THE GENERALIZED BACKWARD KOLMOGOROV EQUATION IN ABSTRACT SPACE

BY

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In this paper we consider questions which arise in the study of Markov processes with stationary transitions where the random variables assume values in an abstract space. By $(\mathfrak{F}, X)$ we denote the abstract space X together with a Borel field of subsets $\mathfrak{F}$ which contain X and all one-point sets. We discuss here some properties of nonnegative "transition" functions $P_t(x, E)$, defined for $t \geqq 0$, $x \in X$, and $E \in \mathfrak{F}$ which describe the probability of a Markov process being in state E at time $t + \tau$ conditioned by the process being in state x at time τ (stationarity implies that this conditional probability is independent of τ).

The transition functions may then be assumed to satisfy the following conditions for any $x \in X$, $E \in \mathfrak{F}$, and $t, s \geqq 0$:

I $P_t(x, \cdot)$ is a probability measure on $\mathfrak{F}$,

II $P_t(\cdot, E)$ is measurable $\mathfrak{F}$,

III $$P_{t+s}(x, E) = \int_X P_t(\cdot, E)\, dP_s(x, \cdot).$$

We shall also assume that for some $x = x^*$,

IV $\lim_{t \downarrow 0} [1 - P_t(x^*, \{x^*\})]/t < \infty$.

Probabilistic and analytic implications of IV have been discussed by Doob [3] (assuming X to be a linear Borel set) and by Kendall [4], and by Chung, Doob, Lévy, and others for the chain case, where it is assumed that X is countable. Doob's arguments with X linear can be generalized to the abstract-space case, and they essentially contain our Theorem 2 (see [3; p. 270]). The countability of X is an essential restriction however, and it is the purpose of this paper to rephrase certain of the known analytical results for that case and to prove them for the abstract-space case.

Throughout this paper it will be necessary for us to assume IV for only one x^*. We shall, however, assume that the following condition, weaker than IV, is satisfied:

IV′ $\lim_{t \downarrow 0} P_t(x, \{x\}) = 1 = P_0(x, \{x\})$.

Kendall [4] has shown that IV′ is sufficient to insure the continuity of $P_\cdot(x, E)$ for each x in X and E in $\mathfrak{F}$. (I, II, and III are not sufficient; see Doob [2].)

Let us now state several known results for the case X countable in such a

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way that there is an abstract-space analogue. We assume X to be the space of positive integers and adopt the conventional notation $P_t(i, \{j\}) = p_{ij}(t)$ $(i, j = 1, 2, \cdots)$; $\mathfrak{F}$ is the Borel field consisting of all collections of integers, and if J is a collection of integers, then $P_t(i, J) = p_{iJ}(t) = \sum_{j \in J} p_{ij}(t)$. Then if I, II, III, and IV′ hold, and if IV holds for some $x^* = i$, we have that the t-derivative $p'_{ij}(t)$ exists for $j = 1, 2, \cdots$, $t \geqq 0$ (see Doob [2] and Austin [1]). It is easy to extend the results of [1] to show that $p'_{iJ}(t)$ exists for all $J \in \mathfrak{F}$ if $t > 0$; however, $p'_{iJ}(t) \mid_{t=0}$ does not in general exist. In fact the existence of $p'_{iJ}(0)$ for each $J \in \mathfrak{F}$ implies that $p'_{iJ}(0)$ is, for fixed i, a signed measure on $\mathfrak{F}$, so that $\sum_j p'_{ij}(0) = 0$. Such processes are called conservative and are discussed in detail by Reuter in [5], where examples of nonconservative processes are given.

There are obvious abstract-case analogues to the results stated in the last paragraph, and we shall establish those analogues. The countable-case arguments are not applicable; in fact to carry out the generalizations we have found it necessary to add an additional assumption, one which causes only esthetic discomfort:

V $P_t(x, \{x\})$ is for each t an $\mathfrak{F}$-measurable function of $x \in X$.

This condition is discussed by Kendall in [4]; in particular Kendall shows that V is satisfied if $\mathfrak{F}$ contains all the open sets of a Hausdorff topology satisfying the second axiom of countability. Kendall also found V necessary in generalizing differentiation results proved by Kolmogorov for the chain case. Kendall's main result is the following:

If $\lim_{t \downarrow 0} P_t(x, \{x\}) = 1$ uniformly on a set $E \in \mathfrak{F}$, and if $x^* \notin E$, then $\lim_{t \downarrow 0} P_t(x^*, E)/t$ exists and is finite.

