INEQUALITIES OF COMPOUND AND INDUCED MATRICES WITH APPLICATIONS TO COMBINATORIAL ANALYSIS[1]

BY

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1. Introduction

In this paper we study inequalities involving the elementary symmetric functions and the homogeneous product sums of the characteristic roots of a nonnegative hermitian matrix. The inequalities obtained for nonnegative hermitian matrices are applied to problems in combinatorial analysis dealing with matrices all of whose entries are 0's and 1's.

Let A be a matrix with elements in the real or complex field. Throughout the discussion A^T denotes the transpose of A, det A the determinant of A, A^{-1} the inverse of A for det $A \neq 0$, tr (A) the trace of A, $C_r(A)$ the r^{th} compound or adjugate of A, $P_r(A)$ the r^{th} induced or power matrix of A. I denotes the identity matrix, and S denotes the matrix all of whose entries are 1's.

Now let H be a nonnegative hermitian matrix of order v, where $v > 1$. Let the characteristic roots of H be $\lambda_1 \geqq \cdots \geqq \lambda_v$, and let k and λ' satisfy

$$\text{tr}\,(H) = kv, \tag{1.1}$$

$$\lambda_v \leqq k + (v - 1)\lambda' \leqq \lambda_1 . \tag{1.2}$$

Define the matrix B' of order v by

$$B' = (k - \lambda')I + \lambda' S. \tag{1.3}$$

Then we prove that

$$\text{tr}\,(C_r(H)) \leqq \text{tr}\,(C_r(B')). \tag{1.4}$$

Equality holds for $r = 1$. If equality holds for an $r > 1$ and $k + (v - 1)\lambda' \neq 0$, then there exists a unitary U such that $H = U^{-1}B'U$. Let k be defined by (1.1) and let $SHS = \mu S$, where

$$\mu = (k + (v - 1)\lambda^*)v. \tag{1.5}$$

The inequalities (1.2) are valid for $\lambda' = \lambda^*$. Thus if the matrix B^* of order v is defined by

$$B^* = (k - \lambda^*)I + \lambda^* S, \tag{1.6}$$

then

$$\text{tr}\,(C_r(H)) \leqq \text{tr}\,(C_r(B^*)). \tag{1.7}$$

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Moreover, we show that if equality holds in (1.7) for an $r > 1$ and $k + (v - 1)\lambda^* \neq 0$, then $H = B^*$. Analogous results hold for the r^{th} induced or power matrix $P_r(H)$ of H, where for this case $\operatorname{tr}(P_r(H)) \geqq \operatorname{tr}(P_r(B'))$ and $\operatorname{tr}(P_r(H)) \geqq \operatorname{tr}(P_r(B^*))$.

The results described in the preceding paragraph are derived in Section 4. Section 2 summarizes the pertinent literature on compound and induced matrices. In Section 3 we establish some algebraic inequalities involving symmetric functions and homogeneous product sums. These inequalities are essential to the derivations in Section 4. Section 5 is concerned with combinatorial analysis. Let Q be a matrix of order v, all of whose entries are 0's and 1's. The matrix QQ^T is nonnegative symmetric, and the integers $\operatorname{tr}(C_r(QQ^T))$ and $\operatorname{tr}(P_r(QQ^T))$ reflect combinatorial properties of Q. Applications to incidence matrices of v, k, λ configurations are studied in detail, and the author's Theorem 3 on maximal determinants [12] is obtained as a special case of a more general result.

2. Compound and induced matrices

Let A be an n by n matrix with elements in the real or complex field, and let r be an integer such that $1 \leqq r \leqq n$. Let $\{n_r\}$ be the collection of all subsets of r elements chosen from the set $1, \cdots, n$. If σ and τ belong to $\{n_r\}$, and if in the matrix A all rows are deleted whose indices do not belong to σ and all columns are deleted whose indices do not belong to τ, then there remains an r by r submatrix of A, which we denote by $A_{\sigma\tau}$. Let the elements of $\{n_r\}$ be $\sigma_1, \cdots, \sigma_N$, where

$$N = \binom{n}{r} = \frac{n!}{r!\,(n-r)!},$$

and for convenience, let the σ's be ordered lexicographically. The N by N matrix

$$C_r(A) = [\det A_{\sigma_i \sigma_j}] \qquad (i, j = 1, \cdots, N)$$

is called the r^{th} *compound* or the r^{th} *adjugate* of A. We state without proof some of the fundamental properties of $C_r(A)$ [3; 8; 9; 13; 14]:

