ALGEBRAIC LIE ALGEBRAS AND REPRESENTATIVE FUNCTIONS

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1. Introduction

We are concerned with the analogues for Lie algebras of the problems centering around the Tannaka duality theorem, such as those treated in [4] and [6]. The analogue of the Tannaka theorem for semisimple Lie algebras was established by Harish-Chandra in 1950. More recently, P. Cartier has sketched (without proofs) a general duality theory for algebraic groups and Lie algebras which absorbs Harish-Chandra's result. The essence of most of the results of this type is that they establish connections between abstract Lie groups (or abstract Lie algebras) and their representations on the one hand, and algebraic groups (or algebraic Lie algebras) and their rational representations on the other. Just as, in the case of groups, the vehicle for these connections is the algebra of representative functions on the group, so, in the case of Lie algebras, it is the algebra of representative functions *on the universal enveloping algebra* of the Lie algebra. Although this has been indicated by Cartier in [1], no systematic development of the relevant techniques and results is available, and it is our purpose here to carry out such a development.

Section 2 gives the requisite mechanism of representative functions. In Section 3, this is applied to obtain a direct construction for the algebraic group and Lie algebra hulls of a linear Lie algebra. The main purpose of Section 4 is to give a characterization (Theorem 3) of the differentials of the rational representations of an algebraic group, involving only the universal enveloping algebra of the Lie algebra. In order to make this directly intelligible, we have included a sketch of the relevant portion of Chevalley's theory of algebraic groups, casting it into a form especially adapted to our purpose. Section 5 reviews the auxiliary results for semisimple Lie algebras. Sections 6 and 7 give the general results concerning the structure of the algebra of representative functions.

It should be pointed out that most of the results presented here constitute a direct outgrowth of previous work done in collaboration with G. D. Mostow ([4], [5], [6]). I wish to thank him here for permitting me to make such free use of his contributing ideas.

2. Representative functions

Let L be a finite-dimensional Lie algebra over a field F, and let U denote the universal enveloping algebra of L. We consider finite-dimensional representations of L or, which amounts to the same thing, finite-dimensional

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representations of the associative algebra U mapping the identity element of U onto the identity transformation of the representation space. Thus, if M is the representation space of such a representation, M is a unitary left U-module of finite dimension over F. Let E denote the algebra of all linear endomorphisms of M, and let $E' = \mathrm{Hom}_F(E, F)$ denote the dual space of E. If t is any element of E' and if ρ is the given representation $U \to E$, the composite map $t \circ \rho \in U' = \mathrm{Hom}_F(U, F)$ is called a *representative function* on U, associated with the representation ρ. Evidently, these functions make up a finite-dimensional vector space $\mathbf{R}(\rho)$. The kernel, I, of ρ in U is a two-sided ideal of finite codimension in U, i.e., U/I is finite-dimensional, and the functions belonging to $\mathbf{R}(\rho)$ vanish on I.

Conversely, every linear function on U that vanishes on some two-sided ideal of finite codimension in U is a representative function. In fact, U' has the structure of a left U-module, the operations being (with f in U' and x, y in U) $f \to x \cdot f$, where $(x \cdot f)(y) = f(yx)$. If f vanishes on an ideal of finite codimension, then f generates a finite-dimensional U-submodule of U', and f is associated with the corresponding representation ρ of $U: f = t \circ \rho$, where $t(e) = e(f)(1)$, for every linear endomorphism e of the U-module generated by f.

Let d denote the unitary homomorphism of U into the tensor product $U \otimes U$ that is determined by the condition that, for every $x \in L$, $d(x) = x \otimes 1 + 1 \otimes x$. If A and B are unitary U-modules, the tensor product $A \otimes B$ is given the structure of a unitary $U \otimes U$-module such that, for $u_i \in U$, $a \in A$ and $b \in B$,

$$(u_1 \otimes u_2) \cdot (a \otimes b) = u_1 \cdot a \otimes u_2 \cdot b.$$

Composing this with d, we obtain the unitary U-module structure of $A \otimes B$ that corresponds to the usual tensor product of the representations of L in A and in B:

$$u \cdot (a \otimes b) = d(u) \cdot (a \otimes b).$$

If f and g are any two elements of U', we denote by $f \otimes g$ the linear function on $U \otimes U$ such that $(f \otimes g)(u_1 \otimes u_2) = f(u_1)g(u_2)$. Then we define the product fg as an element of U' by

$$(fg)(u) = (f \otimes g)(d(u)).$$

This is the multiplication pointed out by P. Cartier in [1]. Clearly, U' is an associative and commutative algebra with this multiplication. If f and g are representative functions associated with representations ρ and σ, respectively, then fg is a representative function associated with the tensor product $\rho \otimes \sigma$. In fact, if $f = s \circ \rho$ and $g = t \circ \sigma$, we have

$$fg = (s \otimes t) \circ (\rho \otimes \sigma).$$

Thus the representative functions on U constitute a subalgebra of U'. This can also be seen directly by noting that if f vanishes on the ideal I and g

on the ideal J, then fg vanishes on the ideal $K = d^{-1}(I \otimes U + U \otimes J)$, and that d yields a monomorphism of U/K into $U/I \otimes U/J$, so that U/K is finite-dimensional whenever U/I and U/J are finite-dimensional.

For every nonnegative integer p, denote by U_p the subspace of U consisting of the elements that can be written as (noncommutative) polynomials of degree not exceeding p in the elements of L. Each U_p is finite-dimensional, U is the union of all the U_p, and $U_p U_q \subset U_{p+q}$. The graded algebra that is associated with U by this filtration is canonically isomorphic with the symmetric algebra built over L.

Now let $g_1, \cdots, g_n$ be elements of U' such that each g_i vanishes on U_0 ($=F$) and that the restrictions of the g_i to L constitute a basis for the dual space L' of L. Then every monomial of degree greater than p in the g_i vanishes on U_p. On the other hand, there is a basis $x_1, \cdots, x_n$ for L such that $g_i(x_j) = \delta_{ij}$. The elements of U_p can be written uniquely as linear combinations of the ordered monomials $x_1^{e_1} \cdots x_n^{e_n}$, with $e_1 + \cdots + e_n \leqq p$. Now let $u_1, \cdots, u_n$ be nonnegative integers such that $u_1 + \cdots + u_n \geqq e_1 + \cdots + e_n$. Then it is immediately verified that

$$(g_1^{u_1} \cdots g_n^{u_n})(x_1^{e_1} \cdots x_n^{e_n}) = \delta_{u_1 e_1} \cdots \delta_{u_n e_n} e_1! \cdots e_n!.$$

Hence we see that, if F is of characteristic 0, every element of U' coincides on U_p with a unique polynomial of degree $\leqq p$ in the g_i. In particular, this shows that, *if F is of characteristic* 0, *U' is an integral domain.*

We have already defined the structure of a left U-module on U': the operations are the left translations $f \to x \cdot f$, where $(x \cdot f)(y) = f(yx)$. We define the right translations similarly: $f \to f \cdot x$, where $(f \cdot x)(y) = f(xy)$. Thus U' has been given the structure of a double U-module. If $x \in U_0 = F$, then both the right and the left translation by x coincide with the scalar multiplication by x. If $x \in L$, the translations by x are derivations of U'. The linear functions on U that vanish on LU are called the *constants.* The multiplication by a constant $f \in U'$ coincides with the scalar multiplication by the value $f(1)$; we identify the constants with their values at $1 \in U$. The right and left translations by an element of L annihilate the constants. If ρ is a representation of L, then $\mathbf{R}(\rho)$ is evidently stable under the right and left translations. In particular, the algebra $\mathbf{R}$ of all representative functions on U is stable under the right and left translations.

LEMMA 1. *Let S be a subspace of $\mathbf{R}$ that is stable under the right translations. Let T be a subspace of S that is stable under the left translations. Then every linear map of S into U' that commutes with the right translations maps T into itself.*

Proof. Let $t \in T$, and let $[t]$ denote the space spanned by the left translates of t. Since t is a representative function, $[t]$ is finite-dimensional. Hence we can find a basis $t_1, \cdots, t_n$ of $[t]$ and elements $u_1, \cdots, u_n$ in U such that $t_i(u_j) = \delta_{ij}$. Then, for every $u \in U$,

$$u \cdot t = \sum_{i=1}^{n} t(u_i u) t_i.$$

Applying this to an element $v \in U$, we obtain

$$t(vu) = \sum_{i=1}^{n} t(u_i u) t_i(v),$$

whence

$$t \cdot v = \sum_{i=1}^{n} t_i(v) t \cdot u_i .$$

Now let e be any linear map of S into U' that commutes with the right translations. Then we have

$$e(t) \cdot v = \sum_{i=1}^{n} t_i(v) e(t) \cdot u_i .$$

If we evaluate this at the identity element 1 of U, we find that

$$e(t) = \sum_{i=1}^{n} e(t)(u_i) t_i \in [t].$$

This completes the proof of Lemma 1.

Let Q be any subalgebra of $\mathbf{R}$ that contains the constants and is stable under the right and left translations. We shall say that a derivation of Q is a *proper derivation* if it annihilates the constants and commutes with the right translations. By a *differentiation* of Q we shall mean an F-linear map δ of Q into F such that

$$\delta(fg) = \delta(f)g(1) + f(1)\delta(g),$$

for all f and g in Q. If D is a proper derivation of Q, we define a differentiation D' of Q by putting $D'(f) = D(f)(1)$, for every $f \in Q$.

