ABSORBING BARRIER PROCESSES CONNECTED WITH THE SYMMETRIC STABLE DENSITIES

BY

JOANNE ELLIOTT[1]

Introduction

To each α, $0 < \alpha \leqq 2$, there corresponds a symmetric stable density K_α defined by

$$(0.1) \qquad K_\alpha(x) = \pi^{-1} \int_0^\infty \cos xz \cdot e^{-|z|^\alpha}\, dz.$$

Associated with K_α are the transition probabilities

$$(0.2) \qquad \mathfrak{P}_\alpha(t, x, S) = t^{-1/\alpha} \int_S K_\alpha(t^{-1/\alpha}[x - y])\, dy,$$

where $t > 0$, $-\infty < x < \infty$, and S is a Borel subset of the real line. These transition probabilities define a separable Markovian process $\{X(t)\}$ if the distribution of $X(0)$ is prescribed. With probability 1 the path functions are continuous except for jumps (cf. [2] p. 422), and we shall assume them continuous on the right.

Let (a, b) be a finite open interval. For any path function $X(t)$ with $a < X(0) = x < b$, let T_X be the first t for which $X(t)$ assumes a value outside of the interval (a, b).

The "α-absorbing barrier process" on (a, b) is derived from the α-process by stopping it at time T_X. Its path functions are given by

$$(0.3) \qquad X_{\text{abs}}(t) = X(t), \qquad t < T_X .$$

In this paper we shall consider the interval $(-1, 1)$ for simplicity. A simple change of scale is involved in translating these results to an arbitrary (a, b). We were led to consider the α-absorbing barrier process by a paper of M. Kac [4], in which he discusses a method of calculating the distribution of the random variable T_X. D. Ray, in a forthcoming paper, has obtained results closely connected with ours, for a semi-infinite interval.

We shall denote the transition probabilities of the α-absorbing barrier process by $\mathfrak{A}_\alpha(t, x, S)$ and the corresponding densities (which will be shown to exist) by $\mathcal{A}_\alpha(t, x, y)$. Furthermore, we shall also use the notation

$$(0.4) \qquad \mathcal{P}_\alpha(t, z) = t^{-1/\alpha} K_\alpha(t^{-1/\alpha} z),$$

where K_α is given in (0.1).

Our main results may be outlined as follows:

(a) *Relation between the stable processes and the absorbing barrier processes*

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for $0 < \alpha < 1$. In the α-process a particle can move from a point $x \in (-1, 1)$ to a Borel subset S of $(-1, 1)$ in one of two ways:[2] either the particle moves from x to S in time t without leaving the interval $(-1, 1)$, or at some intermediate time τ it jumps out of the interval and returns in time $t - \tau$. The probability of the first event is $\mathfrak{A}_\alpha(t, x, S)$, the α-absorbing barrier transition probability. To derive the probability of the second event heuristically, we note (cf. Lemma 1.3) that for $0 < \alpha < 1$,

$$\lim_{t\to 0} t^{-1}\mathcal{P}_\alpha(t, x - y) = C(\alpha)| x - y |^{-1-\alpha}, \tag{0.5}$$

where $C(\alpha)$ is a constant. This makes it plausible that the following relationship should hold between $\mathfrak{P}_\alpha$ and $\mathfrak{A}_\alpha$:

$$\begin{aligned} \mathfrak{P}_\alpha(t, x, S) &= \mathfrak{A}_\alpha(t, x, S) \\ &+ C(\alpha) \int_0^t d\tau \int_{-1}^1 \mathfrak{A}_\alpha(\tau, x, du) \int_{|z|>1} \mathfrak{P}_\alpha(t - \tau, z, S)\cdot| z - u |^{-1-\alpha}\, dz. \end{aligned} \tag{0.6}$$

In Theorem 1.2 we derive (0.6) in terms of the Laplace transforms

$$P_\alpha(\lambda; x - y) = \int_0^\infty e^{-\lambda t}\mathcal{P}_\alpha(t, x - y)\, dt \tag{0.7}$$

and

$$A_\alpha(\lambda; x, y) = \int_0^\infty e^{-\lambda t}\, \mathcal{A}_\alpha(t, x, y)\, dt, \tag{0.8}$$

where $\mathcal{P}_\alpha$ and $\mathcal{A}_\alpha$ are the densities corresponding to $\mathfrak{P}_\alpha$ and $\mathfrak{A}_\alpha$, respectively. We derive this relationship by a limiting process, using (1.1) and (1.2) as a starting point. In [3], we studied the absorbing barrier process for $\alpha = 1$ from a slightly different point of view. There we began by solving the integral equation (1.51) and then proved that the solution must be the Laplace transform of the absorbing barrier process. In the case $0 < \alpha < 1$, which is considered in this paper, we find it more convenient to derive the integral equation directly. The method used is outlined in (d).

(b) *Backward and forward equations for the α-absorbing barrier process and the semigroups associated with them.* If $g \in L(-1, 1)$, the transformation U_t^* defined by (2.46) defines a linear transformation from $L(-1, 1)$ to itself. If g is the probability density of $X_{\text{abs}}(0)$ then

$$\Pr\{X_{\text{abs}}(t) \in S\} = \int_S U_t^* g(y)\, dy \tag{0.9}$$

for $t > 0$ and S a Borel subset of $(-1, 1)$.

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[2] Actually, there is a third possibility, namely that the particle proceeds to ± 1 without jumping from an interior point of the interval. In other words, there may exist a τ such that $| X(t) | < 1$ for $t < \tau$ and $\lim_{t\to\tau^-} X(t) = \pm 1$. However, it will follow from our rigorous derivation of (0.6) that the probability of this event is 0.

It is to be expected that $\mathfrak{A}_\alpha$ satisfies the Chapman-Kolmogoroff equation,

$$\mathfrak{A}_\alpha(t + s, x, y) = \int_{-1}^{1} \mathfrak{A}_\alpha(t, x, u)\mathfrak{A}_\alpha(s, u, y)\, du, \tag{0.10}$$

and hence that $\{U_t^*\}$ forms a semigroup. In Section 2 we show that this is indeed the case. We also determine the infinitesimal generator $\bar{\Omega}_\alpha^*$ of this semigroup. For any $g \in$ domain $\bar{\Omega}_\alpha^*$ this gives the relation

$$\begin{aligned} \frac{\partial}{\partial t} U_t^* g(x) &= \bar{\Omega}_\alpha^* \; U_t^* g(x) \\ &= \pi^{-1}\Gamma(\alpha) \sin \frac{\alpha\pi}{2} \int_{-1}^{1} U_t^* g(y) \cdot |y - x|^{-\alpha} \operatorname{sgn}(y - x)\, dy. \end{aligned} \tag{0.11}$$

This is the "forward equation" associated with the α-absorbing barrier process and is the analogue of the well-known diffusion equation for $\alpha = 2$. We begin in Section 2, however, not with the forward equation and $\{U_t^*\}$, but with the semigroup $\{U_t\}$ given by (2.38) on the space $C_0[-1, 1]$ of continuous functions on $[-1, 1]$ which vanish at the endpoints. Using the fact that U_t^* is the adjoint of U_t restricted to $L(-1, 1)$ we obtain the result mentioned above. The equation (0.11) with U_t^* replaced by U_t and $g \in C_0[-1, 1]$ is the "backward equation" for the α-absorbing barrier process.