$$C_r(A)C_r(B) = C_r(AB) \qquad (A \text{ and } B \text{ of order } n), \tag{2.1}$$

$$C_r(A^T) = (C_r(A))^T, \tag{2.2}$$

$$C_r(A^{-1}) = (C_r(A))^{-1} \qquad (\det A \neq 0), \tag{2.3}$$

$$\det C_r(A) = (\det A)^M \qquad \left(M = \binom{n-1}{r-1}\right). \tag{2.4}$$

Let $\alpha_1, \cdots, \alpha_n$ denote the characteristic roots of A. Then the characteristic roots of $C_r(A)$ are

$$\Sigma_1, \Sigma_2, \cdots, \Sigma_N, \tag{2.5}$$

where the Σ_i are the terms in the r^{th} elementary symmetric function of $\alpha_1, \cdots, \alpha_n$. Let

$$a_r(A) = \textstyle\sum_\sigma \det A_{\sigma\sigma},$$

where σ runs through $\{n_r\}$. The $a_r(A)$'s are the coefficients, apart from the signs, of the characteristic polynomial of A

$$\det (xI - A) = x^n - a_1(A)x^{n-1} + a_2(A)x^{n-2} - \cdots + (-1)^n a_n(A).$$

Thus if $\alpha_1, \cdots, \alpha_n$ are the characteristic roots of A, then

$$\prod_{i=1}^{n}(x + \alpha_i) = x^n + a_1(A)x^{n-1} + a_2(A)x^{n-2} + \cdots + a_n(A). \tag{2.6}$$

Note that $a_1(A) = \operatorname{tr}(A)$, $a_n(A) = \det A$, and for r an integer such that $1 \leqq r \leqq n$,

$$a_r(A) = \operatorname{tr}(C_r(A)). \tag{2.7}$$

Let

$$y_i = a_{i1} x_1 + \cdots + a_{in} x_n \qquad (i = 1, \cdots, n), \tag{2.8}$$

where x_i and y_i are indeterminates. Let r be a positive integer, and form the $N^* = \binom{n+r-1}{r}$ products of the y_i's

$$y_1^{\gamma_1} y_2^{\gamma_2} \cdots y_n^{\gamma_n}, \tag{2.9}$$

where $\sum \gamma_i = r$. Order the products (2.9) lexicographically in the sense that the product $y_1^{\gamma_1} y_2^{\gamma_2} \cdots y_n^{\gamma_n}$ stands before the product $y_1^{\delta_1} y_2^{\delta_2} \cdots y_n^{\delta_n}$ provided that the first nonvanishing difference $\gamma_1 - \delta_1, \gamma_2 - \delta_2, \cdots, \gamma_n - \delta_n$ is positive. Denote the products (2.9) written in this order by

$$Y_1, Y_2, \cdots, Y_{N^*},$$

and denote the corresponding products of the x_i's written in the same order by

$$X_1, X_2, \cdots, X_{N^*}.$$

Let X be the column vector with components $X_1, \cdots, X_{N^*}$, and let Y be the column vector with components $Y_1, \cdots, Y_{N^*}$. Then by (2.8),

$$Y = P_r(A)X,$$

where $P_r(A)$ is a matrix of order N^*. This matrix is called the r^{th} *induced matrix* or *power matrix* of A. Many theorems on compound matrices have analogues for induced matrices, and we list the essential formal properties of $P_r(A)$ [7; 8; 9; 13; 14]:

$$P_r(A)P_r(B) = P_r(AB) \qquad (A \text{ and } B \text{ of order } n), \tag{2.10}$$

$$P_r(A^{-1}) = (P_r(A))^{-1} \qquad (\det A \neq 0), \tag{2.11}$$

$$\det P_r(A) = (\det A)^{M^*} \qquad \left(M^* = \binom{n+r-1}{n}\right). \tag{2.12}$$