Kendall also showed that, under IV′,

$$q(x^*) = \lim_{t \downarrow 0} [1 - P_t(x^*, \{x^*\})]/t \text{ exists.}$$

We first state a lemma without proof which is an extraction of that part of the existence proof for derivatives, [1], in the countable-space case which does generalize readily to the abstract-space case.

LEMMA. *If* I, II, III, *and* IV′ *hold, and if* IV *holds for some $x = x^*$, then $P_t(x^*, E)$ is, for each $E \in \mathfrak{F}$, a Lipschitzian function of t with Lipschitz constant $q(x^*)$.*

We now proceed to our main result. Hereafter we assume I, II, III, IV′, and V, and that IV holds for the fixed $x = x^*$.

THEOREM 1. *The derivative $P'_t(x^*, E)$ exists for $t > 0$ and $E \in \mathfrak{F}$; $P'_t(x^*, \cdot)$ is a uniformly bounded signed measure on $\mathfrak{F}$ which satisfies*

$$(1) \qquad P'_{t+s}(x^*, E) = \int_X P_s(\cdot, E)\, dP'_t(x^*, \cdot).$$

Proof. Let us denote by $P_{t,h}(x^*, E)$ the difference quotient

$$[P_{t+h}(x^*, E) - P_t(x^*, E)]/h$$

for $h \neq 0$, $t \geqq 0$, $t + h \geqq 0$. By I and the lemma, $P_{t,h}(x^*, \cdot)$ is a uniformly bounded signed measure on $\mathfrak{F}$. By IV′ the Hahn decomposition of $P_{0,h}(x^*, \cdot)$ is effected by the sets $\{x^*\}$ and $X - \{x^*\}$. Using III and the lemma we find that, for $t \geqq 0$, $h > 0$, $E \in \mathfrak{F}$,

$$\begin{aligned} P_{t,h}(x^*, E) &= \int_X P_t(\cdot, E)\, dP_{0,h}(x^*, \cdot) \geqq \int_{\{x^*\}} P_t(\cdot, E)\, dP_{0,h}(x^*, \cdot) \\ &= P_t(x^*, E) P_{0,h}(x^*, \{x^*\}) \geqq -q(x^*) P_t(x^*, E). \end{aligned} \tag{2}$$

Now consider the auxiliary function

$$\tilde{P}_t(x^*, E) = P_t(x^*, E) + q(x^*) \int_0^t P_s(x^*, E)\, ds. \tag{3}$$

We observe that for each set $E \in \mathfrak{F}$, $\tilde{P}'_t(x^*, E)$ exists except on a set of (Lebesgue) measure 0; this follows from the lemma and the fact that $\tilde{P}'_t(x^*, E)$ exists whenever $P'_t(x^*, E)$ exists. In general there is ambiguity in the definition of $\tilde{P}'_t(x^*, E)$; however we shall use only Lebesgue integrals of this function, and there the ambiguity disappears. In particular the function

$$\tilde{P}_{t,h}(x^*, E) = \frac{1}{h} \int_0^h \tilde{P}'_{t+t_1}(x^*, E)\, dt_1$$

on $t \geqq 0$, $h > 0$ is defined unambiguously and is nonnegative; that

$$P'_t(x^*, E) \geqq 0$$

wherever defined follows from (2). Furthermore, one readily observes that $\tilde{P}_{t,h}(x^*, \cdot)$ is a bounded measure on $\mathfrak{F}$ and that

$$\begin{aligned} &\int_X P_s(\cdot, E)\, d\tilde{P}_{t,h}(x^*, \cdot) \\ &\quad = P_{t+s,h}(x^*, E) + \frac{q(x^*)}{h} \int_X P_s(\cdot, E)\, d \int_0^h P_{t+t_1}(x^*, \cdot)\, dt_1 \\ &\quad = P_{t+s,h}(x^*, E) + \frac{q(x^*)}{h} \int_0^h dt_1 \int_X P_s(\cdot, E) P_{t+t_1}(x^*, \cdot) \\ &\quad = P_{t+s,h}(x^*, E) + \frac{q(x^*)}{h} \int_0^h P_{t+s+t_1}(x^*, E)\, dt_1 = \tilde{P}_{t+s,h}(x^*, E); \end{aligned}$$

the interchange in order of integration is easily justified by first considering characteristic functions of sets in $\mathfrak{F}$.