(c) *Mean absorption time.* In Theorem 2.7 we show that if a particle starts from some point $x \in (-1, 1)$ and continues according to the α-process until the time T_X when it first leaves $(-1, 1)$, the expected value of T_X is given by (2.47). In other words, (2.47) gives the expected duration of the α-absorbing barrier process.

(d) *Outline of method.* The method used may be briefly described as follows: we first consider an approximation to the α-absorbing barrier transition probabilities $\mathfrak{A}_\alpha(t, x, S)$. Let $\mathfrak{A}_\alpha(n, \Delta; x, S)$ be the probability that in the α-process, a particle starting at some point $x \in (-1, 1)$ at time $t = 0$ will be in the Borel set $S \subset (-1, 1)$ at time $t = n\Delta$, *and will be inside the interval* $(-1, 1)$ *at times* $t_k = k\Delta$, $1 \leqq k < n$. If n and Δ are fixed, then the sequence $\{\mathfrak{A}_\alpha(2^k n, 2^{-k}\Delta; x, S)\}$ is a decreasing sequence as $k \to \infty$. The limiting function, call it $\bar{\mathfrak{A}}_\alpha(t, x, S)$ is the probability that the particle goes from x to S in time t, remaining inside $(-1, 1)$ at all times $\tau < t$ which are of the form $\tau = m\Delta \cdot 2^{-k}$ for some m and k. The set of all τ having this form is dense on $(0, \infty)$. Since the path functions are continuous on the right, we see that $\bar{\mathfrak{A}}_\alpha$ coincides with $\mathfrak{A}_\alpha$.

As an approximation to the integral equation (0.6) we employ the relation between $\mathfrak{A}_\alpha(n, \Delta; x, S)$ and the $\mathfrak{P}_\alpha(n\Delta; x, S)$ given by (1.2). This equation follows directly from the probabilistic definition of $\mathfrak{A}_\alpha$. In Section 1, however, we actually *define* the sequence $\{\mathfrak{A}_\alpha(n, \Delta; x, S)\}$ inductively by (1.1) and (1.2). From this point on we have then to show that in the limit (1.2) gives rise to the integral equation of Theorem 1.2.[3] In Section 2, we make a study of the semigroups $\{U_t\}$ and $\{U_t^*\}$ mentioned above.

In this paper we restrict ourselves to the case $0 < \alpha < 1$. The result for $1 < \alpha < 2$ seems to be similar, but the analytic difficulties are greater, and we have not carried through this case in all detail.

1. Convergence of the approximations

For the probabilistic motivation and interpretation of this and the following section, the reader is referred back to the *Introduction*. From now on, we shall be concerned with the purely analytic side of the problem stated there.

Suppose that $x \in (-1, 1)$ and S is a Borel subset of $(-\infty, \infty)$; for each $\Delta > 0$ and $0 < \alpha < 1$, we define a sequence $\{\mathfrak{A}_\alpha(n, \Delta; x, S)\}$ inductively as follows:

$$
\begin{aligned}
\mathfrak{A}_\alpha(0, \Delta; x, S) &= 1, \qquad && x \in S, \\
&= 0, && x \notin S,
\end{aligned}
\tag{1.1}
$$

and

$$
\begin{aligned}
\mathfrak{P}_\alpha(n\Delta; x, S) &= \mathfrak{A}_\alpha(n, \Delta; x, S) \\
&+ \sum_{k=0}^{n-1} \int_{-1}^{1} \mathfrak{A}_\alpha(k, \Delta; x, du) \int_{|z|>1} \mathfrak{P}_\alpha(\Delta, u, dz)\mathfrak{P}_\alpha[(n-k-1)\Delta, z, S]\, dz,
\end{aligned}
\tag{1.2}
$$

where $\mathfrak{P}_\alpha$ is defined in (0.2).

Since $\mathfrak{P}_\alpha$ satisfies

$$
\mathfrak{P}_\alpha(t + \tau, x, S) = \int_{-\infty}^{\infty} \mathfrak{P}_\alpha(t, x, du)\mathfrak{P}_\alpha(\tau, u, S)\, du,
$$

we easily obtain the following properties of $\mathfrak{A}_\alpha(n, \Delta; x, S)$:

$$
\begin{aligned}
\mathfrak{A}_\alpha(1, \Delta; x, S) &= \mathfrak{P}_\alpha(\Delta, x, S), \qquad && S \subset (-1, 1), \\
&= 0, && S \cap (-1, 1) = 0,
\end{aligned}
\tag{1.3}
$$

and

$$
\mathfrak{A}_\alpha(n + 1, \Delta; x, S) = \int_{-1}^{1} \mathfrak{A}_\alpha(n, \Delta; x, dz)\mathfrak{A}_\alpha(1, \Delta; z, S).
\tag{1.4}
$$

Furthermore, it follows from (1.3) and (1.4) by induction that

$$
\mathfrak{A}_\alpha(n, \Delta; x, S) = 0
\tag{1.5}
$$

when $S \cap (-1, 1) = 0$, that

$$
\mathfrak{A}_\alpha(n, \Delta; x, S) \geqq 0
\tag{1.6}
$$

for all Borel sets S, and that for $n \geqq 1$, $\mathfrak{A}_\alpha(n, \Delta; x, S)$ defines an absolutely continuous measure whose density we denote by $\mathfrak{a}_\alpha(n, \Delta; x, y)$. In the

[3] If we knew beforehand that the event described in footnote 2 was a null event, then the derivation of (0.6) might be approached more directly using results of G. A. Hunt [6].