Let $\alpha_1, \cdots, \alpha_n$ denote the characteristic roots of A. Then the characteristic roots of $P_r(A)$ are

$$\Sigma_1^*, \cdots, \Sigma_{N^*}^*, \tag{2.13}$$

where the Σ_i^* are the terms in the r^{th} homogeneous product sum of $\alpha_1, \cdots, \alpha_n$. Thus if

$$\frac{1}{\prod_{i=1}^{n}(1 - \alpha_i x)} = 1 + h_1 x + h_2 x^2 + \cdots + h_r x^r + \cdots,$$

then for every positive integer r,

$$h_r = \operatorname{tr}(P_r(A)). \tag{2.14}$$

3. Algebraic inequalities

Let $f(x) = \sum a_i x^i$ and $g(x) = \sum b_i x^i$ be polynomials of degree n, where the coefficients a_i and b_i are nonnegative reals. If $a_i \leqq b_i$ $(i = 0, 1, \cdots, n)$, then $f(x)$ is *majorized* by $g(x)$, written

$$f \prec g \quad \text{or} \quad g \succ f. \tag{3.1}$$

If $f(x) = \sum a_i x^i$ and $g(x) = \sum b_i x^i$ are formal power series, we write $f \prec g$ or $g \succ f$ provided $0 \leqq a_i \leqq b_i$ $(i = 0, 1, 2, \cdots)$. It is clear that $f \prec g$ and $f_1 \prec g_1$ imply $ff_1 \prec gg_1$. We now prove the inequalities required in Section 4. Specifically, we study the expressions $\prod(x + \alpha_i)$ and $1/\prod(1 - \alpha_i x)$ for the α_i's nonnegative reals.

LEMMA 3.1. *If $\alpha \geqq \beta \geqq 0$ and $\varepsilon \geqq 0$, then*

$$(x + \alpha + \varepsilon)(x + \beta) \prec (x + \alpha)(x + \beta + \varepsilon).$$

Equality holds for the coefficients of x^2 and x. Equality holds for the coefficient of x^0 if and only if $\alpha = \beta$ or $\varepsilon = 0$.

The proof is immediate.

Lemma 3.1 implies the following well known inequality [6].

LEMMA 3.2. *If $e = (\alpha_1 + \cdots + \alpha_n)/n$ and $\alpha_i \geqq 0$, then*

$$\prod_{i=1}^{n} (x + \alpha_i) \prec (x + e)^n.$$

Equality holds for the coefficients of x^n and x^{n-1}. If equality holds for one of the other coefficients, then each $\alpha_i = e$, and equality holds throughout.

For let $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n \geqq 0$, and in Lemma 3.1, set $\alpha = e$, $\varepsilon = \alpha_1 - e$, $\beta = \alpha_n$. Then

$$(x + \alpha_1)(x + \alpha_n) \prec (x + e)(x + \alpha_1 + \alpha_n - e)$$

and

$$\prod_{i=1}^{n} (x + \alpha_i) \prec (x + e)(x + \alpha_1 + \alpha_n - e) \prod_{i=2}^{n-1} (x + \alpha_i) \prec (x + e)^n. \tag{3.2}$$

Suppose that equality holds throughout (3.2) for some coefficient of x^r, where $r < n - 1$. Then since each $\alpha_i \geqq 0$,

$$\alpha_1 \alpha_n = e(\alpha_1 + \alpha_n - e),$$

whence $\alpha_1 = \cdots = \alpha_n = e$.

Next we derive analogues of Lemmas 3.1 and 3.2 for the formal power series of the form $1/\prod(1 - \alpha_i x)$.

LEMMA 3.3. *If $\alpha \geqq \beta \geqq 0$ and $\varepsilon \geqq 0$, then*

$$\frac{1}{(1 - (\alpha + \varepsilon)x)(1 - \beta x)} \succ \frac{1}{(1 - \alpha x)(1 - (\beta + \varepsilon)x)}.$$

Equality holds for the coefficients of x^0 and x. If equality holds for one of the other coefficients, then $\alpha = \beta$ or $\varepsilon = 0$, and equality holds throughout.

By direct multiplication,

$$\frac{1}{(1 - (\alpha + \varepsilon)x)(1 - \beta x)} = \sum_{r=0}^{\infty} \sum_{i=0}^{r} (\alpha + \varepsilon)^i \beta^{r-i} x^r$$

and

$$\frac{1}{(1 - \alpha x)(1 - (\beta + \varepsilon)x)} = \sum_{r=0}^{\infty} \sum_{i=0}^{r} \alpha^i (\beta + \varepsilon)^{r-i} x^r.$$

