5′}$$

$$\Psi_i = \{\sigma(k) \mid k \in \Pi_i\}. \tag{6′}$$

COROLLARY. *Let* $e_i\,\mathfrak{A}$ *be weakly subordinate to a set of dominant ideals. Then* Π_i *has a single element* k, *i.e.*, $\Pi_i = \{k\}$, *if and only if* $e_i\,\mathfrak{A}$ *is weakly subordinate to the single dominant ideal* $e_{\sigma(k)}\,\mathfrak{A}$.

As before, the relationships among the $\tilde{g}_{ij}$ and $\tilde{h}_{ij}$ are

$$\tilde{g}_{ij} = \tilde{h}_{i\sigma(j)} \quad \text{and} \quad \tilde{h}_{ij} = \tilde{g}_{i\pi(j)}. \tag{7′}$$

3. Cartan invariants for left ideals

DEFINITION. If $\mathfrak{A}e_i$ is weakly subordinate to $\{\mathfrak{A}e_k \mid k \in \Phi_i\}$, then let d_{ij} be the number of irreducible constituents equivalent to $\mathfrak{F}_j$ of the representation whose module is $\mathfrak{M}_i/\mathfrak{M}_i^*$.

THEOREM 2. *If $\mathfrak{A}e_i$ is weakly subordinate to the set of dominant ideals $\{\mathfrak{A}e_k \mid k \in \Phi_i\}$, then for any $j = 1, \cdots, n$*

(8) 1. $c_{ij} + d_{ij} = \sum_{k\in\Phi_i} h_{ik}\, c_{kj}$;

(9) 2. $c_{ij} \leqq \sum_{k\in\Phi_i} h_{ik}\, c_{kj}$;

(10) 3. $c_{ij} = \sum_{k\in\Phi_i} h_{ik}\, c_{kj}$ *for all j if and only if $i \in \Pi$.*

Proof. 1. Since $\mathfrak{F}_j$ appears c_{kj} times in $\mathfrak{A}e_k$ (or, more precisely, $\mathfrak{F}_j$ appears c_{kj} times as an irreducible constituent in the representation $\mathfrak{B}_k$ generated by $\mathfrak{A}e_k$), $\mathfrak{F}_j$ appears $h_{ik}\, c_{kj}$ times in $\mathfrak{M}_{i,k} = \sum_{r=1}^{h_{ik}} \mathfrak{A}e_k$. Thus, since

$$\mathfrak{M}_i = \sum_{k\in\Phi_i} \mathfrak{M}_{i,k} ,$$

$\mathfrak{F}_j$ appears $\sum_{k\in\Phi_i} h_{ik}\, c_{kj}$ times in $\mathfrak{M}_i$. Since $\mathfrak{M}_i^* \cong \mathfrak{A}e_i$, $\mathfrak{F}_j$ appears c_{ij} times in $\mathfrak{M}_i^*$. Since d_{ij} is the number of times that $\mathfrak{F}_j$ appears in $\mathfrak{M}_i/\mathfrak{M}_i^*$, it follows that $c_{ij} + d_{ij} = \sum_{k\in\Phi_i} h_{ik}\, c_{kj}$.

2. Since for all i and j, $d_{ij} \geqq 0$, (9) follows from (8).

3. If $c_{ij} = \sum_{k\in\Phi_i} h_{ik}\, c_{kj}$ for all j, then $d_{ij} = 0$ for all j. Thus, $\mathfrak{M}_i/\mathfrak{M}_i^* = 0$, and hence $\mathfrak{M}_i^* = \mathfrak{M}_i$. But, since $\mathfrak{A}e_i$ is a primitive ideal, it cannot be isomorphic to a sum of more than one primitive ideal. Therefore, $\mathfrak{M}_i$ is a single ideal $\mathfrak{A}e_k$, and $\mathfrak{A}e_i \cong \mathfrak{A}e_k$, which implies $i = k$. Thus, $i \in \Pi$, i.e., $\mathfrak{A}e_i$ is dominant. The converse of (10) is immediate.

In each of the following special cases a set of relations similar to (8), (9), and (10) can be obtained as corollaries to Theorem 2. However, only the formulas corresponding to (8) will be explicitly stated.