sequel we shall use these densities primarily; consequently, for convenience we shall state (1.2) in terms of densities: if $|x| < 1$, $|y| < 1$, then

$$(1.7)\qquad \begin{aligned} &\mathcal{P}_\alpha(n\Delta; x-y) \\ &= \mathcal{Q}_\alpha(n, \Delta; x, y) + \int_{|z|>1} \mathcal{P}_\alpha(\Delta; x-z)\mathcal{P}_\alpha[(n-1)\Delta, z-y] \\ &+\sum_{k=1}^{n-2}\int_{-1}^{1} \mathcal{Q}_\alpha(k, \Delta; x, u)\, du \int_{|z|>1} \mathcal{P}_\alpha(\Delta, u-z)\mathcal{P}_\alpha[(n-k-1)\Delta, z-y]\, dz, \end{aligned}$$

where $\mathcal{P}_\alpha$ is the density defined in (0.4). We have used here the fact that $\mathfrak{A}_\alpha$ and $\mathfrak{P}_\alpha$ both satisfy (1.1). On the other hand, if $|x| < 1$ and $|y| > 1$, then

$$(1.8)\qquad \begin{aligned} \mathcal{P}_\alpha(n\Delta; x-y) &= \int_{|z|>1} \mathcal{P}_\alpha(\Delta, x-z)\mathcal{P}_\alpha[(n-1)\Delta, z-y] \\ &+\sum_{k=1}^{n-2}\int_{-1}^{1} \mathcal{Q}_\alpha(k, \Delta; x, du) \int_{|z|>1} \mathcal{P}_\alpha(\Delta, u-z)\mathcal{P}_\alpha[(n-k-1)\Delta, z-y]\, dz \\ &+\int_{-1}^{1} \mathcal{Q}_\alpha(n-1, \Delta; x, u)\mathcal{P}_\alpha(\Delta, u-y)\, du. \end{aligned}$$

Before proceeding to the main result of this section, we state some preliminary lemmas:

LEMMA 1.1. *For $n \geqq 1$, $\Delta > 0$ and $x \in (-1, 1)$ we have*

$$(1.9)\qquad \mathcal{Q}_\alpha\left(2n, \frac{\Delta}{2}; x, y\right) \leqq \mathcal{Q}_\alpha(n, \Delta; x, y).$$

Proof. By (1.3) and (1.4), we have

$$(1.10)\qquad \begin{aligned} &\mathcal{Q}_\alpha\left(2, \frac{\Delta}{2}; x, y\right) \\ &= \int_{-1}^{1} \mathcal{P}_\alpha\left(\frac{\Delta}{2}, x-z\right)\mathcal{P}_\alpha\left(\frac{\Delta}{2}, z-y\right) dz \leqq \mathcal{P}_\alpha(\Delta, x-y) = \mathcal{Q}_\alpha(1, \Delta; x, y). \end{aligned}$$

The theorem then follows by induction by using (1.4) for densities.

LEMMA 1.2. *If $\beta > 0$, $0 \leqq \gamma < 1$, and $n \geqq 0$, then*

$$(1.11)\qquad \left| \int_0^\infty e^{(ixv - tv^\beta)}(t^n v^{n\beta-\gamma})\, dv \right| \leqq M|x|^{\gamma-1}$$

where M is a constant independent of t and x.

Proof. We rewrite the integral as

$$(1.12)\qquad \int_0^\infty e^{ixv - tv^\beta}\{t^n v^{n\beta} + C_n\}v^{-\gamma}\, dv - C_n \int_0^\infty e^{ixv-tv^\beta} \cdot v^{-\gamma}\, dv,$$

where $C_n = (n-1)^{n-1}$ if $n \neq 1$ or 0, and $C_1 = 1$, $C_0 = 0$. We need only show that the estimate holds for the first integral, since the second is a special case of the first for $n = 0$. After a change of variable, the first integral becomes for $x > 0$,

$$x^{\gamma-1} \int_0^\infty e^{iv} f(t, x, v)\, dv \tag{1.13}$$

where

$$f(t, x, v) = e^{-tv^\beta x^{-\beta}} \cdot \{t^n v^{n\beta} x^{-n\beta} + C_n\} \cdot v^{-\gamma}. \tag{1.14}$$

For $\beta > 0$ and $t > 0$, $f(t, x, v)$ is a decreasing function of v for $v > 0$ (since $e^{-v}\{v^n + C_n\}$ and $v^{-\gamma}$ are decreasing for $v > 0$), and

$$0 \leqq f(t, x, v) \leqq C_n v^{-\gamma} \tag{1.15}$$

when $n \neq 0$, and

$$0 \leqq f(t, x, v) \leqq v^{-\gamma} \tag{1.16}$$

when $n = 0$. Now, taking an arbitrary $A > 0$ and using the second mean value theorem on the real and imaginary parts of (1.13) in the region of integration $v > A$, we see that the absolute value of (1.13) is bounded by

$$\left\{\int_0^A f(t, x, v)\, dv + 4f(t, x, A)\right\} x^{\gamma-1}, \tag{1.17}$$

for $x > 0$. Now an application of (1.15) and (1.16) proves the theorem for $x > 0$. The proof for $x < 0$ is entirely similar.

LEMMA 1.3. *If $\mathcal{P}_\alpha(t, z)$ is defined as in* (0.4), *then*

$$\left|\frac{\partial}{\partial t} \mathcal{P}_\alpha(t, z)\right| < M |z|^{-1-\alpha}, \qquad t > 0, \tag{1.18}$$

where M is a constant, and

$$\frac{\partial}{\partial t} \mathcal{P}_\alpha(t, z)\Big|_{t=0} = C(\alpha) |z|^{-1-\alpha} \tag{1.19}$$

with

$$C(\alpha) = \pi^{-1} \cdot \Gamma(\alpha + 1) \cdot \sin(\pi\alpha/2). \tag{1.20}$$

Proof. The first part, (1.18), follows as a corollary to Lemma 1.2, for if $t > 0$

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{P}_\alpha(t, z) &= -\pi^{-1} \int_0^\infty \cos vz (v^\alpha e^{-tv^\alpha})\, dv \\ &= \alpha (z\pi)^{-1} \int_0^\infty \sin vz \{v^{\alpha-1}[1 - tv^\alpha] e^{-tv^\alpha}\}\, dv. \end{aligned} \tag{1.21}$$

The result follows immediately since $0 < \alpha < 1$. From (1.21), it follows easily that

$$\lim_{t\to 0} \frac{\partial}{\partial t} \mathcal{P}_\alpha(t, z) = C(\alpha) \left| z \right|^{-1-\alpha} \tag{1.22}$$

(see [1], p. 23).