PROPOSITION 1. *Let Q be a subalgebra of $\mathbf{R}$ that contains the constants and is stable under the right and left translations. Then the map $D \rightarrow D'$ is a linear isomorphism of the space of all proper derivations of Q onto the space of all differentiations of Q.*

Proof. Let δ be any differentiation of Q. We define a linear map δ^* of Q into U' by setting $\delta^*(f)(u) = \delta(f \cdot u)$, for every $f \in Q$ and every $u \in U$. It is clear from the definition that δ^* commutes with the right translations. Hence we conclude from Lemma 1 that $\delta^*(Q) \subset Q$.

Let φ denote the multiplication $U' \otimes U' \rightarrow U'$. Let us regard $U' \otimes U'$ as a right module for $U \otimes U$ such that, for f and g in U' and a and b in U,

$$(f \otimes g) \cdot (a \otimes b) = f \cdot a \otimes g \cdot b.$$

Then, for every $u \in U$, we have

$$(fg) \cdot u = \varphi((f \otimes g) \cdot d(u)).$$

If f and g are in Q, we have

$$\delta((f \cdot a)(g \cdot b)) = \delta(f \cdot a)g(b) + f(a)\delta(g \cdot b)$$
$$= (\delta^*(f) \otimes g + f \otimes \delta^*(g))(a \otimes b).$$

This gives

$$(\delta^*(f) \otimes g + f \otimes \delta^*(g))(d(u)) = \delta(\varphi((f \otimes g) \cdot d(u))) = \delta((fg) \cdot u),$$

i.e., $(\delta^*(f)g + f\delta^*(g))(u) = \delta^*(fg)(u)$. Thus we have shown that δ^* is a derivation of Q. It is clear that δ^* is therefore a proper derivation of Q, and one verifies immediately that $(\delta^*)' = \delta$. If D is any proper derivation of Q, one sees at once that $(D')^* = D$. Hence Proposition 1 is proved.

3. Algebraic Lie algebras

From now on, *we shall assume throughout that our base field F is of characteristic* 0. Let ρ be a representation of the Lie algebra L in a finite-dimensional vector space M over F. We shall always use the same letter ρ to denote the corresponding representation of the universal enveloping algebra U of L. Let E be the algebra of all linear endomorphisms of M, and let P be the algebra of all polynomial functions on E. Let S_ρ be the subalgebra of $\mathbf{R}$ that is generated by the constants and the elements of $\mathbf{R}(\rho)$. The map $t \to t \circ \rho$ is a linear map of E' onto $\mathbf{R}(\rho)$. It can be extended uniquely to a unitary algebra epimorphism of P onto S_ρ. Let Q denote the kernel of this epimorphism. We shall call Q *the ideal* (of polynomial functions) *associated with* ρ. Since S_ρ is an integral domain, Q is a prime ideal.

THEOREM 1. *Let ρ be a finite-dimensional representation of the Lie algebra L, with representation space M. Let Q be the ideal associated with ρ. Then the set G of all automorphisms of M that are zeros of Q is the smallest algebraic group whose Lie algebra contains $\rho(L)$, and Q is the ideal of all polynomial functions vanishing on G.*

Proof. If f is any function on E, and $e \in E$, we define the left and right e-translates $e\cdot f$ and $f\cdot e$ of f by $(e\cdot f)(x) = f(xe)$ and $(f\cdot e)(x) = f(ex)$. Let H be the group of all linear automorphisms h of M for which $Q\cdot h = Q$. We indicate the canonical epimorphism $P \to S_\rho$ by $p \to p'$. If $p \in P$, then $p'(1)$ coincides with the value taken by p at the identity automorphism of M. Hence every element of Q vanishes at the identity automorphism of M. It follows that every element of H is a zero of the ideal Q. We shall show that, conversely, every automorphism of M that is a zero of Q belongs to H.

For every $e \in E$, let D_e denote the derivation of P that annihilates the constants and coincides with the left translation by e on E'. Clearly, D_e commutes with the right translations on P. We regard P as a left U-module such that the linear endomorphism of P corresponding to an element x of L is $D_{\rho(x)}$. With this U-module structure of P, the epimorphism $p \to p'$ of P onto S_ρ is evidently a U-epimorphism, i.e., if $u \in U$ and $p \in P$, and if $u(p)$ denotes the transform of p by u, $(u(p))' = u\cdot p'$. Hence Q is a U-submodule of P.

Now let k be any linear automorphism of M that is a zero of Q. Let $q \in Q$ and $u \in U$. Then we have

$$\begin{aligned}(q\cdot k)'(u) = (u\cdot(q\cdot k)')(1) &= (u(q\cdot k))'(1) \\ &= (u(q)\cdot k)'(1) = u(q)(k) = 0.\end{aligned}$$

Hence $q\cdot k \in Q$, and so $Q\cdot k \subset Q$. Since k is an automorphism, and since the right translation by k preserves degrees in P, it follows that $Q \cdot k = Q$, i.e., that $k \in H$. Thus we have shown that H is precisely the set of all automorphisms of M that are zeros of Q. Hence H is an algebraic group, and $H = G$.

Now let T be any algebraic group of automorphisms of M whose Lie algebra contains $\rho(L)$, and let A be the ideal of all polynomial functions vanishing on T. Since the Lie algebra of T contains $\rho(L)$, A is a U-submodule of P. Hence, for every $u \in U$ and every $p \in A$, $u(p)$ vanishes at the identity automorphism of M. Hence we have $u(p)'(1) = 0$, i.e., $p'(u) = 0$. We conclude that $A \subset Q$, whence $G \subset T$.

Let x be any element of $\rho(L)$, and let G_x be the intersection of all algebraic groups of automorphisms of M whose Lie algebras contain x. By [2, Theorem 10, p. 165],[1] G_x is an irreducible algebraic group whose Lie algebra contains x, and $\exp(tx)$ is a generic point of G_x over the field of the formal power series in t with coefficients in F. It is easily seen that, for every $p \in P$,

$$\exp(tx) \cdot p = \exp(tD_x)(p).$$

If $p \in Q$, the right-hand side is evidently a power series in t with coefficients in Q. Hence it vanishes at the identity automorphism, so that

$$p(\exp(tx)) = 0.$$

Thus $\exp(tx)$ is a generalized zero of Q. Since every element of G_x is a specialization of $\exp(tx)$, we conclude that $G_x \subset G$. It follows, as in the proof of Corollary 1 below, that the Lie algebra of G_x, and in particular x, is contained in the Lie algebra of G. Thus we have shown that the Lie algebra of G contains $\rho(L)$. As we have seen above, this implies that the ideal of all polynomial functions vanishing on G is contained in Q, and thus coincides with Q. This completes the proof of Theorem 1.

Corollary 1. *The Lie algebra K of all $e \in E$ for which $D_e(Q) \subset Q$ is the smallest algebraic Lie algebra containing $\rho(L)$. The ideal associated with the identity representation of K coincides with Q.*

Proof. K is the Lie algebra of G and thus is algebraic. We know also that $\rho(L) \subset K$. Now let C be any algebraic Lie algebra contained in E and containing $\rho(L)$. Then C is the Lie algebra of an algebraic group H of linear automorphisms of M, and $G \subset H$. Let $x \in K$, and let p be a polynomial function vanishing on H. For every $h \in H$, $p \cdot h$ is a polynomial function vanishing on H, and a fortiori on G. Hence $D_x(p \cdot h)$ vanishes on G, and in particular at the identity automorphism of M. But $D_x(p \cdot h) = D_x(p) \cdot h$, so that we conclude that $D_x(p)(h) = 0$. Thus D_x maps the ideal of polynomial functions vanishing on H into itself, so that $x \in C$. Thus $K \subset C$, and the first part of the corollary is proved.

[1] A direct proof of the next assertion, proceeding from the point of view adopted here, will appear in Proc. Amer. Math. Soc.

Now let J be the ideal associated with the identity representation of K. By Theorem 1, J is the ideal of all polynomial functions vanishing on the smallest algebraic group H whose Lie algebra contains K. Since $\rho(L) \subset K$, we have $G \subset H$, whence $J \subset Q$. Now let $q \in Q$. Then $v(q) \in Q$, for every element v of the universal enveloping algebra of K. In particular, $v(q)$ vanishes at the identity automorphism of M. As we have seen in proving Theorem 1, this implies that $q \in J$. Thus we have $Q \subset J$, and so $Q = J$. This completes the proof of Corollary 1.

We have defined the structure of a left U-module on P, using the derivations D_e, such that the epimorphism $P \to S_\rho$ is a module epimorphism for the left U-module structures. Replacing left translations by right translations throughout, we define the structure of a right U-module on P such that the epimorphism $P \to S_\rho$ is a module epimorphism also for the right U-module structures. Clearly, the right U-operations on P commute with the left U-operations. Moreover, all the derivations D_e, as e ranges over E, commute with the right U-operations. If $D_e(Q) \subset Q$, then D_e induces a derivation D_e' on S_ρ and, since D_e commutes with the right U-operations on P, D_e' commutes with the right translations on S_ρ. Thus D_e' is a proper derivation of S_ρ. The map $e \to D_e'$ is therefore a homomorphism of the Lie algebra K of Corollary 1 into the Lie algebra of the proper derivations of S_ρ. If $D_e' = 0$, then $D_e(P) \subset Q$. In particular, for every $p \in E'$, $D_e(p)$ vanishes at the identity automorphism of M, which means that $p(e) = 0$. Hence $e = 0$, and we have shown that the map $e \to D_e'$ is a monomorphism.