COROLLARY. *If $\mathfrak{A}e_i$ is subordinate to $\{\mathfrak{A}e_k \mid k \in \Phi_i\}$, $\mathfrak{A}e_k$ dominant, then $c_{ij} + d_{ij} = \sum_{k\in\Phi_i} c_{kj}$, $j = 1, \cdots, n$. If $\mathfrak{A}e_i$ is weakly subordinate to a single dominant ideal $\mathfrak{A}e_k$, then $c_{ij} + d_{ij} = h_{ik}\, c_{kj}$, $j = 1, \cdots, n$. If $\mathfrak{A}e_k$ is subordinate to a single dominant ideal $\mathfrak{A}e_k$, then $c_{ij} + d_{ij} = c_{kj}$, $j = 1, \cdots, n$.*

3′. Cartan invariants for right ideals

If $e_i\,\mathfrak{A}$ is weakly subordinate to $\{\mathfrak{A}e_k \mid k \in \Psi_i\}$, then let $\tilde{d}_{ij}$ be the number of irreducible constituents equivalent to $\mathfrak{F}_j$ of the representation whose module is $\tilde{\mathfrak{M}}_i/\tilde{\mathfrak{M}}_i^*$. If, in the preceding section, c_{ij} is replaced by $\tilde{c}_{ij}$, d_{ij} by $\tilde{d}_{ij}$, h_{ij} by $\tilde{h}_{ij}$, etc., then the results hold for a primitive right ideal $e_i\,\mathfrak{A}$ weakly subordinate to $\{e_k\,\mathfrak{A} \mid k \in \Psi_i\}$. For example, (8) of Theorem 2 becomes $\tilde{c}_{ij} + \tilde{d}_{ij} = \sum_{k\in\Psi_i} \tilde{h}_{ik}\, \tilde{c}_{kj}$.

However, for every i and j, $\tilde{c}_{ij} = c_{ji}$ [1, p. 106]. But it is not true in general that $\tilde{h}_{ij} = h_{ji}$ or $\tilde{d}_{ij} = d_{ji}$. For example, let $\mathfrak{A}$ be the algebra of all matrices of the form

$$\begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix},$$

where

$$V_1 = \begin{bmatrix} \alpha_1 & 0 & 0 \\ \alpha_5 & \alpha_2 & 0 \\ \alpha_6 & \alpha_7 & \alpha_3 \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} \alpha_4 & 0 & 0 \\ \alpha_8 & \alpha_1 & 0 \\ \alpha_9 & \alpha_5 & \alpha_1 \end{bmatrix},$$

and the α_i are elements of the field K. Let x_i denote the matrix in which $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$. Then $\{x_i\}_{i=1}^{9}$ form a basis of $\mathfrak{A}$, and $\{x_i\}_{i=1}^{4}$ are a maximal set of nonisomorphic primitive idempotents. Then, with respect to this maximal set, the numbers c_{ij}, d_{ij}, etc., can be easily calculated. It is seen that $d_{31} = 1$ while $\tilde{d}_{13} = 0$, and $h_{31} = 1$ while $\tilde{h}_{13} = 0$.

By using the numbers d_{ij} and h_{ij} along with the Cartan invariants c_{ij} the results for right ideals are as given below.

THEOREM 2′. *If $e_i\,\mathfrak{A}$ is weakly subordinate to a set of dominant ideals $\{e_k\,\mathfrak{A} \mid k \in \Psi_i\}$, then for any $j = 1, \cdots, n$*

(8′) 1. $c_{ji} + \tilde{d}_{ij} = \sum_{k \in \Psi_i} \tilde{h}_{ik}\, c_{jk}$;

(9′) 2. $c_{ji} \leqq \sum_{k \in \Psi_i} \tilde{h}_{ik}\, c_{jk}$;

(10′) 3. $c_{ji} = \sum_{k \in \Psi_i} \tilde{h}_{ik}\, c_{jk}$ *for all j if and only if $i \in \Sigma$.*

As in §3, only the formulas corresponding to (8′) of Theorem 2′ will be stated in the corollary.