LEMMA 1.4. *For each* $0 < \alpha < 1$, *there exists a constant* $M(\alpha)$ *such that*

$$\begin{aligned} &| \mathcal{Q}_\alpha(n + 1, \Delta; x, y) - \mathcal{Q}_\alpha(n, \Delta; x, y) | \\ &\qquad \leqq \Delta M(\alpha)[| x - y |^{-1-\alpha} + \min\{(1 - x^2)^{-1-\alpha}, (1 - y^2)^{-1-\alpha}\}]. \end{aligned} \tag{1.23}$$

Proof. For simplicity, we shall use the abbreviation

$$G_\alpha(t_1, t_2 ; z) = \mathcal{P}_\alpha(t_1, z) - \mathcal{P}_\alpha(t_2, z). \tag{1.24}$$

Then by (1.7),

$$\begin{aligned} \mathcal{Q}_\alpha(n + 1, \Delta; x, y) - \mathcal{Q}_\alpha(n, \Delta; x, y) &= G_\alpha(\{n + 1\}\Delta, n\Delta; x - y) \\ &- \sum_{k=1}^{n-2} \int_{-1}^{1} \mathcal{Q}_\alpha(k, \Delta; x, u)\, du \\ &\cdot \int_{|z|>1} \mathcal{P}_\alpha(\Delta, u - z)G(\{n - k\}\Delta, \{n - k - 1\}\Delta; z - y)\, dz \\ &- \int_{|z|>1} \mathcal{P}_\alpha(\Delta, u - z)G_\alpha(n\Delta, \{n - 1\}\Delta; z - y)\, dz \\ &- \int_{-1}^{1} \mathcal{Q}_\alpha(n - 1, \Delta; x, u)\, du \int_{|z|>1} \mathcal{P}_\alpha(\Delta, u - z)\mathcal{P}_\alpha(\Delta, z - y)\, dz. \end{aligned} \tag{1.25}$$

By (1.18),

$$| G_\alpha(t_1, t_2 ; z) | < M| t_1 - t_2 |\cdot| z |^{-1-\alpha}. \tag{1.26}$$

Therefore, using (1.25) and the fact that

$$\int_{-\infty}^{\infty} \mathcal{P}_\alpha(t, z)\, dz = 1, \tag{1.27}$$

we have

$$\begin{aligned} &\left| \mathcal{Q}_\alpha(n + 1, \Delta; x, y) - \mathcal{Q}_\alpha(n, \Delta; x, y) \right| \\ &\qquad \leqq K_1(\alpha)\Delta \cdot \{\left| x - y \right|^{-1-\alpha} + \left| 1 - y^2 \right|^{-1-\alpha}\} \\ &\qquad + K_2(\alpha) \cdot \Delta \cdot (1 - y^2)^{-1-\alpha} \\ &\qquad \cdot \sum_{k=1}^{n-1} \int_{-1}^{1} \mathcal{Q}_\alpha(k, \Delta; x, u) \int_{|z|>1} \mathcal{P}_\alpha(\Delta, u - z)\, dz. \end{aligned} \tag{1.28}$$

But it follows from (1.2) with $S = (-\infty, \infty)$ that

$$\sum_{k=1}^{n-1} \int_{-1}^{1} \mathcal{Q}_\alpha(k, \Delta; x, u)\, du \int_{|z|>1} \mathcal{P}_\alpha(\Delta, u - z)\, dz < 1. \tag{1.29}$$

Since $\mathcal{Q}_\alpha(n, \Delta; x, y) = \mathcal{Q}_\alpha(n, \Delta; y, x)$, it follows that the term $(1 - y^2)^{-1-\alpha}$ in (1.28) can be replaced by $\min[(1 - x^2)^{-1-\alpha}, (1 - y^2)^{-1-\alpha}]$. This completes the proof.

DEFINITION 1.1. *We define*

$$P_\alpha(\lambda, \Delta; z) = \sum_{n=1}^{\infty} e^{-\lambda n \Delta} \mathcal{P}_\alpha(n\Delta, z) \cdot \Delta \tag{1.30}$$

and

$$A_\alpha(\lambda, \Delta; x, y) = \sum_{n=1}^{\infty} e^{-\lambda n \Delta} \mathcal{Q}_\alpha(n, \Delta; x, y) \cdot \Delta \tag{1.31}$$

for each $\lambda \geqq 0$, $\Delta > 0$ *and* $x \neq y$.

That the series in (1.30) and (1.31) are convergent follows from the inequality

$$0 \leqq \mathcal{Q}_\alpha(n, \Delta; x, y) \leqq \mathcal{P}_\alpha(n\Delta, x - y) \tag{1.32}$$

and the following lemma.

LEMMA 1.5. *For all* $\lambda \geqq 0$ *and* $0 < \alpha < 1$,

$$P_\alpha(\lambda, \Delta; z) < C|z|^{\alpha-1} \tag{1.33}$$

where C *is a constant independent of* Δ *and* λ.

Proof. We have

$$\begin{aligned} P_\alpha(\lambda, \Delta, z) &< \pi^{-1}\Delta \sum_{n=1}^{\infty} \int_0^\infty \cos vz e^{-n\Delta v^\alpha}\, dv \\ &= \pi^{-1}\Delta \int_0^\infty \cos vz (e^{\Delta v^\alpha} - 1)^{-1}\, dv. \end{aligned} \tag{1.34}$$

If we let $\Delta^{1/\alpha} v = w$,

$$P_\alpha(\lambda, \Delta, z) < \pi^{-1}\Delta^{1-1/\alpha} \int_0^\infty \cos(zw\Delta^{-1/\alpha})(e^{w^\alpha} - 1)^{-1}\, dw. \tag{1.35}$$

Since $(e^{w^\alpha} - 1)^{-1}$ is positive and decreasing for $w > 0$, we may use the second mean value theorem to obtain an estimate for the integral on the right:

$$\begin{aligned} \int_0^\infty \cos zw(e^{w^\alpha} - 1)^{-1}\, dw &= |z|^{-1} \int_0^\infty \cos w(e^{w^\alpha |z|^{-\alpha}} - 1)^{-1}\, dw \\ &< |z|^{-1}\left\{\int_0^A (e^{w^\alpha |z|^{-\alpha}} - 1)^{-1}\, dw + 2(e^{A^\alpha |z|^{-\alpha}} - 1)^{-1}\right\} < C|z|^{\alpha-1}, \end{aligned} \tag{1.36}$$

where A is an arbitrary positive number.

The next two theorems incorporate the main results of this section.