If

$$w_r = \textstyle\sum_{k=0}^{r} [(\alpha + \varepsilon)^{r-k}\beta^k - (\beta + \varepsilon)^{r-k}\alpha^k],$$

then $w_0 = w_1 = 0$, and we must prove that $w_r \geqq 0$. Let

$$w_{ik} = \tbinom{r-k}{i}\varepsilon^i(\alpha^{r-k-i}\beta^k - \beta^{r-k-i}\alpha^k) \qquad (k = 0, 1, \cdots, r, \quad i = 0, 1, \cdots, r - k).$$

Then

$$w_r = \sum w_{ik},$$

$$2w_r = \sum (w_{ik} + w_{i,r-k-i}) \qquad (k = 0, 1, \cdots, r, \quad i = 0, 1, \cdots, r - k).$$

If $r - k \geqq k + i$, then

$$w_{ik} + w_{i,r-k-i} = \varepsilon^i(\alpha\beta)^k \left(\tbinom{r-k}{i} - \tbinom{k+i}{i}\right)(\alpha^{r-2k-i} - \beta^{r-2k-i}) \geqq 0,$$

and if $r - k < k + i$, then

$$w_{ik} + w_{i,r-k-i} = \varepsilon^i(\alpha\beta)^{r-k-i}\left(\tbinom{r-k}{i} - \tbinom{k+i}{i}\right)(\beta^{-r+2k+i} - \alpha^{-r+2k+i}) \geqq 0,$$

whence $w_r \geqq 0$.

LEMMA 3.4. *If $e = (\alpha_1 + \cdots + \alpha_n)/n$ and $\alpha_i \geqq 0$, then*

$$\frac{1}{\prod_{i=1}^{n} (1 - \alpha_i x)} \succ \frac{1}{(1 - ex)^n}.$$

Equality holds for the coefficients of x^0 and x. If equality holds for one of the other coefficients, then each $\alpha_i = e$, and equality holds throughout.

For let $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n \geqq 0$, and in Lemma 3.3, set $\alpha = e$, $\varepsilon = \alpha_1 - e$, $\beta = \alpha_n$. Then

$$\frac{1}{(1 - \alpha_1 x)(1 - \alpha_n x)} \succ \frac{1}{(1 - ex)(1 - (\alpha_1 + \alpha_n - e)x)}$$

and

$$\frac{1}{\prod_{i=1}^{n}(1-\alpha_i x)} > \frac{1}{(1-ex)(1-(\alpha_1+\alpha_n-e)x)\prod_{i=2}^{n-1}(1-\alpha_i x)},$$

whence the result follows.

4. Hermitian matrices

We now study inequalities involving tr $(C_r(H))$ and tr $(P_r(H))$, where the matrix H is nonnegative hermitian. Define the matrix B of order v by the equation

$$B = (k - \lambda)I + \lambda S. \tag{4.1}$$

Here k and λ are real numbers, I is the identity matrix, and S is the matrix all of whose entries are 1's. We select $v > 1$ and note that B is the matrix with k in the main diagonal and λ in all other positions. The characteristic polynomial of B is easily computed by subtracting column one of det $(xI - B)$ from each of the other columns, and then adding to row one each of the remaining rows. Thus

$$\det (xI - B) = (x - (k + (v - 1)\lambda))(x - (k - \lambda))^{v-1}, \tag{4.2}$$

and hence the v characteristic roots of B are $k + (v - 1)\lambda$ taken once and $k - \lambda$ taken $v - 1$ times. Note that

$$\det B = (k + (v - 1)\lambda)(k - \lambda)^{v-1}. \tag{4.3}$$

It is now easy to evaluate tr $(C_r(B))$ and tr $(P_r(B))$ explicitly. Evidently,

$$\operatorname{tr} (C_r(B)) = \binom{v}{r} (k + (r - 1)\lambda)(k - \lambda)^{r-1} \tag{4.4}$$

and

$$\operatorname{tr} (P_r(B)) = \sum_{i=0}^{r} \binom{v+i-2}{i} (k + (v - 1)\lambda)^{r-i} (k - \lambda)^{i}. \tag{4.5}$$

Let H be a nonnegative hermitian matrix of order v, where $v > 1$. Let the characteristic roots of H be $\lambda_1, \cdots, \lambda_v$, where

$$\lambda_1 \geqq \cdots \geqq \lambda_v \geqq 0.$$

Let k and λ' satisfy

$$\operatorname{tr} (H) = kv, \tag{4.6}$$

$$\lambda_v \leqq k + (v - 1)\lambda' \leqq \lambda_1 . \tag{4.7}$$

Now define the matrix B' of order v by

$$B' = (k - \lambda')I + \lambda' S. \tag{4.8}$$

Note that by (4.7) the matrix B' is nonnegative hermitian.