We fix $\bar{h} > 0$ and $\bar{t} > 0$ and introduce a uniformizing measure as follows:

$$P(x^*, \cdot) = \int_0^{\bar{t}} \int_0^{\bar{h}} [\tilde{P}_{t,h}(x^*, \cdot) + \tilde{P}'_t(x^*, \cdot)]\, dt\, dh; \tag{4}$$

clearly $P(x^*, \cdot)$ is a bounded measure on $\mathfrak{F}$. Employing V we see that if T is any dense set on $(0, \infty)$ and $\delta > 0$, then

$$f_\delta(x) = \text{glb}\,[P_t(x, \{x\});\, t \in T,\, t < \delta]$$

is $\mathfrak{F}$-measurable; and, in view of the continuity of $P_t(x, \{x\})$ as a function of t, $\lim_{\delta \downarrow 0} f_\delta(x) = 1$ uniformly on a set E in $\mathfrak{F}$ implies that

$$\lim_{t \downarrow 0} P_t(x, \{x\}) = 1$$

uniformly on E. This observation together with the lemma enables us to apply the Egorov theorem repeatedly in order to obtain a monotone decreasing sequence of sets $G_n \in \mathfrak{F}$, $n = 1, 2, \cdots$, so that

(a) $P(x^*, \cap G_n) = 0$,

(b) $\lim_{t \downarrow 0} P_t(x, \{x\}) = 1$ uniformly for $x \notin G_n$ $(n = 1, 2, \cdots)$.

In view of our definition of the uniformizing measure we have that

$$\lim_n \tilde{P}'_t(x^*, G_n) = 0$$

in measure on $0 \leqq t \leqq \bar{t}$; hence we may find a set $T_1 \in [0, \bar{t}]$ which contains almost all points of $[0, \bar{t}]$ and a subsequence n_i $(i = 1, 2, \cdots)$ such that for $t \in T_1$

(i) $\tilde{P}'_t(x^*, G_{n_i})$ exists for all n_i,

(ii) $\tilde{P}'_t(x^*, \cap G_{n_i})$ exists, and

(iii) $\lim_i \tilde{P}'_t(x^*, G_{n_i}) = \tilde{P}'_t(x^*, \cap G_{n_i})$.

Taking note again of the definition of $P(x^*, \cdot)$, we apply Fubini's theorem to find a set $T_2 \subset T_1$ such that if $t \in T_2$ we have for almost all h in $[0, \bar{h}]$

$$\tilde{P}_{t,h}(x^*, \cap G_{n_i}) = 0. \tag{5}$$

But for fixed t, $\tilde{P}_{t,h}(x^*, \cap G_{n_i})$ is a continuous function of h, so that (5) holds for all h on $(0, \bar{h}]$; and in view of (ii) the formula (5) is valid on the compact set $I = [h;\, 0 \leqq h \leqq \bar{h}]$ where, of course, $\tilde{P}_{t,0}(x^*, \cap G_{n_i}) = \tilde{P}'_t(x^*, \cap G_{n_i})$. By (i) each of the functions $\tilde{P}_{t,h}(x^*, G_{n_i})$ is continuous on I, and by (iii)

$$\lim_i \tilde{P}_{t,h}(x^*, G_{n_i}) = \tilde{P}_{t,h}(x^*, \cap G_{n_i}) \qquad \text{for } h \in I. \tag{6}$$

All of the conditions of Dini's monotone convergence theorem are satisfied, and we may apply that theorem to conclude that the convergence in (6) is uniform on I.