THEOREM 1.1. *Define*

$$F_\alpha^{(k)}(t,\Delta,x,y) = \mathcal{A}_\alpha(n, 2^{-k}\Delta; x, y) \tag{1.37}$$

for $2^{-k}n\Delta \leqq t < (n+1)2^{-k}\cdot\Delta$ *when* $n > 1$ *and*

$$F_\alpha^{(k)}(t,\Delta,x,y) = 0 \tag{1.38}$$

for $0 \leqq t < 2^{-k}\Delta$. *Then*

$$\lim_{k\to\infty} F_\alpha^{(k)}(t,\Delta,x,y) = \mathcal{A}_\alpha(t,x,y) \tag{1.39}$$

exists and is finite when $x \neq y$. *The limiting function* $\mathcal{A}_\alpha$ *satisfies*

$$|\mathcal{A}_\alpha(t_1, x, y) - \mathcal{A}_\alpha(t_2, x, y)| < f(x,y)|t_1 - t_2|, \tag{1.40}$$

where $f(x,y) \geqq 0$ *is finite for* $x \neq y$, *and*

$$\mathcal{A}_\alpha(t,x,y) = O(t^{-1/\alpha}) \tag{1.41}$$

as $t \to \infty$. *Also for* $\lambda \geqq 0$

$$\begin{aligned}\lim_{k\to\infty}\int_0^\infty e^{-\lambda t}F_\alpha^{(k)}(t,\Delta,x,y)\,dt &= \lim_{k\to\infty} A_\alpha(\lambda, \Delta\cdot 2^{-k}; x, y)\\ &= \int_0^\infty e^{-\lambda t}\mathcal{A}_\alpha(t,x,y)\,dt.\end{aligned} \tag{1.42}$$

Proof. First suppose that $t = 2^{-l}n\Delta$ for some l, n, and Δ. Then

$$F_\alpha^{(k)}(t,\Delta,x,y) = \mathcal{A}_\alpha(2^{k-l}n, 2^{-k}\Delta; x, y) \tag{1.43}$$

for $k \geqq l$. Also by (1.9),

$$F_\alpha^{(k+1)}(t,\Delta,x,y) \leqq F_\alpha^{(k)}(t,\Delta,x,y) \tag{1.44}$$

for $k \geqq l$. Hence, for fixed Δ, x, y ($x \neq y$),

$$\lim_{k\to 0} F_\alpha^{(k)}(t,\Delta,x,y) = \mathcal{A}_\alpha(t,x,y) \tag{1.45}$$

exists and is finite for t of the above form. Such t's are dense on the real line. To show that the limit in (1.45) exists for all t, we shall prove that

$$|F_\alpha^{(k)}(t_1,\Delta,x,y) - F_\alpha^{(k)}(t_2,\Delta,x,y)| < f(x,y)|t_1 - t_2 + 2^{-k}\Delta|, \tag{1.46}$$

where $0 \leqq f(x,y) < \infty$, when $x \neq y$. Suppose that

$$2^{-k}n_i\,\Delta \leqq t_i < 2^{-k}(n_i+1)\Delta, \qquad i = 1, 2. \tag{1.47}$$

Then

$$\begin{aligned}&|F_\alpha^{(k)}(t_1,\Delta,x,y) - F_\alpha^{(k)}(t_2,\Delta,x,y)|\\ &\qquad = |\mathcal{A}_\alpha(n_1, 2^{-k}\Delta; x, y) - \mathcal{A}_\alpha(n_2, 2^{-k}\Delta; x, y)|,\end{aligned} \tag{1.48}$$

and (1.46) follows after an application of (1.23). This completes the proof of (1.39). Clearly, (1.40) also follows from (1.46).

To prove (1.41), we note that

$$\mathcal{A}_\alpha(t, x, y) \leqq \mathcal{P}_\alpha(t, x - y) \leqq \pi^{-1} t^{-1/\alpha} \int_0^\infty e^{-|w|^\alpha}\, dw. \tag{1.49}$$

Finally, to show the convergence of the Laplace transforms in (1.42), we need only show that given $\varepsilon > 0$, we can find a T independent of k, such that

$$\int_T^\infty F^{(k)}(t, \Delta, x, y)\, dt < \varepsilon. \tag{1.50}$$

THEOREM 1.2. *If P_α and A_α are defined as in* (0.7) *and* (0.8) *respectively, then*

$$\begin{aligned} P_\alpha(\lambda; x - y) &= A_\alpha(\lambda; x, y) \\ &\quad + C(\alpha) \int_{-1}^{1} A_\alpha(\lambda; x, u)\, du \int_{|z|>1} P_\alpha(\lambda; z - y) \,|\, z - u \,|^{-1-\alpha}\, dz \end{aligned} \tag{1.51}$$

with $C(\alpha)$ given in (1.20).

Proof. For each Δ, it follows from (1.7) and Definition 1.1 that when $|\, x \,| < 1$, $|\, y \,| < 1$ we have

$$\begin{aligned} P_\alpha(\lambda, \Delta; x - y) &= A_\alpha(\lambda, \Delta; x, y) \\ &+ e^{-\lambda\Delta} \int_{-1}^{1} A_\alpha(\lambda, \Delta; x, u)\, du \int_{|z|>1} \Delta^{-1} \cdot \mathcal{P}_\alpha(\Delta, u - z) P_\alpha(\lambda, \Delta; z - y)\, dz \\ &+ e^{-\lambda\Delta} \int_{|z|>1} \mathcal{P}_\alpha(\Delta, x - z) P_\alpha(\lambda, \Delta; z - y)\, dz. \end{aligned} \tag{1.52}$$

When $|\, x \,| < 1$, $|\, y \,| > 1$, it follows from (1.8) and Definition 1.1 that

$$\begin{aligned} &P_\alpha(\lambda, \Delta; x - y) \\ &= e^{-\lambda\Delta} \Big\{ \int_{-1}^{1} A_\alpha(\lambda, \Delta; x, u)\, du \int_{|z|>1} \Delta^{-1} \mathcal{P}_\alpha(\Delta, u - z) P_\alpha(\lambda, \Delta; z - y)\, dz \\ &\qquad + \int_{-1}^{1} A_\alpha(\lambda, \Delta; x, u) \mathcal{P}_\alpha(\Delta, u - y)\, du \\ &\qquad + \int_{|z|>1} \mathcal{P}_\alpha(\Delta, x - z) P_\alpha(\lambda, \Delta; z - y)\, dz + \Delta \mathcal{P}_\alpha(\Delta, x - y) \Big\}. \end{aligned} \tag{1.53}$$