Now let D be any proper derivation of S_ρ. Then the map $p \to D(p')(1)$ is a linear function on E', so that there is an element $e \in E$ such that

$$D(p')(1) = p(e),$$

for every $p \in E'$. It is seen at once that then $D_e' = D$. Thus the map $e \to D_e'$ is an isomorphism of K onto the Lie algebra of all proper derivations of S_ρ. Now we know that $\rho(L)$ is algebraic if and only if it coincides with K. Hence we have the following result.

THEOREM 2. *The Lie algebra $\rho(L)$ is algebraic if and only if every proper derivation of S_ρ is the left translation by an element of L, or, equivalently (by Proposition 1), if and only if, for every differentiation δ of S_ρ, there is an $x \in L$ such that $\delta(f) = f(x)$, for every $f \in S_\rho$.*

4. Rational representations

Let M, E, P have the same meanings as before, and let G be an irreducible algebraic group of linear automorphisms of M. Let Q be the ideal of all polynomial functions vanishing on G. Let M_1 be another finite-dimensional vector space over F, E_1 the algebra of all linear endomorphisms of M_1, P_1 the algebra of all polynomial functions on E_1. Let φ be a homomorphism of G into the group of the linear automorphisms of M_1. Such a homomorphism φ is called a *rational representation* of G if, for every $f \in E_1'$, the composite $f \circ \varphi$ is a rational function on G (necessarily defined at every point of G).

Now let L denote the Lie algebra of G. The *differential* φ^* of the rational representation φ is a homomorphism of L into E_1, which is defined as follows. Let $x \in L$, and consider the derivation of P/Q that is induced by D_x. This extends uniquely to a derivation d_x of the field of quotients of P/Q, i.e., of the field of the rational functions on G. Let T be the subring of this field consisting of the functions that are defined at the identity element 1 of G. The elements of T are representable as quotients a/b, where a and b are elements of P and $b(1) \neq 0$. Hence we see at once that d_x maps T into itself, because

$$d_x(a/b) = (D_x(a)b - aD_x(b))/b^2.$$

Hence we can define a differentiation d_x' of T by setting

$$d_x'(f) = d_x(f)(1), \qquad \text{for every} \quad f \in T.$$

If $g \in E_1'$, then $g \circ \varphi \in T$, and the map $g \to d_x'(g \circ \varphi)$ is evidently linear. Hence there is one and only one element $x_1 \in E_1$ such that $g(x_1) = d_x'(g \circ \varphi)$, for every $g \in E_1'$. We put $\varphi^*(x) = x_1$. Evidently, φ^* is a linear map of L into E_1.

Let $x \in L$, $g \in E_1'$, and $a \in G$. Then we have

$$\begin{aligned} d_x(g \circ \varphi)(a) &= d_x((g \circ \varphi)\cdot a)(1) = d_x'((g \circ \varphi)\cdot a) = d_x'((g\cdot\varphi(a)) \circ \varphi) \\ &= (g\cdot\varphi(a))(x_1) = (D_{x_1}(g) \circ \varphi)(a). \end{aligned}$$

Hence $d_x(g \circ \varphi) = D_{x_1}(g) \circ \varphi$, and this extends immediately to every $g \in P_1$.

Now let x and y be any two elements of L. We have

$$[D_x, D_y] = D_{[x,y]}, \quad \text{whence} \quad [d_x, d_y] = d_{[x,y]}.$$

Hence we obtain

$$d_{[x,y]}' = d_x' \circ d_y - d_y' \circ d_x.$$

With $g \in E_1'$, this gives

$$\begin{aligned} g([x_1, y_1]) &= g(x_1 y_1) - g(y_1 x_1) = D_{y_1}(g)(x_1) - D_{x_1}(g)(y_1) \\ &= d_x'(D_{y_1}(g) \circ \varphi) - d_y'(D_{x_1}(g) \circ \varphi) \\ &= (d_x' \circ d_y - d_y' \circ d_x)(g \circ \varphi) \\ &= d_{[x,y]}'(g \circ \varphi) = g([x, y]_1). \end{aligned}$$

Thus $[x_1, y_1] = [x, y]_1$, and we have shown that φ^* is a Lie algebra homomorphism.

Now let G_1 denote the smallest algebraic group containing $\varphi(G)$, and let Q_1 be the ideal of the polynomial functions vanishing on G_1. Let $g \in Q_1$. Then we have $D_{x_1}(g)(1) = d_x'(g \circ \varphi) = 0$. Hence we conclude that x_1 belongs to the Lie algebra L_1 of G_1. Thus $\varphi^*(L) \subset L_1$.

Observe that the map $g \to g \circ \varphi$ is a homomorphism of P_1 into T, and that the kernel of this homomorphism is precisely Q_1. Indeed, if $g \circ \varphi = 0$, then g vanishes on $\varphi(G)$ and hence on G_1, so that $g \in Q_1$. Since T is an integral domain, it follows that Q_1 is a prime ideal, i.e., that G_1 is irreducible. If d_{x_1} denotes the derivation of the field of rational functions on G_1 that is induced by D_{x_1}, we see at once from the above that, for every rational function f on G_1,

$$d_{x_1}(f) \circ \varphi = d_x(f \circ \varphi).$$

Actually, the same result evidently holds when G_1 is replaced by any other irreducible algebraic group containing $\varphi(G)$.

The most important fact for our purposes is that, actually, $\varphi^*(L) = L_1$. The proof of this fact, as given in [2], is bound up with a number of considerably more difficult questions, so that it seems worth while to isolate it here. Let K, K_1 denote the fields of the rational functions on G, G_1, respectively. Let $z \in L_1$, and let d_z denote the corresponding derivation of K_1. The map $f \to f \circ \varphi$ is a monomorphism of K_1 into K, and, since K is of characteristic 0, we can extend the derivation $f \circ \varphi \to d_z(f) \circ \varphi$ to a derivation of K. Thus there is a derivation δ of K such that $\delta(f \circ \varphi) = d_z(f) \circ \varphi$, for every $f \in K_1$.

Let u be the element of $E \otimes K$ such that, for every $p \in E'$ (regarded also as a linear function on $E \otimes K$, in the natural fashion), $p(u)$ is the canonical image p' of p in P/Q. Then we shall have $p(u) = p'$, for every $p \in P$. Hence $p(u) = 0$ if and only if $p \in Q$. In particular, this implies that the determinant function does not vanish at u, so that u is an automorphism of $M \otimes K$. Since u is a zero of $Q \otimes K$, it belongs to the smallest algebraic group G^K of linear automorphisms of $M \otimes K$ that contains G.

The derivation δ of K defines a derivation δ^* of $E \otimes K$ such that, for $e \in E$ and $k \in K$, $\delta^*(e \otimes k) = e \otimes \delta(k)$. We claim that the element $u^{-1}\delta^*(u)$ of $E \otimes K$ actually belongs to $L \otimes K$, which is the Lie algebra of G^K. Put $v = u^{-1}\delta^*(u)$, and let $p \in E'$. Then we have

$$D_v(p)(u) = p(uv) = p(\delta^*(u)) = \delta(p(u)),$$

and this extends at once to give $D_v(p)(u) = \delta(p(u))$, for every $p \in P$. Hence, if $p \in Q$, $D_v(p \cdot u)(1) = D_v(p)(u) = 0$. Since $u \in G^K$ and since $Q \otimes K$ is the ideal of all polynomial functions vanishing on G^K, we have $(Q \otimes K) \cdot u = Q \otimes K$, i.e., $(Q \cdot u)K = Q \otimes K$. Hence we conclude that $D_v(q)(1) = 0$, for every $q \in Q$, which implies that $D_v(Q) \subset Q \otimes K$, whence $v \in L \otimes K$.

Our rational representation φ of G in M_1 extends canonically to a rational representation of G^K in $M_1 \otimes K$, which we shall still denote by φ (see [2], p. 109). The differential of this extended representation is easily seen to be the canonical K-linear extension of φ^*, and we shall denote this extension still by φ^*. We shall also retain the symbol δ^* to denote the derivation of $E_1 \otimes K$ that is induced by δ.

Let $h \in E_1'$. Then we have $h(\delta^*(\varphi(u))) = \delta(h(\varphi(u)))$. But $h(\varphi(u))$ is simply the rational function $h \circ \varphi$ on G, and, since δ is an extension of the derivation $f \circ \varphi \to d_z(f) \circ \varphi$ $(f \in K_1)$, we have

$$\delta(h \circ \varphi) = D_z(h) \circ \varphi = D_z(h)(\varphi(u)) = h(\varphi(u)z).$$

Thus $h(\delta^*(\varphi(u))) = h(\varphi(u)z)$, for every $h \in E_1'$, whence

$$\delta^*(\varphi(u)) = \varphi(u)z.$$

On the other hand, consider the image $\varphi^*(v)$ of v in the Lie algebra $L_1 \otimes K$ of G_1^K. As we have seen earlier, we have

$$d_{\varphi^*(v)}(f) \circ \varphi = d_v(f \circ \varphi),$$

for every rational function f on G_1^K. Hence

$$d_{\varphi^*(v)}(f)(\varphi(u)) = d_v(f \circ \varphi)(u).$$

Now let f be the function on G_1^K that corresponds to an element $h \in E_1'$. Then the expression on the right of the last equation is equal to $\delta(h(\varphi(u)))$, as follows from the fact that, for every $p \in P$, $D_v(p)(u) = \delta(p(u))$. Hence our above equation gives

$$h(\varphi(u)\varphi^*(v)) = \delta(h(\varphi(u))) = h(\delta^*(\varphi(u))).$$

Hence we have

$$\varphi(u)\varphi^*(v) = \delta^*(\varphi(u)) = \varphi(u)z, \quad \text{whence} \quad \varphi^*(v) = z.$$

Now we can write $v = \sum_{i=1}^n x_i \otimes k_i$, where the x_i are elements of L and the k_i are elements of K that are linearly independent over F, and $k_1 = 1$. Then our last result shows that $\varphi^*(x_1) = z$. Thus we have shown that φ^* maps L *onto* the Lie algebra L_1 of G_1. In particular, $\varphi^*(L)$ is therefore an algebraic Lie algebra.