COROLLARY. *If $e_i\,\mathfrak{A}$ is subordinate to $\{e_k\,\mathfrak{A} \mid k \in \Psi_i\}$, $e_k\,\mathfrak{A}$ dominant, then $c_{ji} + \tilde{d}_{ij} = \sum_{k \in \Psi_i} c_{jk}$, $j = 1, \cdots, n$. If $e_i\,\mathfrak{A}$ is weakly subordinate to a single dominant ideal $e_k\,\mathfrak{A}$, then $c_{ji} + \tilde{d}_{ij} = \tilde{h}_{ik}\, c_{jk}$, $j = 1, \cdots, n$. If $e_k\,\mathfrak{A}$ is subordinate to a single dominant ideal $e_k\,\mathfrak{A}$, then $c_{ji} + \tilde{d}_{ij} = c_{jk}$, $j = 1, \cdots, n$.*

4. Cartan invariants of UMFR algebras

In a previous paper [7] properties of various subclasses of UMFR algebras (algebras with unique minimal faithful representations) have been studied. The definitions of some of these classes will be repeated here, and the Cartan invariants of algebras in these classes will be studied in this section.

1. $\mathfrak{A}$ is UMFR if and only if every primitive ideal (left or right) is weakly subordinate to a set of dominant ideals of $\mathfrak{A}$.

This characterization of the UMFR algebras was given by Thrall [6, Theorem 5]. It has been shown that the dominant ideals may be chosen mutually nonisomorphic [7, Theorem 1]. In the language of Theorems 1 and 1′, another characterization of the UMFR algebras can be obtained: $\mathfrak{A}$ is UMFR if and only if for every i, $\Sigma_i \subset \Sigma$ and $\Pi_i \subset \Pi$.

2. $\mathfrak{A}$ is type A if and only if every primitive ideal is subordinate to a set of dominant ideals, i.e., is subordinate to an ideal which is the direct sum of dominant ideals. The dominant ideals in the set cannot necessarily be chosen nonisomorphic.

If $\mathfrak{A}$ is a UMFR algebra, then $\mathfrak{A}$ is type A if and only if for all i and k,

$$h_{ik} \leqq f_k \quad \text{and} \quad \tilde{h}_{ik} \leqq f_k \qquad (\text{hence, also } g_{ik} \leqq f_{\pi(k)} \text{ and } \tilde{g}_{ik} \leqq f_{\sigma(k)}), \tag{11}$$

where f_k, given by (1), is the number of primitive idempotents isomorphic to e_k in any decomposition of the unit element [7, Theorem 5].

3. $\mathfrak{A}$ is type AC if and only if every primitive ideal is subordinate to a set of mutually nonisomorphic dominant ideals of $\mathfrak{A}$.

4. $\mathfrak{A}$ is type B if and only if every primitive ideal is weakly subordinate to a dominant ideal of $\mathfrak{A}$.

5. $\mathfrak{A}$ is type AB if and only if every primitive ideal of $\mathfrak{A}$ is subordinate to a set of isomorphic dominant ideals of $\mathfrak{A}$, i.e., is subordinate to an ideal of $\mathfrak{A}$ which is the direct sum of isomorphic dominant ideals of $\mathfrak{A}$.

6. $\mathfrak{A}$ is type ABC if and only if every primitive ideal is subordinate to a dominant ideal of $\mathfrak{A}$.

From the corollaries to Theorems 1 and 1′ it follows that if $\mathfrak{A}$ is a UMFR algebra, then $\mathfrak{A}$ is type B if and only if, for every i, Σ_i and Π_i are sets with one element. Thus, for an algebra of type B two functions can be defined.

Definition. If $\mathfrak{A}$ is type B, then define functions σ and π from $\{1, \cdots, n\}$ into itself as follows: $\sigma : i \to \sigma(i)$; $\pi : i \to \pi(i)$; where $\sigma(i)$ and $\pi(i)$ are the unique elements of Σ_i and Π_i, respectively.

This generalizes the functions σ and π defined by Thrall [6] for algebras of type ABC.