To prove the theorem, we shall let Δ run through a sequence $\{\Delta_k\}$ with $\Delta_k = \eta \cdot 2^{-k}$, for some fixed η, in (1.52) and (1.53). It will be sufficient to show that

$$\begin{aligned} \lim_{k\to\infty} &\int_{-1}^{1} A_\alpha(\lambda, \Delta_k; x, u)\, du \int_{|z|>1} \Delta_k^{-1} \mathcal{P}_\alpha(\Delta_k, u - z) P_\alpha(\lambda, \Delta_k; z - y)\, dz \\ &= C(\alpha) \int_{-1}^{1} A_\alpha(\lambda; x, u)\, du \int_{|z|>1} P_\alpha(\lambda; z - y) \,|\, z - u \,|^{-1-\alpha}\, dz, \end{aligned} \tag{1.54}$$

since $\lim_{k\to\infty} P_\alpha(\lambda, \Delta_k ; x - y) = P_\alpha(\lambda; x - y)$ and the extraneous terms in (1.52) and (1.53) are $O(\Delta_k)$. The result then follows from dominated convergence by using Lemmas 1.3 and 1.4 with the inequality

$$A_\alpha(\lambda; x, y) \leqq P_\alpha(\lambda; x - y). \tag{1.55}$$

2. The backward and forward equations

Associated with each of the stable symmetric densities of order α, there is a semigroup $\{T_t\}$ on the space of totally finite measures to itself given by

$$T_t\, \mu(S) = \int_S dx \int_{-\infty}^{\infty} \mathcal{P}_\alpha(t, x - y)\mu(dy), \tag{2.1}$$

where S is a Borel subset of the real line. The resolvent R_λ of the semigroup is the transformation defined by

$$M_\lambda(S) = R_\lambda\, \mu(S) = \int_0^\infty e^{-\lambda t} T_t\, \mu(S)\, dt. \tag{2.2}$$

Both $T_t\, \mu$ and $R_\lambda\, \mu$ are absolutely continuous measures.

If μ is a totally finite measure, we shall use the notation

$$\tilde{\mu}(x) = \int_{-\infty}^{\infty} e^{ixt} \mu(dt), \tag{2.3}$$

and if $f \in L(-\infty, \infty)$, we shall write

$$\tilde{f}(x) = \int_{-\infty}^{\infty} e^{ixt} f(t)\, dt. \tag{2.4}$$

In the next three theorems we state some properties of the semigroups associated with the symmetric stable densities.

THEOREM 2.1. *If μ is a totally finite measure, and M_λ is defined as in* (2.2), *then for $\lambda \geqq 0$*

$$\tilde{M}_\lambda(x) = \tilde{\mu}(x)(\lambda + |x|^\alpha)^{-1}. \tag{2.5}$$

Proof. This follows directly from the definitions of R_λ and T_t by using (0.4) and (0.1).

As a matter of notation we let

$$\mu(x) = \mu\{(-\infty, x)\} \tag{2.6}$$

whenever μ is a totally finite measure on the real line.

THEOREM 2.2. *If x_1 and x_2 are points of continuity of the totally finite measure μ, then for each $\lambda \geqq 0$*

$$\begin{aligned} &\lambda[R_\lambda\, \mu(x_2) - R_\lambda\, \mu(x_1)] - [\mu(x_2) - \mu(x_1)] \\ &\qquad = -i(2\pi)^{-1} \lim_{T\to\infty} \int_{-T}^{T} (e^{-ix_2 u} - e^{-ix_1 u}) \cdot \tilde{M}_\lambda(u) \cdot |u|^{\alpha-1} \operatorname{sgn} u\, du, \end{aligned} \tag{2.7}$$

where M_λ is defined as in (2.2).

Proof. We have from (2.5),

$$\lambda \tilde{M}_\lambda(x) - \tilde{\mu}(x) = -|x|^\alpha(\lambda + |x|^\alpha)^{-1}\tilde{\mu}(x) = -\tilde{M}_\lambda(x)\cdot|x|^\alpha. \tag{2.8}$$

Since $R_\lambda\,\mu$ is an absolutely continuous measure, and x_1 and x_2 are points of continuity of μ, we can apply the Lévy inversion formula to (2.8) to obtain (2.7).

COROLLARY 2.1. *If μ is an absolutely continuous measure with density $f \in L(-\infty, \infty)$, then for each $\lambda \geqq 0$*

$$\begin{aligned}\lambda \frac{d}{dx} R_\lambda\,\mu(x) &- f(x)\\ &= i(2\pi)^{-1}\frac{d}{dx}\lim_{T\to\infty}\int_{-T}^{T}(e^{-ixu} - 1)\cdot\tilde{M}_\lambda(x)\cdot|u|^{\alpha-1}\operatorname{sgn} u\,du.\end{aligned} \tag{2.9}$$

In the sequel we shall mainly use the special case in which μ is an absolutely continuous measure. To this end we shall state the following theorem.

THEOREM 2.3. *The transformations*

$$T_t\,f(x) = \int_{-\infty}^{\infty}\mathcal{P}_\alpha(t, x - y)f(y)\,dy \tag{2.10}$$

define a semigroup from $L(-\infty, \infty)$ to itself, $\mathcal{P}_\alpha$ being defined by (0.4). *The infinitesimal generator is of the form*

$$\Omega_\alpha^*\,\Phi(x) = \pi^{-1}\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\frac{d}{dx}\int_{-\infty}^{\infty}\Phi(u)\cdot|u - x|^{-\alpha}\operatorname{sgn}(u - x)\,du. \tag{2.11}$$

Proof. It follows from Corollary 2.1 that the infinitesimal generator of the semigroup (2.10) is

$$\Omega_\alpha^*\,\Phi(x) = -i(2\pi)^{-1}\frac{d}{dx}\lim_{T\to\infty}\int_{-T}^{T}(e^{-ixv} - 1)\tilde{\Phi}(v)\,|v|^{\alpha-1}\operatorname{sgn} v\,dv. \tag{2.12}$$

We shall first show that

$$\lim_{T\to\infty}\int_{-T}^{T}\tilde{\Phi}(v)\,|v|^{\alpha-1}\operatorname{sgn} v\,dv \tag{2.13}$$

exists.

If $\Phi \in$ domain Ω_α^*, then for some $\lambda > 0$, Φ is also in the range of the resolvent of the semigroup. Therefore, by Theorem 2.1, we have $\tilde{\Phi}(v) = \tilde{f}(v)(\lambda + |v|^\alpha)^{-1}$ for some $f \in L(-\infty, \infty)$ and some $\lambda > 0$. Hence,

$$\begin{aligned}\int_{-T}^{T}\tilde{\Phi}(v)\cdot|v|^{\alpha-1}\operatorname{sgn} v\,dv &= 2i\int_{-\infty}^{\infty}f(x)\,dx\int_{0}^{T}\sin vx(\lambda + v^\alpha)^{-1}v^{\alpha-1}\,dv\\ &= 2i\int_{-\infty}^{\infty}f(x)G_T(x)\,dx.\end{aligned} \tag{2.14}$$

Since $|G_T(x)| < M < \infty$ and $\lim_{T\to\infty} G_T(x)$ exists, we conclude by dominated convergence that the limit in (2.13) exists as $T \to \infty$. The infinitesimal generator can thus be written

$$\begin{aligned}\Omega_\alpha^* \Phi(x) &= (2\pi i)^{-1} \frac{d}{dx} \lim_{T\to\infty} \int_{-T}^{T} e^{-ixv}\tilde{\Phi}(v)\cdot|v|^{\alpha-1} \operatorname{sgn} v \, dv \\ &= (2\pi)^{-1} \frac{d}{dx} \lim_{T\to\infty} \int_{-\infty}^{\infty} \Phi(u)\, du \int_{-T}^{T} \sin v(u-x)\cdot|v|^{\alpha-1} \operatorname{sgn} v\, dv \\ &= \pi^{-1}\Gamma(\alpha)\cdot \sin\frac{\alpha\pi}{2} \frac{d}{dx} \int_{-\infty}^{\infty} \Phi(u)\,|u-x|^{-\alpha} \operatorname{sgn}(u-x)\, du\end{aligned} \tag{2.15}$$

for almost all x.