If we pass to the power series field over F in an auxiliary variable t, say F^*, we have $\exp(tx) \in G^{F^*}$, for an element x of E, if and only if $x \in L$. The proof is as follows. Note first that, for every $p \in P$,

$$\exp(tx) \cdot p = \exp(tD_x)(p).$$

Now suppose that $\exp(tx) \in G^{F^*}$. Then, for all $q \in Q$, $\exp(tx) \cdot q \in Q \otimes F^*$, i.e., $\exp(tD_x)(q) \in Q \otimes F^*$, which evidently implies that $D_x(q) \in Q$. Thus, if $\exp(tx) \in G^{F^*}$, then $x \in L$.

Conversely, if $x \in L$ we have $\exp(tx) \cdot Q = \exp(tD_x)(Q) \subset Q \otimes F^*$. Since $\exp(tx)$ is an automorphism of $M \otimes F^*$, this implies that $\exp(tx) \in G^{F^*}$.

Now we shall show that, if φ is a rational representation of G, we have $\varphi(\exp(tx)) = \exp(t\varphi^*(x))$, for every $x \in L$. In order to prove this, we shall use the derivation with respect to t, applying it to formal power series whose coefficients may be in K, $E \otimes K$, or $E_1 \otimes K$. We denote this derivation by τ throughout. We have $\tau(\exp(tx)) = \exp(tx)x$, whence, for every

$p \in E'$, $\tau(p(\exp(tx))) = D_x(p)(\exp(tx))$. This result extends immediately to give $\tau(f(\exp(tx))) = d_x(f)(\exp(tx))$, for every $f \in K$ such that f is defined at every point of G^{F^*}. In particular, we may take $f = h \circ \varphi$, where $h \in E_1'$. Then we obtain

$$h(\tau(\varphi(\exp(tx)))) = \tau((h \circ \varphi)(\exp(tx))) = d_x(h \circ \varphi)(\exp(tx))$$
$$= D_{\varphi^*(x)}(h)(\varphi(\exp(tx))) = h(\varphi(\exp(tx))\varphi^*(x)),$$

so that

$$\tau(\varphi(\exp(tx))) = \varphi(\exp(tx))\varphi^*(x).$$

This shows, first of all, that $\varphi(\exp(tx))$ is an *integral* power series in t. Hence the constant term is obtained by putting $t = 0$, so that the constant term is $\varphi(1) = 1$. But then the above differential equation for $\varphi(\exp(tx))$ shows that we must have $\varphi(\exp(tx)) = \exp(t\varphi^*(x))$.

Now let V be a subspace of the representation space M_1 of the rational representation φ. Then V is stable under $\varphi(G)$ if and only if it is stable under $\varphi^*(L)$. We can prove this from the above results as follows. Suppose first that V is stable under $\varphi(G)$. Then, for every $x \in L$, $\varphi(\exp(tx))$ must map V into $V \otimes F^*$, because $\exp(tx) \in G^{F^*}$, so that $\varphi(\exp(tx)) \in G_1^{F^*}$, which is the smallest algebraic group of automorphisms of $M_1 \otimes F^*$ that contains $\varphi(G)$. Thus $\exp(t\varphi^*(x))$ maps V into $V \otimes F^*$, whence $\varphi^*(x)$ maps V into itself. Conversely, suppose that V is stable under $\varphi^*(L)$. Then $\varphi(\exp(tx))$ maps V into $V \otimes F^*$. Let $x_1, \cdots, x_n$ be a basis of L, and let $t_1, \cdots, t_n$ be algebraically independent elements over F. Then $\varphi(\exp(t_1 x_1) \cdots \exp(t_n x_n))$ maps V into $V \otimes F^*$, where now F^* denotes the field of quotients of the ring of power series in the t_i over F. If $s \in G$, the condition that $\varphi(s)$ map V into itself is that s be a zero of a certain set of polynomial functions. If p is any polynomial function on E we have

$$p(\exp(t_1 x_1) \cdots \exp(t_n x_n)) = \exp(t_1 D_{x_1}) \cdots \exp(t_n D_{x_n})(p)(1).$$

We see from this that if $p(\exp(t_1 x_1) \cdots \exp(t_n x_n)) = 0$ then the image p' of p in the algebra of representative functions on the universal enveloping algebra of L that is associated with the identity representation of L is 0. Indeed, the coefficient of $t_1^{e_1} \cdots t_n^{e_n}$ on the right side of the above equation is $(e_1! \cdots e_n!)^{-1}p'(x_1^{e_1} \cdots x_n^{e_n})$. Thus we conclude that if

$$p(\exp(t_1 x_1) \cdots \exp(t_n x_n)) = 0,$$

then p vanishes at every point of G. Hence it follows from what we have said above that V is stable under $\varphi(G)$.

Let L be an algebraic Lie algebra of linear endomorphisms of M, and let S denote the algebra of representative functions on the universal enveloping algebra U of L that is generated by the constants and the representative functions associated with the identity representation of L. We wish to give an intrinsic characterization of the representations of L that are the differentials of rational representations of the corresponding irreducible algebraic group G.

Let d denote the determinant function on E, and let s denote the trace function. It is easy to see that, for every $e \in E$, we have $D_e(d) = s(e)d$. We see from this that the image d' of d in S is given by

$$d'(y_1 \cdots y_m) = s(y_1) \cdots s(y_m), \qquad d'(1) = 1,$$

where $y_1, \cdots, y_m$ are arbitrary elements of L. There is an element z in U' such that

$$z(y_1 \cdots y_m) = (-1)^m s(y_1) \cdots s(y_m), \quad \text{and} \quad z(1) = 1.$$

One checks immediately that $d'z = 1$. Thus z is the element of the field of quotients of S that corresponds to d^{-1}. For $x \in L$, we have

$$x \cdot z = z \cdot x = -s(x)z.$$

In particular, this shows that $z \in \mathbf{R}$. We are now in a position to state our result.

THEOREM 3. *A representation ρ of an algebraic Lie algebra L is the differential of a rational representation of the corresponding algebraic group G if and only if $\mathbf{R}(\rho) \subset S[z]$.*

Proof. Let M_ρ denote the representation space of ρ, and suppose first that $\rho = \varphi^*$, where φ is a rational representation of G in M_ρ. Let E_ρ be the algebra of all linear endomorphisms of M_ρ, and let $f \in E_\rho'$. Consider the representative function $f \circ \varphi$ on G. Since it is rational and defined at every point of G, it follows (see Lemma 10.1 in [4]) that there is a nonnegative integer k such that $d^k(f \circ \varphi)$ is the restriction to G of some $p \in P$. Hence $d^k(f \circ \varphi) - p$ vanishes at the generalized point $u = \exp(t_1 x_1) \cdots \exp(t_n x_n)$ of G, where $x_1, \cdots, x_n$ is a basis for L. Let $\rho(u)$ stand for the power series whose coefficients are the images by ρ of the coefficients of u. Then, since $\varphi(\exp(tx)) = \exp(t\rho(x))$, we have $\varphi(u) = \rho(u)$. Now write v for $\exp(t_1 D_{x_1}) \cdots \exp(t_n D_{x_n})$. Then we have

$$0 = d^k(u)f(\varphi(u)) - p(u) = v(d^k)(1)f(\rho(u)) - v(p)(1).$$

When the last expression is written out as a power series, the coefficient of $t_1^{e_1} \cdots t_n^{e_n}$ is equal to $(e_1! \cdots e_n!)^{-1} ((d^k)'(f \circ \rho) - p')(x_1^{e_1} \cdots x_n^{e_n})$. Since each coefficient must be zero, we find that $(d^k)'(f \circ \rho) - p' = 0$, whence $f \circ \rho = z^k p'$. Thus our condition on $\mathbf{R}(\rho)$ is necessary.

Now suppose that $\mathbf{R}(\rho) \subset S[z]$. The group G operates by left translations on P, and Q is G-stable. Hence we get induced left G-translations on S, and hence also on $S[z]$. In fact, if $x \in G$, we have $x \cdot d^{-1} = d(x)^{-1}d^{-1}$, whence $x \cdot z = d(x)^{-1}z$. The G-operations on $S[z]$ are algebra automorphisms and commute with the right U-translations. Since $\mathbf{R}(\rho)$ and $S[z]$ are stable under the right and left U-translations, it follows at once from Lemma 1 that $\mathbf{R}(\rho)$ is stable under the left G-translations. Thus we have a representation of G by left translations in $\mathbf{R}(\rho)$. The representative functions on G that are associated with this representation are the linear combinations of the maps $x \to (x \cdot f)(u)$, where u ranges over U and f ranges over $\mathbf{R}(\rho)$. However, we have

$$(x \cdot f)(u) = ((x \cdot f) \cdot u)(1) = (x \cdot (f \cdot u))(1),$$

and $f \cdot u \in \mathbf{R}(\rho)$. Hence the representative functions on G that are associated with our representation of G in $\mathbf{R}(\rho)$ are precisely the maps $x \to (x \cdot f)(1)$, where f ranges over $\mathbf{R}(\rho)$. But these are precisely the rational functions on G that correspond to the elements of $\mathbf{R}(\rho)$ under the canonical isomorphism between $S[z]$ and a subalgebra of the algebra of the rational functions on G. Hence our representation of G in $\mathbf{R}(\rho)$ is a rational representation. Let φ denote this rational representation.