THEOREM 4.1. *The matrices H and B' satisfy*

$$\operatorname{tr} (C_r(H)) \leqq \operatorname{tr} (C_r(B')).$$

Equality holds for $r = 1$. *If equality holds for an* $r > 1$ *and* $k + (v - 1)\lambda' \neq 0$, *then there exists a unitary* U *such that*

$$H = U^{-1}B'U.$$

In Lemma 3.1, let $\varepsilon = \lambda_1 - (k + (v - 1)\lambda')$, $\alpha = \lambda_1 - \varepsilon$, and $\beta = \lambda_v$. Then

$$(x + \lambda_1)(x + \lambda_v) \prec (x + k + (v - 1)\lambda')(x + \lambda_v + \varepsilon).$$

Now

$$(\lambda_v + \varepsilon + \lambda_2 + \cdots + \lambda_{v-1})/(v - 1) = (kv - \lambda_1 + \varepsilon)/(v - 1) = k - \lambda'.$$

Thus by Lemma 3.2,

$$\begin{aligned} &(x + \lambda_1)(x + \lambda_v) \textstyle\prod_{i=2}^{v-1} (x + \lambda_i) \\ (4.9) \qquad &\prec (x + (k + (v - 1)\lambda'))(x + \lambda_v + \varepsilon)\textstyle\prod_{i=2}^{v-1} (x + \lambda_i) \\ &\prec (x + (k + (v - 1)\lambda'))(x + (k - \lambda'))^{v-1}, \end{aligned}$$

whence the first conclusion of the theorem follows.

Suppose now that $k + (v - 1)\lambda' \neq 0$ and that equality holds throughout (4.9) for some coefficient of x^r, where $r \neq v, v - 1$. Consider the case in which $k - \lambda' > 0$. Then equality must hold for some coefficient of x^r in

$$(x + \lambda_v + \varepsilon)\textstyle\prod_{i=2}^{v-1} (x + \lambda_i) \prec (x + (k - \lambda'))^{v-1},$$

where $r \neq v - 1, v - 2$. By Lemma 3.2,

$$\lambda_2 = \cdots = \lambda_{v-1} = \lambda_v + \varepsilon = k - \lambda'.$$

Moreover, we must have

$$\lambda_1 \lambda_v = (k + (v - 1)\lambda')(k - \lambda')$$

and

$$\lambda_1 + \lambda_v = (k - \lambda') + (k + (v - 1)\lambda'),$$

whence $\lambda_1 = k + (v - 1)\lambda'$ and $\lambda_v = k - \lambda'$, or $\lambda_1 = k - \lambda'$ and $\lambda_v = k + (v - 1)\lambda'$. If $k - \lambda' = 0$, then $\varepsilon = \lambda_2 = \cdots = \lambda_v = 0$ and $\lambda_1 = kv$. Thus under all possibilities the characteristic roots of H must be $k + (v - 1)\lambda'$ taken once and $k - \lambda'$ taken $v - 1$ times. This means that H and B' have the same characteristic roots, and hence there exists a unitary U such that $H = U^{-1}B'U$.

THEOREM 4.2. *The matrices* H *and* B' *satisfy*

$$\operatorname{tr}(P_r(H)) \geqq \operatorname{tr}(P_r(B')).$$

Equality holds for $r = 1$. *If equality holds for an* $r > 1$, *then there exists a unitary* U *such that*

$$H = U^{-1}B'U.$$

In Lemma 3.3, let $\varepsilon = \lambda_1 - (k + (v - 1)\lambda')$, $\alpha = \lambda_1 - \varepsilon$, and $\beta = \lambda_v$.