Theorem 3. *If $\mathfrak{A}$ is a UMFR algebra, then for every i and j*

$$c_{ij} \leqq \sum_{k=1}^{n} g_{ik}\, c_{jk}\,, \tag{12}$$

with equality holding for all j if and only if $i \in \Pi$; and

$$c_{ij} \leqq \sum_{k=1}^{n} \tilde{g}_{jk}\, c_{ki}\,, \tag{13}$$

with equality holding for all i if and only if $j \in \Sigma$.

Proof. By (9), (7), (2), and (5) it follows that $c_{ij} \leqq \sum_{k \in \Phi_i} h_{ik}\, c_{kj} = \sum_{k \in \Phi_i} g_{i\sigma(k)}\, c_{kj} = \sum_{k \in \Phi_i} g_{i\sigma(k)}\, c_{j\sigma(k)} = \sum_{k \in \Sigma_i} g_{ik}\, c_{jk}$. Since $g_{ik} = 0$ for $k \notin \Sigma_i$, the final summation can be extended, and thus $c_{ij} \leqq \sum_{k=1}^{n} g_{ik}\, c_{jk}$. The condition for equality follows immediately from (10).

The proof of (13) is dual to that of (12) and uses Theorem 2′ with suitable changes in subscripts.

COROLLARY 1. *If $\mathfrak{A}$ is type* A, *then for each i and j*

$$c_{ij} \leqq \sum\nolimits_{k\epsilon\Phi_i} f_k\, c_{kj} = \sum\nolimits_{k\epsilon\Sigma_i} f_{\pi(k)}\, c_{jk}\,, \tag{14}$$

$$c_{ij} \leqq \sum\nolimits_{k\epsilon\Psi_j} f_k\, c_{ik} = \sum\nolimits_{k\epsilon\Pi_j} f_{\sigma(k)}\, c_{ki}\,. \tag{15}$$

Proof. The first part of (14) follows from (9) and (11), and the second part follows from (12) and (11) summing only over Σ_i. The proof of (15) is dual to that of (14).

Although the summations in Corollary 1 could be extended to run from 1 to n, the resulting inequalities would in general be less accurate estimates for c_{ij} since $f_i \geqq 1$ for all i. Similarly, the estimates in Corollary 1 may be less accurate than those in Theorem 3, but (14) and (15) involve only the c_{ij}'s and the f_i's.

COROLLARY 2. *If $\mathfrak{A}$ is type* AC, *then for every i and j*

$$c_{ij} \leqq \sum\nolimits_{k\epsilon\Sigma_i} c_{jk}\,, \tag{16}$$

with equality holding for all j if and only if $i \,\epsilon\, \Pi$; and

$$c_{ij} \leqq \sum\nolimits_{k\epsilon\Pi_j} c_{ki}\,, \tag{17}$$

with equality holding for all i if and only if $j \,\epsilon\, \Sigma$.

Proof. (16) can be proved either from Theorem 3 or by use of Theorem 1 and the corollary to Theorem 2. The proof of (17) is dual.

Note that for algebras of type AC, (16) and (17) give relationships among the c_{ij} alone.

COROLLARY 3. *If $\mathfrak{A}$ is type* B, *then for every i and j*

$$c_{ij} \leqq g_{i\sigma(i)}\, c_{j\sigma(i)}\,, \tag{18}$$

with equality for all j if and only if $i \,\epsilon\, \Pi$; and

$$c_{ij} \leqq \tilde{g}_{j\pi(j)}\, c_{\pi(j)i}\,, \tag{19}$$

with equality for all i if and only if $j \,\epsilon\, \Sigma$.

Proof. From Theorem 3 and the corollary to Theorem 2, it follows that $c_{ij} \leqq g_{i\sigma(k)}\, c_{j\sigma(k)}$, where $\mathfrak{A}e_k$ is the dominant ideal to which $\mathfrak{A}e_i$ is weakly subordinate. But, if $\mathfrak{A}e_i$ is weakly subordinate to $\mathfrak{A}e_k$, $\sigma(i) = \sigma(k)$. Thus, (18) is proved. The proof of (19) is dual.