Let F^* denote the power series field in one variable t over F, and extend φ in the canonical fashion to a rational representation of G^{F^*} in $\mathbf{R}(\rho) \otimes F^*$. We have $\varphi(\exp(tx)) = \exp(t\varphi^*(x))$, for every $x \in L$. On the other hand, the left translation by $\exp(tx)$ on $\mathbf{R}(\rho) \otimes F^*$ coincides with $\exp(t\tau_x)$, where τ_x denotes the left translation by x (regarded as an element of U) on $\mathbf{R}(\rho)$. It follows that $\varphi^*(x) = \tau_x$.

Let m be the dimension of M_ρ, and let $m \cdot \mathbf{R}(\rho)$ denote the direct sum of m copies of $\mathbf{R}(\rho)$. Our representation φ yields a rational representation ψ of G in $m \cdot \mathbf{R}(\rho)$ in the natural fashion. Let $h_1, \cdots, h_m$ be a basis for M_ρ'. For $v \in M_\rho$, let h_i/v be the element of $\mathbf{R}(\rho)$ that is given by

$$(h_i/v)(u) = h_i(\rho(u)(v)), \qquad \text{for every} \quad u \in U.$$

Then the map $v \to (h_1/v, \cdots, h_m/v)$ is easily seen to be a left U-module monomorphism of M_ρ into $m \cdot \mathbf{R}(\rho)$. The image of M_ρ in $m \cdot \mathbf{R}(\rho)$ is therefore stable under $\psi^*(L)$, and hence stable under $\psi(G)$. Hence, by restriction, ψ yields a rational representation of G in M_ρ, and it is clear that the differential of this representation is precisely ρ. This completes the proof of Theorem 3.

Let L be any Lie algebra over F, and let S be a finite-dimensional subspace of $\mathbf{R}$ that is stable under the right and left translations. Let S^* denote the subalgebra of $\mathbf{R}$ that is generated by the constants and the elements of S. Let ρ be the representation of L by left translations in S. Then we have evidently $S_\rho = S^*$. Let L_S be the restriction to S of the Lie algebra of all proper derivations of S^*. It follows at once from Theorem 2 that L_S is the smallest algebraic Lie algebra of linear endomorphisms of S containing $\rho(L)$. Let T be another subspace of $\mathbf{R}$ satisfying the same conditions as S and such that $S \subset T$. The representative functions on the universal enveloping algebra of L_T that are associated with the identity representation of L_T are the maps $u \to u(t)(1)$, where t ranges over T. Now S is stable under L_T, so that we get a representation of L_T in S. The representative functions associated with this representation are the maps $u \to u(s)(1)$, where s ranges over S. Hence it follows from Theorem 3 and the fact that the differential of a rational group representation sends the Lie algebra onto an algebraic Lie algebra that the restriction of L_T to S is an algebraic Lie algebra. Since it contains $\rho(L)$, it must therefore contain L_S. On the other hand, the restriction of L_T to S is evidently contained in L_S. Hence it coincides with L_S.

Now we have an inverse system of Lie algebra epimorphisms $L_T \to L_S$, for all pairs $S \subset T$. By Proposition 2.10 of [4], the natural homomorphism of the inverse limit of our system into L_S is an *epimorphism*, for each S. The inverse limit of our system may evidently be identified with the Lie algebra of all proper derivations of $\mathbf{R}$. Hence we have the following result.

THEOREM 4. *Let A be any finitely generated subalgebra of* $\mathbf{R}$ *that contains the constants and is stable under the left and right translations. Then every proper derivation of A is the restriction to A of a proper derivation of* $\mathbf{R}$.

5. Semisimple Lie algebras

Let M be a finite-dimensional vector space over F, and let E be the algebra of all linear endomorphisms of M. We consider a tensor space

$$T = M \otimes \cdots \otimes M \otimes M' \otimes \cdots \otimes M'.$$

Regarding E as a Lie algebra, we consider the usual tensor representation of E in T. Let A and B be subspaces of T such that $A \subset B$. It is a standard result that the Lie algebra consisting of all elements of E that map B into A is an algebraic Lie algebra. In fact, let φ denote the tensor representation of the full linear group over M in T. Let G be the group of all linear automorphisms s of M such that $\varphi(s)(b) - b \in A$, for every $b \in B$. Evidently, G is an algebraic group. Extend the base field F to the field F^* of power series in one variable t. Then the extended group G^{F^*} is the group of all automorphisms s of $M \otimes F^*$ such that $\varphi(s)(b) - b \in A \otimes F^*$, for every $b \in B$. Now it is easily checked quite directly that, if $e \in E$, $\exp(te) \in G^{F^*}$ if and only if, in the Lie algebra representation of E in T, e sends B into A. By what we have seen in Section 4, this means that the Lie algebra of all elements of E that send B into A is precisely the Lie algebra of G. and thus is an algebraic Lie algebra.

Now let L be any Lie subalgebra of E, and apply the above with

$$T = M \otimes M' = E,$$

$A = [L, L]$, and $B = L$. We conclude that, if S denotes the smallest algebraic Lie algebra containing L, $[S, L] = [L, L]$. Now take $A = [L, L]$ and $B = S$. We find that $[S, S] = [L, L]$.

Now suppose that L is semisimple. Then M is semisimple as an L-module, and it follows at once from the above, with $T = M$, that M is semisimple also as an S-module. As is well known, this implies that S is the direct sum of $[S, S] = [L, L] = L$ and its center, C say, and that M is semisimple with respect to every element $c \in C$. We wish to prove that $C = (0)$. In order to do this, we extend the base field F to its algebraic closure. Noting that the smallest algebraic Lie algebra containing the natural extension of L is the natural extension of S, we see that we may suppose without loss of generality that F is algebraically closed. Assuming this, let $c \in C$, and let W be the c-submodule of M that corresponds to a characteristic value γ of c. Let $w_1, \cdots, w_n$ be a basis for W. Since c is in the center of S, W is an L-submodule of M. Let T be the tensor product of n copies of M. Let b denote the element

$$\sum_\sigma (\sigma) w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)} ,$$

where the summation is over all permutations σ of $(1, \cdots, n)$, and (σ) is 1 or -1 according to whether σ is even or odd. Let $B = Fb$, $A = (0)$. If $x \in L$, we have $x \cdot b = T_W(x)b$, where $T_W(x)$ denotes the trace of the restriction of x to W. Since $L = [L, L]$, it follows that $T_W(x) = 0$. Hence L maps B into $A = (0)$. By the general result of the beginning of this section, we conclude that S maps B into (0). Hence $c \cdot b = 0$. But evidently $c \cdot b = n\gamma b$. Thus $\gamma = 0$. Thus every characteristic value of c is 0, so that $c = 0$. We have shown that $C = (0)$, whence $S = L$. Our conclusion is the well known result that *every semisimple linear Lie algebra is algebraic.*

The other result we shall need is that, *for every semisimple Lie algebra L, the algebra $\mathbf{R}$ of all representative functions on the universal enveloping algebra of L is finitely generated.* This result, proved first (for an *arbitrary* base field of characteristic 0) by Harish-Chandra, is not elementary. Using the theory of weights for the representations of a semisimple Lie algebra (as contained, for instance, in [7]), one can prove the result quite rapidly, as follows. Let F^* denote the algebraic closure of F. Then $\mathbf{R} \otimes F^*$ is the algebra of representative functions on the universal enveloping algebra of $L \otimes F^*$. If $\mathbf{R} \otimes F^*$ is finitely generated, then the same must evidently hold for $\mathbf{R}$. Since $L \otimes F^*$ is still semisimple, we may therefore suppose without loss of generality that F is algebraically closed. In that case, every simple representation of L is determined (up to an isomorphism) by its highest weight with respect to some Cartan subalgebra of L. It is known that the highest weights belonging to the simple representations of L are all the sums, with repetitions allowed, of a finite set $\zeta_1, \cdots, \zeta_r$ of weights (where r is the rank of L). If V and W are L-modules, the weights of their tensor product $V \otimes W$ are all the sums $\gamma + \delta$, where γ ranges over the weights of V and δ ranges over the weights of W. Hence the highest weight of $V \otimes W$ is the sum of the highest weight of V and the highest weight of W. For each i, let V_i be a simple L-module with highest weight ζ_i. Then the tensor product of e_1 copies of $V_1, \cdots, e_r$ copies of V_r has $e_1 \zeta_1 + \cdots + e_r \zeta_r$ for its highest weight, and hence must contain a simple component with that highest weight. It follows that if V is the direct sum $V_1 + \cdots + V_r$, then every simple L-module is a component of some tensor power of V. Hence, if ρ is the representation of L in V, $\mathbf{R}$ is generated by the constants and the representative functions associated with ρ. This completes the proof.

6. The algebra of representative functions

Let L be a finite-dimensional Lie algebra over F, and suppose that L is a semidirect sum $H + K$, where K is an ideal of L and H a subalgebra. Let $U(L)$, $U(K)$, $U(H)$ denote the universal enveloping algebras L, K, H, respectively. For any universal enveloping algebra U, we denote the projection map $U \to U_0 = F$, whose kernel is the ideal generated by the elements of the Lie algebra, by $u \to u_0$. Let $\mathbf{R}(L)$, etc., denote the algebra of representative functions on $U(L)$, etc. The epimorphism $L \to H$ with kernel K extends uniquely to a unitary epimorphism $U(L) \to U(H)$ with kernel $KU(L)$. This yields an isomorphism of $\mathbf{R}(H)$ onto a subalgebra $\mathbf{R}^H(L)$ of $\mathbf{R}(L)$. If $f \in \mathbf{R}(H)$, the corresponding element of $\mathbf{R}^H(L)$ is denoted f^+. Identifying $U(H)$ and $U(K)$ with their canonical images in $U(L)$, we have $U(L) = U(H)U(K) \approx U(H) \otimes U(K)$, and $f^+(uv) = f(uv_0)$, for all $u \in U(H)$and all $v \in U(K)$.