We now turn to the semigroups associated with the α-absorbing barrier process.

THEOREM 2.4. *For each $\lambda \geqq 0$, $0 < \alpha < 1$ and $f \in C[-1, 1]$, the equation*

$$\lambda F_\lambda(x) - \pi^{-1}\cdot\Gamma(\alpha)\cdot\sin\frac{\pi\alpha}{2}\frac{d}{dx}\int_{-1}^{1} F_\lambda(u)|u-x|^{-\alpha}\operatorname{sgn}(u-x)\,du = f(x) \tag{2.16}$$

has a unique solution $F_\lambda \in C[-1, 1]$ given by

$$F_\lambda(x) = \int_{-1}^{1} A_\alpha(\lambda; x, y) f(y)\, dy. \tag{2.17}$$

Furthermore,

$$\int_{-1}^{1} A_\alpha(0; x, y)\, dy = [\Gamma(\alpha+1)]^{-1}(1-x^2)^{\alpha/2}, \tag{2.18}$$

and for all $\lambda \geqq 0$,

$$|F_\lambda(x)| \leqq \|f\|\cdot[\Gamma(\alpha+1)]^{-1}(1-x^2)^{\alpha/2}, \tag{2.19}$$

if we use the norm of f in $C[-1, 1]$.

Proof. Let

$$\Phi_\lambda(x) = \int_{-1}^{1} P_\alpha(\lambda; x-y) f(y)\, dy. \tag{2.20}$$

From (1.51) by defining F_λ as in (2.17),

$$\Phi_\lambda(y) = F_\lambda(y) + C(\alpha)\int_{|z|>1} P_\alpha(\lambda; z-y)\, dz \int_{-1}^{1} F_\lambda(x)\,|x-z|^{-1-\alpha}\, dx. \tag{2.21}$$

For simplicity, we put

$$\begin{aligned} g(z) &= 0, & |z| < 1, \\ &= \int_{-1}^{1} \frac{F_\lambda(x)\, dx}{|x-z|^{1+\alpha}}, & |z| > 1. \end{aligned} \tag{2.22}$$

It is easily seen that $g \in L(-\infty, \infty)$, since by (1.55) and Lemma 1.5, $F_\lambda(x)$ is bounded for all $\lambda \geqq 0$. It follows from Theorem 2.3 that whenever

$h \in L(-\infty, \infty)$ the function ψ defined by

$$\psi(x) = \int_{-\infty}^{\infty} P_\alpha(\lambda; x - y)h(y)\, dy \tag{2.23}$$

satisfies $\lambda\psi - \Omega_\alpha^* \psi = h$. We conclude, therefore, from (2.21) that

$$f(x) = \lambda\Phi_\lambda(x) - \Omega_\alpha^* \Phi_\lambda(x) = \lambda F_\lambda(x) - \Omega_\alpha^* F_\lambda(x) \tag{2.24}$$

for $| x | < 1$. This establishes that F_λ is a solution of (2.16).

We next show that F_λ is continuous in the open interval $(-1, 1)$. We shall do this by showing that the left and right members of (2.21) are in $C(-1, 1)$. It is easily shown that $\Phi_\lambda \in C(-1, 1)$ by dominated convergence with the use of Lemma 1.5. That the right member is also in $C(-1, 1)$ follows from dominated convergence with Lemma 1.5 and the fact that the function g defined in (2.22) is integrable over $(-\infty, \infty)$.

So far we have shown that for $\lambda \geqq 0$, F_λ is a bounded solution of (2.16), continuous in $(-1, 1)$. We shall have to use a rather circuitous argument to show the continuity at the endpoints. We shall first prove that when $\lambda = 0$, there is only one bounded solution of (2.16) in $C(-1, 1)$; in other words

$$\frac{d}{dx} \int_{-1}^{1} F(\xi) \,| \xi - x |^{-\alpha} \operatorname{sgn} (\xi - x)\, d\xi = 0 \tag{2.25}$$

has no nontrivial solutions in $C(-1, 1)$. Suppose that such an F exists, and let F^+ and F^- be the positive and negative parts of F. The function ϕ defined by

$$\phi(x) = \int_{-1}^{1} F^+(\xi) \,| \xi - x |^{-\alpha} \operatorname{sgn} (\xi - x)\, d\xi \tag{2.26}$$

is in $C[-1, 1]$. Let x be interior to an interval in which $F(x) \leqq 0$. We may assume without loss of generality that $F^+(x) \not\equiv 0$. Then

$$\phi'(x) = \alpha \int_{-1}^{1} F^+(\xi) \,| x - \xi |^{-1-\alpha}\, d\xi > 0. \tag{2.27}$$

On the other hand, if x is interior to an interval in which $F(x) \geqq 0$, then

$$\frac{d}{dx} \int_{-1}^{1} F^-(\xi) \,| \xi - x |^{-\alpha} \operatorname{sgn} (\xi - x)\, d\xi \leqq 0, \tag{2.28}$$

and therefore $\phi'(x) \geqq 0$. Since $\phi \in C[-1, 1]$, it follows that ϕ is a nondecreasing function, but this is impossible since

$$\phi(-1) = \int_{-1}^{1} F^+(\xi) \cdot | \xi + 1 |^{-\alpha}\, d\xi > 0 \tag{2.29}$$

and

$$\phi(1) = -\int_{-1}^{1} F^+(\xi) \,| \xi - 1 |^{-\alpha}\, d\xi < 0. \tag{2.30}$$

We are now ready to prove (2.18) and (2.19), which will also establish that

$$\lim_{x\to\pm1} F_\lambda(x) = 0, \tag{2.31}$$

thus proving that $F_\lambda \in C[-1, 1]$.

It is proved in [5] that

$$\begin{aligned} &-\{\Gamma(\alpha+1)\pi\}^{-1}\Gamma(\alpha)\sin\frac{\alpha\pi}{2} \\ &\qquad\cdot\frac{d}{dx}\int_{-1}^{1}(1-u^2)^{\alpha/2}\cdot|u-x|^{-\alpha}\cdot\operatorname{sgn}(u-x)\,du \equiv 1, \end{aligned} \tag{2.32}$$

when $|x| \leqq 1$. By our uniqueness proof above, it follows that (2.18) must hold. Since $A_\alpha(\lambda; x, y) \geqq 0$ and decreasing in λ for all $\lambda \geqq 0$, the inequality (2.19) is an immediate consequence of (2.18).