We fix a point t^* in T_2; then by (b) and the uniform convergence of (6), we may, for given $\varepsilon > 0$, find a positive integer N and a number $\bar{s} > 0$ such that for $\bar{s} > s$, $E \in \mathfrak{F}$,

$$\begin{aligned} &P_s(x, E) > 1 - \varepsilon && \text{for} \quad x \in E, \quad x \notin G_N, \\ &P_s(x, E) < \varepsilon && \text{for} \quad x \notin E \cup G_N, \\ &\tilde{P}_{t^*,h}(x^*, G_{n_i}) < \varepsilon && \text{for} \quad h \in I, \quad n_i \geqq N. \end{aligned} \tag{7}$$

By (7) we have, for $0 < h < \bar{h}$, $n \geqq N$,

$$\tilde{P}_{t*+s,h}(x^*, E) = \int_X \{P_s(\cdot, E)\, d\tilde{P}_{t*,h}(x^*, \cdot)\}$$
(8)
$$\leqq \int_{G_n} \{\quad\} + \int_{X-(E\cup G_n)} \{\quad\} + \int_E \{\quad\} \leqq 2\varepsilon + \tilde{P}_{t*,h}(x^*, E),$$

and

$$\tilde{P}_{t*+s,h}(x^*, E) \geqq \int_{E-G_n} P_s(\cdot, E)\, d\tilde{P}_{t*,h}(x^*, \cdot)$$
(9)
$$\geqq (1-\varepsilon)\tilde{P}_{t*,h}(x^*, E - G_n) = (1-\varepsilon)\tilde{P}_{t*,h}(x^*, E)$$
$$- (1-\varepsilon)\tilde{P}_{t*,h}(x^*, G_n) \geqq (1-\varepsilon)P_{t*,h}(x^*, E) - \varepsilon.$$

But $\tilde{P}_{t*,h}(x^*, E)$ is a difference quotient for the function $\tilde{P}_t(x^*, E)$ defined in (3); hence by the theorem of Dini which states that the difference quotient and the derivates of a continuous function have the same bounds, we conclude that $\tilde{P}_{t,h}(x^*, E)$ has a right-hand derivative, $\tilde{P}_{t*}^{(r)}(x^*, E)$, at $t = t^*$. Furthermore, since the estimates (7) are independent of E, and $\tilde{P}_{t,h}(x^*, \cdot)$ has a uniform bound over sets $E \in \mathfrak{F}$, we conclude from (8) and (9) that

(10) $$\lim_{h \downarrow 0} \tilde{P}_{t*,h}(x^*, E) = \tilde{P}_{t*}^{(r)}(x^*, E)$$

uniformly with respect to $E \in \mathfrak{F}$, and that, for any $\delta > 0$, there exists an s_δ not dependent on E such that if $0 < s < s_\delta$ and the right-hand derivative $\tilde{P}_{t*+s}^{(r)}(x^*, E)$ exists, then

(11) $$|\tilde{P}_{t*}^{(r)}(x^*, E) - \tilde{P}_{t*+s}^{(r)}(x^*, E)| < \delta.$$

Now for any $t > t^*$ it follows easily, by applying (10) to

$$\lim_{h \downarrow 0} \tilde{P}_{t,h}(x^*, E) = \lim_{h \downarrow 0} \int_X P_{t-t*}(\cdot, E)\, d\tilde{P}_{t*,h}(x^*, \cdot),$$

that the right derivative of $\tilde{P}_t(x^*, E)$ exists for $t > t^*$ and (11) remains valid with t^* replaced by t. Thus the right derivative $\tilde{P}_t^{(r)}(x^*, E)$ exists and is uniformly right continuous on $t \geqq t^*$. Hence the right derivative is continuous on $t > t^*$, and we may apply the Dini derivate theorem for the second time to conclude that the derivative exists and is continuous on $t > 0$.

It is now immediate from (3) that $P_t(x^*, E)$ has a continuous derivative for $t > 0$, $E \in \mathfrak{F}$, and (1) follows on applying the Helly-Bray theorem to

$$P_{t+s,h}(x^*, E) = \lim_{h \downarrow 0} \int_X P_s(\cdot, E)\, dP_{t,h}(x^*, \cdot).$$

The abstract version of the backward Kolmogorov equation now follows immediately from the existence theorem.

THEOREM 2. *If $P_t'(x^*, E)$ exists at $t = 0$ for all $E \in \mathfrak{F}$, then*

$$P_t'(x^*, E) = \int_X P_t(\cdot, E)\, dP_0'(x^*, \cdot) \quad \textit{for all} \quad t \geqq 0.$$

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