On the other hand, we have the restriction epimorphism $f \to f_K$ of $\mathbf{R}(L)$ onto a subalgebra $\mathbf{R}(L)_K$ of $\mathbf{R}(K)$. We shall characterize $\mathbf{R}(L)_K$ as a subalgebra of $\mathbf{R}(K)$ and obtain an algebra monomorphism of $\mathbf{R}(L)_K$ into $\mathbf{R}(L)$ inverse to the restriction epimorphism $\mathbf{R}(L) \to \mathbf{R}(L)_K$. Clearly, the restriction epimorphism is a $U(K)$-epimorphism with respect to the double $U(K)$-module structures of $\mathbf{R}(L)$ and $\mathbf{R}(K)$. Hence $\mathbf{R}(L)_K$ is stable under the right and left translations on $\mathbf{R}(K)$. Since K is an ideal in L, $U(K)$ is stable under the commutations with the elements of L and, in particular, with the elements of H. Hence we may define the structure of an H-module on $U(K)'$ as follows: let $h \in H, f \in U(K)', u \in U(K)$. We define the transform $h(f)$ by putting $h(f)(u) = f(uh - hu)$. If $f = g_K$, with $g \in \mathbf{R}(L)$, we have $h(f) = (h \cdot g - g \cdot h)_K$. Thus $\mathbf{R}(L)_K$ is stable under the H-operations.

The H-operations on $U(K)'$ can be combined with the left translations by the elements of K to yield an L-module structure on $U(K)'$. In fact, given $x \in L$, we have uniquely $x = h + k$, with $h \in H$ and $k \in K$. For $f \in U(K)'$, we define $x(f) = h(f) + k \cdot f$. It is immediately verified that

$$h(k \cdot f) - k \cdot h(f) = [h, k] \cdot f,$$

whence it is seen that our definition gives an L-module structure on $U(K)'$. This, in turn, defines the structure of a left $U(L)$-module on $U(K)'$, the induced $U(K)$-module structure being again the canonical left translation structure. It is clear from the above that every element of $\mathbf{R}(L)_K$ generates a *finite-dimensional* $U(L)$-submodule of $\mathbf{R}(L)_K$.

Conversely, suppose that $f \in U(K)'$ and that f generates a finite-dimensional $U(L)$-submodule T of $U(K)'$. Then we have a finite-dimensional representation of L in T. The map $u \to u(f)(1)$ of $U(L)$ into F is evidently a representative function associated with this representation. Call this representative function f^+. If $u \in U(H)$ and $v \in U(K)$, we have

$$f^+(uv) = f(u_0 v).$$

Hence $f = (f^+)_K \in \mathbf{R}(L)_K$. Thus we have shown that $\mathbf{R}(L)_K$ is precisely the sum of all finite-dimensional $U(L)$-submodules of $U(K)'$, which is also the sum of all finite-dimensional $U(L)$-submodules of $\mathbf{R}(K)$. Observe also that, for every $f \in U(K)'$, there is one and only one $f^+ \in U(L)'$ such that $f^+(uv) = f(u_0 v)$, for every $u \in U(H)$ and every $v \in U(K)$, because

$$U(L) \approx U(H) \otimes U(K).$$

Moreover, the map $f \to f^+$ is evidently an algebra monomorphism.

Our two algebra isomorphisms of $\mathbf{R}(H)$ onto $\mathbf{R}^H(L)$, and of $\mathbf{R}(L)_K$ onto a subalgebra $\mathbf{R}^K(L)$ of $\mathbf{R}(L)$, compose to an algebra homomorphism of $\mathbf{R}(H) \otimes \mathbf{R}(L)_K$ into $\mathbf{R}(L)$. We claim that this is actually an isomorphism of $\mathbf{R}(H) \otimes \mathbf{R}(L)_K$ onto $\mathbf{R}(L)$. Since it is clearly a monomorphism, there remains only to show that $\mathbf{R}(L) = \mathbf{R}^H(L)\mathbf{R}^K(L)$.

Let $f \in \mathbf{R}(L)$, and let $f_1, \cdots, f_n$ be a basis for the space spanned by the left translates of f. Then, for every $x \in U(L)$, we have

$$x \cdot f = \sum_{i=1}^n g_i(x) f_i,$$

where each g_i is an element of $\mathbf{R}(L)$. Take $x = v \in U(K)$. Then we may write

$$(v \cdot f)_H = \sum_{i=1}^n (g_i)_K(v)(f_i)_H.$$

Evaluating this at an element $u \in U(H)$, we find

$$f(uv) = \sum_{i=1}^n (f_i)_H(u)(g_i)_K(v).$$

Put $p_i = ((f_i)_H)^+ \in \mathbf{R}^H(L)$, and $q_i = ((g_i)_K)^+ \in \mathbf{R}^K(L)$. Since

$$p_i(U(L)K) = (0),$$

and $q_i(HU(L)) = (0)$, it follows from the definition of the multiplication in $U(L)'$ that $(p_i q_i)(uv) = p_i(u)q_i(v)$. Hence our last expression for $f(uv)$ shows that $f = \sum_{i=1}^n p_i q_i \in \mathbf{R}^H(L)\mathbf{R}^K(L)$. This completes the proof that the maps $f \to f^+$ give an isomorphism of $\mathbf{R}(H) \otimes \mathbf{R}(L)_K$ onto $\mathbf{R}(L)$.

We shall need a result on extensions of representations, due to Zassenhaus, and we shall also need a by-product of the proof of this result, as given in [5] (Theorem 5.1), where the result is obtained by using the same technique of representative functions that we are using here. The full statement of what we shall need, in terms of the above notation, is as follows. Let ρ be a representation of K, and suppose that ρ is nilpotent on an ideal J of K

that contains $[H, K]$. Then ρ can be extended (allowing extension of the representation space) to a representation σ of L with the following properties:

(1) σ is nilpotent on J;

(2) if f is a representative function associated with ρ, then its image f^+ in $U(L)'$ is a representative function associated with σ;

(3) if H is nilpotent on K under the adjoint representation, then σ is nilpotent on $H + J$.

We are now in a position to begin with an analysis of the structure of $\mathbf{R}(L)$. Let A denote the radical of L. Then L is a semidirect sum $S + A$, where S is a maximal semisimple subalgebra of L. By the above, we have

$$\mathbf{R}(L) = \mathbf{R}^S(L)\mathbf{R}^A(L) \approx \mathbf{R}(S) \otimes \mathbf{R}(L)_A .$$

Let $T = [L, A]$. We shall show first that $\mathbf{R}(L)_A$ consists precisely of the representative functions on $U(A)$ that are associated with representations of A whose restrictions to T are nilpotent. It follows at once from Lie's theorem that every representation of L is nilpotent on T, so that our condition on the elements of $\mathbf{R}(L)_A$ is necessary. Conversely, let f be a representative function on $U(A)$ associated with a representation ρ of A that is nilpotent on T. By what we have said above, ρ can be extended to a representation of L, and it follows that $f \in \mathbf{R}(L)_A$.

Since T is a nilpotent Lie algebra, we can find a basis $x_1, \cdots, x_m$ for T such that each commutator $[x_p, x_q]$ is a linear combination of elements x_i with $i < \min(p, q)$. In particular, the subspace T_p spanned by $x_1, \cdots, x_p$ is an ideal of T for each p. Now we complete this basis to a basis $x_1, \cdots, x_n$ of A. Then the elements of $U(A)$ can be written uniquely as linear combinations of the ordered monomials $x_n^{e_n} \cdots x_1^{e_1}$. Let f_i be the element of $U(A)'$ that takes the value 1 at x_i and the value 0 at every other ordered monomial.

PROPOSITION 2. *$\mathbf{R}(L)_T$ coincides with the algebra generated by $(f_1)_T, \cdots, (f_m)_T$ and the constants, and each $f_i \in \mathbf{R}(L)_A$.*

Proof. Clearly, the restriction of f_1 to $U(T_1)$ is a representative function associated with a nilpotent representation of T_1. Now let g be any representative function on $U(T_1) = F[x_1]$ that is associated with a nilpotent representation. Then there is an exponent q such that g vanishes on $x_1^q U(T_1)$, and, since $f_1^p(x_1^e) = \delta_{pe}\, e!$, we see that g is a polynomial of degree less than q in the restriction of f_1 to $U(T_1)$.

Now observe that A is reached from T_1 by a sequence of semidirect sum constructions passing along the composition series whose terms are the subalgebras A_i spanned by $x_1, \cdots, x_i$; clearly, each A_i is an ideal of A_{i+1}. If, starting with the restriction of f_1 to $U(T_1)$, we apply the maps $f \to f^+$ successively on each level of the composition series, we evidently arrive at the function f_1 in the end. It follows from what we have said above concerning extensions of representations that f_1 is associated with a representation of A that is nilpotent on T, so that $f_1 \in \mathbf{R}(L)_A$.