Then

$$\frac{1}{(1-\lambda_1 x)(1-\lambda_v x)} > \frac{1}{(1-(k+(v-1)\lambda')x)(1-(\lambda_v+\varepsilon)x)}.$$

Furthermore,

$$\begin{aligned}&\frac{1}{(1-\lambda_1 x)(1-\lambda_v x)\prod_{i=2}^{v-1}(1-\lambda_i x)}\\ (4.10)\qquad &> \frac{1}{(1-(k+(v-1)\lambda')x)(1-(\lambda_v+\varepsilon)x)\prod_{i=2}^{v-1}(1-\lambda_i x)}\\ &> \frac{1}{(1-(k+(v-1)\lambda')x)(1-(k-\lambda')x)^{v-1}}.\end{aligned}$$

Suppose that equality holds throughout (4.10) for some coefficient of x^r, where $r \neq 0, 1$. Then equality must hold for some coefficient of x^r in

$$\frac{1}{(1-(\lambda_v+\varepsilon)x)\prod_{i=2}^{v-1}(1-\lambda_i x)} > \frac{1}{(1-(k-\lambda')x)^{v-1}},$$

where $r \neq 0, 1$. Thus $\lambda_2 = \cdots = \lambda_{v-1} = \lambda_v + \varepsilon = k - \lambda'$. Also equality must hold for some coefficient of x^r in

$$\frac{1}{(1-\lambda_1 x)(1-\lambda_v x)} > \frac{1}{(1-(k+(v-1)\lambda')x)(1-(\lambda_v+\varepsilon)x)},$$

where $r \neq 0, 1$. Thus we must have $\varepsilon = 0$, $\lambda_1 = k + (v-1)\lambda'$, and $\lambda_v = k - \lambda'$, or $\alpha = \beta$, $\lambda_1 = k - \lambda'$, and $\lambda_v = k + (v-1)\lambda'$. Hence the characteristic roots of H are $k + (v-1)\lambda'$ taken once and $k - \lambda'$ taken $v - 1$ times. Thus there exists a unitary U such that $H = U^{-1}B'U$.

Consider the matrix $B = (k-\lambda)I + \lambda S$ of order v, where k is fixed by (4.6) and where

$$(4.11)\qquad -k/(v-1) \leqq \lambda \leqq k.$$

The matrix B is nonnegative hermitian, and tr $(C_r(B))$ and tr $(P_r(B))$ are polynomials in λ. Theorems 4.1 and 4.2 imply that for $r > 1$, tr $(C_r(B))$ is strictly decreasing and tr $(P_r(B))$ is strictly increasing in the interval $0 \leqq \lambda \leqq k$. Also tr $(C_r(B))$ is strictly increasing and tr $(P_r(B))$ is strictly decreasing in the interval $-k/(v-1) \leqq \lambda \leqq 0$. For if $\lambda \geqq 0$, let

$$(4.12)\qquad -\lambda/(v-1) \leqq \lambda' \leqq \lambda,$$

and if $\lambda \leqq 0$, let

$$(4.13)\qquad \lambda \leqq \lambda' \leqq -\lambda/(v-1).$$

Then if $B' = (k-\lambda')I + \lambda' S$, it follows that

$$(4.14)\qquad \mathrm{tr}\,(C_r(B)) \leqq \mathrm{tr}\,(C_r(B'))$$

and

$$\text{tr}\,(P_r(B)) \geqq \text{tr}\,(P_r(B')). \tag{4.15}$$

In Theorems 4.1 and 4.2 the λ' is confined to the interval

$$(\lambda_v - k)/(v-1) \leqq \lambda' \leqq (\lambda_1 - k)/(v-1), \tag{4.16}$$

where λ_1 is the maximal and λ_v is the minimal characteristic root of H. The preceding remarks imply that the best selection for λ' in the theorems from the standpoint of sharpness of approximation is either $(\lambda_1 - k)/(v-1)$ or $(\lambda_v - k)/(v-1)$. However, these values require information concerning the characteristic roots of H. In what follows we select a $\lambda' = \lambda^*$ that satisfies the inequalities (4.16) and is determined by the sum of the nondiagonal elements of H. Moreover, if equality holds in the theorems for the case $\lambda' = \lambda^*$, then the matrices themselves are equal.