This completes the proof of the theorem except for the uniqueness when $\lambda > 0$. Suppose that there exists a solution $G \in C[-1, 1]$ of (2.16) with $f \equiv 0$ in $[-1, 1]$. We then define

$$H(x) = \textstyle\sum_{n=0}^{\infty}(-\lambda)^n S_\lambda^n\, G(x), \tag{2.33}$$

where

$$S_\lambda G(x) = {}'\!\int_{-1}^{1} A_\alpha(\lambda; x, y)G(y)\,dy. \tag{2.34}$$

The series in (2.33) is meaningful, since for any $G \in C[-1, 1]$,

$$|S_\lambda G(x)| < \left|\int_{-1}^{1} P_\alpha(\lambda; x-y)G(y)\,dy\right| < \|G\|\cdot\lambda^{-1}. \tag{2.35}$$

Hence, as a transformation from $C[-1, 1]$ to itself,

$$\|S_\lambda\| < \lambda^{-1}. \tag{2.36}$$

Now H satisfies the equation

$$H(x) - \lambda S_\lambda H(x) = G(x), \tag{2.37}$$

and by the first part of the theorem, this implies that H is a nontrivial solution of (2.25) in $C[-1, 1]$, which we have shown to be impossible. Our proof is now complete.

THEOREM 2.5. *The transformation*

$$U_t f(x) = \int_{-1}^{1} \mathfrak{A}_\alpha(t, x, y)f(y)\,dy \tag{2.38}$$

defines a contraction semigroup on the space $C_0[-1, 1]$ of functions continuous on $[-1, 1]$ and vanishing at ±1, to itself. The infinitesimal generator is given by

$$\bar{\Omega}_\alpha F(x) = \pi^{-1}\Gamma(\alpha)\sin\frac{\alpha\pi}{2}\frac{d}{dx}\int_{-1}^{1} F(u)\,|u-x|^{-\alpha}\operatorname{sgn}(u-x)\,du, \tag{2.39}$$

the domain of $\bar{\Omega}_\alpha$ *consisting of those* $F \in C_0[-1, 1]$ *for which the right side of* (2.39) *is also in* $C_0[-1, 1]$. *The resolvent of the semigroup is given by the transformation* S_λ *of* (2.34).

Proof. For each $f \in C_0[-1, 1]$, we have

$$\int_0^\infty e^{-\lambda t} U_t f(x)\, dt = S_\lambda f(x). \tag{2.40}$$

Theorem 2.4 shows that S_λ is a transformation from $C_0[-1, 1]$ to itself, and that $S_\lambda f = F_\lambda$ is the unique solution in $C_0[-1, 1]$ of (2.16). Therefore, by the Hille-Yosida theorem all we need show in order to prove that S_λ is the resolvent of a contraction semigroup is that the domain of $\bar{\Omega}_\alpha$ in (2.39) is dense in $C_0[-1, 1]$. To accomplish this, we shall show that

$$\lim_{\lambda \to \infty} \| \lambda S_\lambda f - f \| = 0 \tag{2.41}$$

for all $f \in C_0[-1, 1]$. We may consider (2.10) as a semigroup from $C[-\infty, \infty]$ to itself, with resolvent transformation $f \to \Phi_\lambda$ given by

$$\Phi_\lambda(x) = \int_{-\infty}^\infty P_\alpha(\lambda; x - y) f(y)\, dy. \tag{2.42}$$

Hence for any $f \in C[-\infty, \infty]$,

$$\lim_{\lambda \to \infty} \| \lambda \Phi_\lambda f - f \| = 0, \tag{2.43}$$

by using the norm of $C[-\infty, \infty]$. Furthermore, using (2.21), (2.30) and the equation

$$\int_{-\infty}^\infty P_\alpha(\lambda; x - y)\, dy = \lambda^{-1}, \tag{2.44}$$

we have

$$\begin{aligned} &\lambda \int_{|z|>1} P_\alpha(\lambda; z - y) \int_{-1}^1 S_\lambda f(u) \cdot | u - z |^{-1-\alpha}\, du \\ &\qquad \leqq \| f \| \int_{|z|>1} P_\alpha(\lambda; z - y) \int_{-1}^1 | u - z |^{-1-\alpha}\, du \leqq \lambda^{-1} \| f \|. \end{aligned} \tag{2.45}$$

Hence, the left side of (2.45) tends uniformly to 0 as $\lambda \to \infty$. Combining this with (2.43), we conclude that (2.41) holds.

THEOREM 2.6. *The transformation*

$$U_t^* g(x) = \int_{-1}^1 \mathfrak{A}_\alpha(t, x, y) g(y)\, dy \tag{2.46}$$

from $L(-1, 1)$ *to itself is a contraction semigroup whose infinitesimal generator* $\bar{\Omega}_\alpha^*$ *satisfies* (2.39) *with* $\bar{\Omega}_\alpha$ *replaced by* $\bar{\Omega}_\alpha^*$.

Proof. T_t^* is the adjoint transformation to T_t, acting on the space $L(-1, 1)$. Therefore T_t^* is a contraction semigroup on $L(-1, 1)$ to itself.

That its resolvent S_λ^* satisfies the same equation as S_λ is easily proved by the method used in the previous theorem.

We wish to make a final comment on (2.18), which has an interesting probabilistic significance:

THEOREM 2.7. *If T_X is the random variable defined in the Introduction, then*

$$E(T_X) = [\Gamma(\alpha + 1)]^{-1}(1 - x^2)^{\alpha/2}; \tag{2.47}$$

this is the expectation of the time taken for a particle starting at some point $x \in (-1, 1)$ to be absorbed at one of the boundaries ± 1 in the α-absorbing barrier process for $0 < \alpha < 1$.

Proof. Let F_X be the distribution of T_X. Then we have

$$\begin{aligned} \int_0^\infty t\, d\, F_X(t) &= -\int_0^\infty t\, d \int_{-1}^1 \mathcal{A}_\alpha(t, x, y)\, dy \\ &= -t \int_{-1}^1 \mathcal{A}_\alpha(t, x, y)\, dy \Big|_{t=0}^{t=\infty} + \int_{-1}^1 A_\alpha(0, x, y)\, dy. \end{aligned} \tag{2.48}$$

It then follows from (1.41) that the next to the last term in (2.48) vanishes.

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BARNARD COLLEGE, COLUMBIA UNIVERSITY
NEW YORK, NEW YORK