Suppose now that $m \geqq p > 1$, and that we have already shown that $f_1, \cdots, f_{p-1}$ belong to $\mathbf{R}(L)_A$, and that their restrictions to $U(T_{p-1})$, together with the constants, generate the algebra of all representative functions associated with nilpotent representations. We have $T_p = Fx_p + T_{p-1}$. Write H for the 1-dimensional Lie algebra Fx_p. The restriction of f_p to $U(T_p)$ evidently belongs to $\mathbf{R}^H(T_p) \approx \mathbf{R}(H)$, and we see as above for f_1 that, together with the constants, it generates the algebra of all elements of $\mathbf{R}^H(T_p)$ that are associated with nilpotent representations.

Now let g be any element of $\mathbf{R}(T_p)$ that is associated with a nilpotent representation. We may write $g = \sum u_i v_i$, where the u_i are linearly independent elements of $\mathbf{R}^H(T_p)$ and the v_i belong to $\mathbf{R}^{T_{p-1}}(T_p)$. If $x \in T_{p-1}$, we have $x \cdot u_i = 0$, so that $x \cdot g = \sum u_i x \cdot v_i$. Clearly, $\mathbf{R}^{T_{p-1}}(T_p)$ is stable under the left translations with the elements of T_{p-1}, and the elements u_i are free over this algebra. Since g is associated with a nilpotent representation, there is an exponent q such that $T_{p-1}^q U(T_{p-1}) \cdot g = (0)$. We see at once from the above that each v_i is annihilated from the left by this same ideal of $U(T_{p-1})$. Hence the restriction of each v_i to $U(T_{p-1})$ is associated with a nilpotent representation. By our inductive hypothesis, this means that these restrictions of the v_i are polynomials in the restrictions to $U(T_{p-1})$ of $f_1, \cdots, f_{p-1}$. It follows that each v_i is a polynomial in the restrictions to $U(T_p)$ of $f_1, \cdots, f_{p-1}$.

Now we can rewrite g as a sum of products as above, but this time with the v_i *linearly independent* polynomials in the restrictions to $U(T_p)$ of $f_1, \cdots, f_{p-1}$, while, of course, we have to abandon the requirement that the u_i be linearly independent. Since $v_i \cdot x_p = 0$, we have $g \cdot x_p = \sum u_i \cdot x_p v_i$. Since g is associated with a nilpotent representation, it is annihilated from the right by some power of x_p. Since now the v_i are free over $\mathbf{R}^H(T_p)$, it follows that the u_i are annihilated from the right by the same power of x_p, so that they are associated with nilpotent representations. Hence we see as above that the u_i are polynomials in the restriction of f_p to $U(T_p)$. The extension theory shows that $f_p \in \mathbf{R}(L)_A$, and we have now completed the inductive step, whence we may conclude that $f_1, \cdots, f_m$ belong to $\mathbf{R}(L)_A$, and that their restrictions to $U(T)$, together with the constants, generate the algebra of all representative functions associated with nilpotent representations of T. Since every nilpotent representation of T can be extended to a representation of L that is still nilpotent on T (because L is obtained from T by a sequence of semidirect sum constructions), this algebra coincides with $\mathbf{R}(L)_T$.

In order to complete the proof of Proposition 2, we must merely observe that the f_i with $i > m$ are also elements of $\mathbf{R}(L)_A$, because they are associated with representations of A that are trivial on T.

Before we proceed with our analysis of $\mathbf{R}(L)_A$ we have to examine the representative functions for the 1-dimensional Lie algebra. If x denotes any nonzero element of the 1-dimensional Lie algebra, the universal enveloping algebra is the polynomial algebra $F[x]$. Let f denote the representative function defined by $f(x^e) = \delta_{1e}$. Let F^* denote the algebraic closure of F, and let us first examine the representative functions on $F[x]$ with values in F^*. These are associated with representations in vector spaces over F^*. For every element c of F^*, there is a unique homomorphism of $F[x]$ into F^* sending 1 onto 1 and x onto c. This homomorphism is given in the natural fashion by the power series $\exp(cf)$. It is evidently a representative function.

Now let V be any representation space for $F[x]$ over F^*. Then we can decompose V into a direct sum of stable subspaces V_i for each of which there is an element $c_i \in F^*$ such that V_i is annihilated by some power of $x - c_i$. Hence it suffices to consider representation spaces V for which there is an element $c \in F^*$ such that some power of $x - c$ annihilates V. Let ρ denote the representation of $F[x]$ in V, m the dimension of V, I the identity map of V onto V. Then we have $\rho(1) = I$, and $\rho(x) = u + cI$, where $u^m = 0$. Hence

$$\rho(x^n) = c^n I + \binom{n}{1} c^{n-1} u + \cdots + \binom{n}{k} c^{n-k} u^k + \cdots,$$

where all the terms after the m^{th} one are zero. Noting that

$$\frac{1}{k!} (\exp(cf) f^k)(x^n) = \binom{n}{k} c^{n-k},$$

we see that the above may be written

$$\rho = \exp(cf) \left(I + fu + \tfrac{1}{2} f^2 u^2 + \cdots + \frac{1}{(m-1)!} f^{m-1} u^{m-1} \right),$$

where the product of a function g on $F[x]$ by a linear transformation e of V denotes the map of $F[x]$ into the algebra of linear transformations of V sending each element a of $F[x]$ onto $g(a)e$. We see from this that the representative functions on $F[x]$ can be written as polynomials in f whose coefficients are F^*-linear combinations of functions $\exp(cf)$, with $c \in F^*$.

The F^*-linear combinations of the $\exp(cf)$ constitute an algebra G^* of F^*-valued representative functions, and the powers of f are free over G^*. Let G be the subalgebra of G^* consisting of the F-valued elements of G^*. Then it is clear that the F-valued representative functions on $F[x]$ are the polynomials in f with coefficients in G.

An element of $U(L)'$ will be called an *elementary function* if it vanishes on $U(L)_0$ and on $L^2U(L)$. Clearly, the elementary functions are representative functions associated with representations of L that are trivial on $[L, L]$. Thus the elementary functions belong to the canonical image, $R^*(L)$ say, of $\mathbf{R}(L/[L, L])$ in $\mathbf{R}(L)$.

If f is an elementary function and c is an element of F^*, then $\exp(cf)$ is defined as a linear map of $U(L)$ into F^*. The F^*-linear combinations of these functions $\exp(cf)$ that are such that their values lie in F are evidently elements of $\mathbf{R}^*(L)$. We shall call these functions the *trigonometric functions.*

Let $g_1, \cdots, g_n$ be the elements of $\mathbf{R}^A(L)$ whose restrictions to $U(A)$ are the functions $f_1, \cdots, f_n$ of Proposition 2. Then $g_{m+1}, \cdots, g_n$ belong to $\mathbf{R}^*(L)$ and span the space of all elementary functions on $U(L)$; in fact, $L/[L, L]$ is canonically isomorphic with A/T, so that the cosets mod. T of the elements $x_{m+1}, \cdots, x_n$ may be identified with the elements of a basis for $L/[L, L]$.

It follows that the trigonometric functions on $U(L)$ can be written as polynomials with coefficients in F^* in the functions $\exp(c_i g_i)$, with $i > m$ and $c_i \in F^*$. We claim that $\mathbf{R}(L)_A$ is generated by the f_i $(i = 1, \cdots, n)$ and the restrictions to $U(A)$ of the trigonometric functions on $U(L)$. Clearly, $\mathbf{R}(L)_A$ contains all these functions. The proof that $\mathbf{R}(L)_A$ is generated by them is by induction on $n - m$. If $n - m = 0$, the result is already established, by Proposition 2. Indeed, in this case, we have $T = A$, whence $L = [L, L]$, and the trigonometric functions are the constants. We base our induction on the sequence of semidirect sum constructions leading from T to A, via the A_i with $i > m$. Since each A_{i+1}/A_i is 1-dimensional, we can use our determination of the representative functions for the 1-dimensional Lie algebra at each level, and the remaining arguments for completing the induction are merely repetitions of parts of our proof of Proposition 2.

Observe also that the case $T = (0)$ gives the result that $\mathbf{R}^*(L)$ is generated by the elementary functions and the trigonometric functions. More precisely, if G is the algebra of the trigonometric functions, we have

$$\mathbf{R}^*(L) = G[g_{m+1}, \cdots, g_n],$$

and the monomials in the g_i $(i > m)$ are free over G.

The algebra generated by $g_1, \cdots, g_m$ is the image in $\mathbf{R}(L)$, by the composition of the maps $f \rightarrow f^+$ corresponding to the levels of our sequence of extensions leading from T to L, of the algebra of all representative functions on $U(T)$ that are associated with nilpotent representations. Denote this last algebra by $\mathbf{R}_0(T)$, and denote its image in $\mathbf{R}(L)$ by $\mathbf{R}_0^T(L)$. Then we can summarize our results as follows.

THEOREM 5. $\mathbf{R}(L) = \mathbf{R}^S(L)\mathbf{R}^*(L)\mathbf{R}_0^T(L) \approx \mathbf{R}^S(L) \otimes \mathbf{R}^*(L) \otimes \mathbf{R}_0^T(L)$; $\mathbf{R}^S(L)$ *is canonically isomorphic with* $\mathbf{R}(S) = \mathbf{R}(L/A)$; $\mathbf{R}^*(L)$ *is canonically isomorphic with* $\mathbf{R}(L/[L, L])$ *and is generated by the elementary functions and the trigonometric functions*; $\mathbf{R}_0^T(L) \approx \mathbf{R}_0(T)$ *and is generated by the m algebraically independent functions* $g_1, \cdots, g_m$; $\mathbf{R}^*(L)\mathbf{R}_0^T(L) = \mathbf{R}^A(L)$.

It is now easy to deduce the following result.