COROLLARY 4. *If $\mathfrak{A}$ is type* AB, *then for every i and j*

$$c_{ij} \leqq f_{\pi(\sigma(i))}\, c_{j\sigma(i)}\,, \tag{20}$$

$$c_{ij} \leqq f_{\sigma(\pi(j))}\, c_{\pi(j)i}\,. \tag{21}$$

Proof. Corollary 4 follows immediately from Corollary 3 and (11).

As noted concerning Corollary 1, the estimates (20) and (21) may be less accurate than (18) and (19), but only the c_{ij}'s and f_i's are involved in (20) and (21).

COROLLARY 5. *If $\mathfrak{A}$ is type* ABC, *then for every i and j*

$$c_{ij} \leqq c_{j\sigma(i)}, \tag{22}$$

with equality for all j if and only if $i \in \Pi$; and

$$c_{ij} \leqq c_{\pi(j)i}, \tag{23}$$

with equality for all i if and only if $j \in \Sigma$.

Proof. The proof is immediate from Corollaries 2 and 3.

Inequalities (22) and (23) were proved for ABC algebras by Thrall [6, Theorem 3]. Theorem 3 and its corollaries generalize these results to the UMFR algebras and its subclasses.

5. Cartan matrices of UMFR algebras

Since the various summations in Theorems 2 and 3 can be extended to run from 1 to n, it is possible to restate these results in matrix form. In addition to the Cartan matrix $C(\mathfrak{A}) = (c_{ij})$, define $D(\mathfrak{A}) = (d_{ij})$, $H(\mathfrak{A}) = (h_{ij})$ and $G(\mathfrak{A}) = (g_{ij})$.

THEOREM 4. *If $\mathfrak{A}$ is a UMFR algebra, then*

$$C(\mathfrak{A}) + D(\mathfrak{A}) = H(\mathfrak{A})C(\mathfrak{A}) = G(\mathfrak{A})C(\mathfrak{A})', \tag{24}$$

$$C(\mathfrak{A}) \leqq H(\mathfrak{A})C(\mathfrak{A}) = G(\mathfrak{A})C(\mathfrak{A})', \tag{25}$$

with equality if and only if $\mathfrak{A}$ is a quasi-Frobenius algebra.

If $\mathfrak{A}$ is quasi-Frobenius, the $G(\mathfrak{A})$ and $H(\mathfrak{A})$ are permutation matrices. The matrix $C(\mathfrak{A})'$ is the transpose of $C(\mathfrak{A})$, and the relation "$\leqq$" is defined elementwise.

The relationship between the matrices $H(\mathfrak{A})$ and $G(\mathfrak{A})$ is given by a matrix $T(\mathfrak{A}) = (t_{ij})$ where

$$t_{ij} = \begin{cases} 1 & \text{if} \quad i \in \Pi,\ s(i) = j \quad \text{or} \quad i \notin \Pi,\ i = j, \\ 0 & \text{if} \quad i \in \Pi,\ s(i) \neq j \quad \text{or} \quad i \notin \Pi,\ i \neq j. \end{cases}$$

Then the relationship is $G(\mathfrak{A})T(\mathfrak{A}) = H(\mathfrak{A})$.

A similar set of matrix relations for $C(\mathfrak{A})$ can be obtained in terms of $\tilde{D}(\mathfrak{A}) = (\tilde{d}_{ij})$, $\tilde{H}(\mathfrak{A}) = (\tilde{h}_{ij})$ and $\tilde{G}(\mathfrak{A}) = (\tilde{g}_{ij})$.

THEOREM 4′. *If $\mathfrak{A}$ is a UMFR algebra, then*

$$C(\mathfrak{A}) + \tilde{D}(\mathfrak{A})' = C(\mathfrak{A})\tilde{H}(\mathfrak{A})' = C(\mathfrak{A})'\tilde{G}(\mathfrak{A})', \tag{24′}$$

$$C(\mathfrak{A}) \leqq C(\mathfrak{A})\tilde{H}(\mathfrak{A})' = C(\mathfrak{A})'\tilde{G}(\mathfrak{A})', \tag{25′}$$

with equality if and only if $\mathfrak{A}$ is a quasi-Frobenius algebra.

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University of North Carolina
Chapel Hill, North Carolina