THEOREM 6. *A proper derivation D of* $\mathbf{R}(L)$ *is the left translation by an element of L if and only if its natural extension to a proper derivation of* $\mathbf{R}(L) \otimes F^*$ *satisfies* $D(\exp(f)) = D(f)\exp(f)$, *for every elementary function f on* $U(L \otimes F^*)$.

Proof. Since $\exp(f)$ is a homomorphism, we have, for all $x \in L$ and all $u \in U(L)$,

$$
\begin{aligned}
(x\cdot\exp(f))(u) &= \exp(f)(ux) = \exp(f)(u)\exp(f)(x) \\
&= \exp(f)(u)f(x) = ((x\cdot f)\exp(f))(u).
\end{aligned}
$$

Hence the condition is necessary.

Now suppose that D is a proper derivation satisfying the condition of our theorem. Let δ be the differentiation of $\mathbf{R}(L)$ defined by $\delta(f) = D(f)(1)$. We must prove that there is an element $x \in L$ such that $\delta(f) = f(x)$, for every $f \in \mathbf{R}(L)$. First consider the restriction of δ to $\mathbf{R}^S(L)$. Since $\mathbf{R}^S(L)$ is isomorphic with $\mathbf{R}(S)$, we know from the end of Section 5 that it is finitely generated. In fact, $\mathbf{R}(S)$ is the algebra generated by the constants and the representative functions associated with a representation ρ of S. We know from Section 5 that $\rho(S)$ is algebraic. Hence it follows from Theorem 2 that, if δ^* is the differentiation of $\mathbf{R}(S)$ that is induced by δ, there is an element $s \in S$ such that $\delta^*(f) = f(s)$, for every $f \in \mathbf{R}(S)$. But this means that $\delta(f) = f(s)$, for every $f \in \mathbf{R}^S(L)$. Now we subtract from D the left translation by s. The property assumed for D is thereby preserved, and the new differentiation δ annihilates $\mathbf{R}^S(L)$. Hence we may now assume that

$$\delta(\mathbf{R}^S(L)) = (0).$$

There is a linear map of A' into F sending the restriction to $U(A)$ of each g_i (as defined above) onto $\delta(g_i)$. Hence there is an element $a \in A$ such that $\delta(g_i) = g_i(a)$, for each i. Since the elements of $\mathbf{R}^S(L)$ vanish on A, we have, using the notation of Theorem 5,

$$\delta(f) = f(a), \qquad \text{for every} \quad f \in \mathbf{R}^S(L)\mathbf{R}_0^T(L).$$

Now subtract from D the left translation by a. Then we may assume that δ annihilates $\mathbf{R}^S(L)\mathbf{R}_0^T(L)$ and also the elementary functions, which are the linear combinations of the g_i with $i > m$. The condition imposed on D means that $\delta(\exp(f)) = \delta(f)$, for every elementary function on $U(L) \otimes F^*$. Hence we have now $\delta(\exp(cf)) = 0$, for every elementary function f on $U(L)$ and every $c \in F^*$. Thus δ annihilates the trigonometric functions on $U(L)$, whence δ annihilates all of $\mathbf{R}^*(L)$. We may now conclude from Theorem 5 that $\delta = 0$. For the originally given derivation D, this means that D is the left translation by $s + a$, and this completes the proof of Theorem 6.

Let D be a proper derivation, and let δ be the corresponding differentiation $f \to D(f)(1)$. One checks immediately that, for every elementary function f, $D(f)$ coincides with the constant $\delta(f)$, and $D(\exp(f)) = \delta(\exp(f))\exp(f)$. It follows that the commutator algebra of the algebra of all proper derivations annihilates the elementary functions and their exponentials. Hence it follows at once from Theorem 6 that this commutator algebra lies in the algebra of the left translations by the elements of L. Hence *the image of L*

in the Lie algebra of all proper derivations of $\mathbf{R}(L)$ *is an ideal, and the factor algebra is abelian.*

7. Algebras with nilpotent radical

Now let us assume that the radical A of the Lie algebra L is nilpotent. Let N denote the subalgebra of $\mathbf{R}(L)$ consisting of the representative functions associated with representations of L that are nilpotent on A. Evidently, $\mathbf{R}^S(L)$ is contained in N, and so are the elementary functions. Since A is nilpotent, the nilpotent representations of T can be extended in the standard fashion to *nilpotent* representations of A and from there to representations of L that are still nilpotent on A. It follows that, in the notation of Theorem 5, $\mathbf{R}_0^T(L)$ is contained in N. Thus N contains $\mathbf{R}^S(L)\mathbf{R}_0^T(L)$ and the elementary functions. We shall prove that N coincides with the algebra generated by $\mathbf{R}^S(L)\mathbf{R}_0^T(L)$ and the elementary functions. It is easily seen that, in doing this, we may suppose without loss of generality that F is algebraically closed.

By Theorem 5, it is clear that $\mathbf{R}(L)$ is generated by N and the trigonometric functions. Let H denote the multiplicative group consisting of the functions $\exp(f)$, where f ranges over all elementary functions. Since we assume that F is algebraically closed, the trigonometric functions become simply the linear combinations of the elements of H. Hence it is clear that, if N were strictly larger than the algebra generated by $\mathbf{R}^S(L)\mathbf{R}^T(L)$ and the elementary functions, the elements of H could not be free over N. Hence it suffices to show that the elements of H are free over N.

We have $H \subset \mathbf{R}^A(L)$, and it is easily seen, using that $A\cdot\mathbf{R}^S(L) = (0)$, that $N = \mathbf{R}^S(L) \otimes M$, where $M = N \cap \mathbf{R}^A(L)$. Hence it suffices to show that the elements of H are free over M. Let H_1 and N_1 denote the images of H and N, respectively, in $\mathbf{R}(A)$, by the restriction map. Then N_1 coincides with the restriction image of M, and since the restriction map is a monomorphism on $\mathbf{R}^A(L)$, we see that it will suffice to prove that the elements of H_1 are free over N_1.

Suppose that the elements of H_1 are not free over N_1, and let

$$\textstyle\sum_{i=1}^n t_i h_i = 0$$

be a nontrivial relation with the minimal number n of terms, where $t_i \in N_1$ and $h_i \in H_1$. We can evidently arrange to have $h_1 = 1$. By the minimality of n, the h_i are all distinct, and the t_i are not 0. Moreover, we must evidently have $n > 1$. Let $x \in A$. Then we have

$$x\cdot(t_i h_i) = (x\cdot t_i)h_i + t_i(x\cdot h_i) = (x\cdot t_i + h_i(x)t_i)h_i\,.$$

The last expression is of the form $s_i h_i$, where $s_i \in N_1$. Repeating this, we find that, for every $u \in U(A)$, there is an $s_i \in N_1$ such that

$$u\cdot(t_i h_i) = s_i h_i\,,$$

and $s_i(1) = (t_i h_i)(u)$. Now there is a positive integer q such that $u \cdot t_1 = 0$, for every $u \in A^q U(A)$. Applying the left translation by such an element u to the given relation, we find, because of the minimality of n, that each s_i must be 0. In particular, $s_2(1) = 0$, i.e., $(t_2 h_2)(u) = 0$. Hence the representation of A by left translations in the space $[t_2 h_2]$ spanned by the left translates of $t_2 h_2$ is nilpotent.

By the above, we have $[t_2 h_2] = [t_2]h_2$, and evidently $(A \cdot [t_2])h_2$ is an A-stable subspace of $[t_2]h_2$. The induced representation of A in the factor space must still be nilpotent. On the other hand, our above result for the translate $x \cdot (t_2 h_2)$ shows that, if v denotes the coset of $t_2 h_2$ mod. $(A \cdot [t_2])h_2$, we have $x \cdot v = h_2(x)v$, for every $x \in A$, and hence also $u \cdot v = h_2(u)v$, for every $u \in U(A)$. Since $v \neq 0$ (because the representation of A in $[t_2]$ is nilpotent), this shows that h_2 must be a constant, i.e., $h_2 = h_1$, which is a contradiction.

Thus we have proved the following result.

THEOREM 7. *Suppose that the radical A of the Lie algebra L is nilpotent. Then the algebra N of the representative functions associated with representations that are nilpotent on A is generated by $\mathbf{R}^S(L)\mathbf{R}_0^T(L)$ and the elementary functions. Moreover, if C is the algebra of the trigonometric functions, we have $\mathbf{R}(L) = CN \approx C \otimes N$.*

Observe that both C and N are stable under the right and left translations, and hence also under every proper derivation. It is clear from the proof of Theorem 6 that *every proper derivation of N is the left translation by an element of L.* Hence we see that *the Lie algebra of all proper derivations of $\mathbf{R}(L)$ contains the Lie algebra of the left translations by elements of L as a direct summand, the complementary summand being isomorphic (by restriction) with the Lie algebra of all proper derivations of C.*

As an easy consequence of our above results, we obtain the following well known result. Let L be an arbitrary Lie algebra, and let ρ be a representation of L such that the restriction of ρ to the radical of L is a nilpotent representation. Then $\rho(L)$ is an algebraic Lie algebra.

In proving this, we may replace L by $\rho(L)$, so that we may assume that the radical of L is a nilpotent Lie algebra. By assumption, we have $S_\rho \subset N$. By Theorem 4, every proper derivation of S_ρ can be extended to a proper derivation of $\mathbf{R}(L)$. By the above, this derivation of $\mathbf{R}(L)$ coincides on N with the left translation by an element of L. Hence every proper derivation of S_ρ is the left translation by an element of L. By Theorem 2, this implies that $\rho(L)$ is algebraic.

It is an immediate corollary that, if L is any linear Lie algebra, $[L, L]$ is an algebraic Lie algebra.

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