Let k be defined by (4.6), and let

$$SHS = \mu S, \tag{4.17}$$

where

$$\mu = (k + (v-1)\lambda^*)v. \tag{4.18}$$

Define the matrix B^* of order v by

$$B^* = (k - \lambda^*)I + \lambda^* S. \tag{4.19}$$

THEOREM 4.3. *The matrices H and B^* satisfy*

$$\text{tr}\,(C_r(H)) \leqq \text{tr}\,(C_r(B^*)).$$

Equality holds for $r = 1$. If equality holds for an $r > 1$ and $k + (v-1)\lambda^* \neq 0$, then

$$H = B^*.$$

Since H is nonnegative hermitian, there exists a matrix P such that

$$H = \bar{P}^T P,$$

where the bar denotes complex conjugate. Let p_i denote the sum of row i of P. Then

$$S\bar{P}^T PS = (p_1 \bar{p}_1 + \cdots + p_v \bar{p}_v)S = SHS = \mu S,$$

whence

$$\mu = p_1 \bar{p}_1 + \cdots + p_v \bar{p}_v\,. \tag{4.20}$$

Now there exists a unitary U such that

$$U \begin{bmatrix} p_1 \\ \vdots \\ p_v \end{bmatrix} = \begin{bmatrix} \sqrt{\mu/v} \\ \vdots \\ \sqrt{\mu/v} \end{bmatrix}.$$

Let $Q = UP$. Then

$$\bar{Q}^T Q = \bar{P}^T \bar{U}^T U P = \bar{P}^T P = H.$$

Moreover,

$$QS = UPS = \sqrt{\mu/v}\, S = \sqrt{k + (v-1)\lambda^*}\, S.$$

Now the characteristic roots of $H = \bar{Q}^T Q$ satisfy

$$\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_v ,$$

and a theorem of Browne [1] asserts that if ρ is a characteristic root of Q, then

$$\lambda_v \leqq \rho\bar{\rho} \leqq \lambda_1 .$$

But since $QS = \sqrt{k + (v-1)\lambda^*}\, S$, we may select $\rho = \sqrt{k + (v-1)\lambda^*}$, whence

$$\lambda_v \leqq k + (v-1)\lambda^* \leqq \lambda_1 . \tag{4.21}$$

Thus by Theorem 4.1,

$$\operatorname{tr}(C_r(H)) \leqq \operatorname{tr}(C_r(B^*)).$$

If equality holds for an $r > 1$ and $k + (v-1)\lambda^* \neq 0$, then there exists a unitary U such that

$$H = \bar{U}^T B^* U = (k - \lambda^*)I + \lambda^* \bar{U}^T S U.$$

Let u_i denote the sum of row i of U and let

$$u = u_1 + \cdots + u_v . \tag{4.22}$$

Then

$$SHS = (k - \lambda^*)vS + \lambda^* u\bar{u} S = (k - \lambda^* + \lambda^* v)vS,$$

and

$$\lambda^* u\bar{u} = \lambda^* v^2.$$

If $\lambda^* = 0$, then $H = B^* = kI$, and if $\lambda^* \neq 0$, then

$$(u_1 + \cdots + u_v)(\bar{u}_1 + \cdots + \bar{u}_v) = v^2. \tag{4.23}$$

Since $\bar{U}^T U = I$,

$$u_1\bar{u}_1 + \cdots + u_v\bar{u}_v = v. \tag{4.24}$$

But Cauchy's inequality implies

$$v^2 = (u_1 + \cdots + u_v)(\bar{u}_1 + \cdots + \bar{u}_v) \leqq (u_1\bar{u}_1 + \cdots + u_v\bar{u}_v)v = v^2.$$

Since equality holds, we must have $u_1 = \cdots = u_v = e$, where $e\bar{e} = 1$. Thus $US = eS$, $e\bar{U}^T S = S$, and $SU = eS = US$. Hence

$$H = (k - \lambda^*)I + \lambda^* \bar{U}^T S U = B^*.$$

THEOREM 4.4. *The matrices H and B^* satisfy*

$$\operatorname{tr}(P_r(H)) \geqq \operatorname{tr}(P_r(B^*)).$$

Equality holds for $r = 1$. If equality holds for an $r > 1$, then

$$H = B^*.$$

This theorem is a consequence of Theorem 4.2 and the preceding discussion.

5. Applications to combinatorial analysis

Let $Q = [q_{ij}]$ be a matrix of order v, all of whose entries are 0's and 1's. Let $v > 1$, and let τ denote the total number of 1's in Q. The matrix Q may be regarded as an incidence matrix for an arrangement of v elements $x_1, \cdots, x_v$ into v sets $S_1, \cdots, S_v$, where $q_{ij} = 1$ if x_j is in S_i, and $q_{ij} = 0$ if x_j is not in S_i. The incidence matrix Q gives a complete description of the combinatorial arrangement of the v elements into the v sets.

With Q we associate the nonnegative symmetric matrix

$$W = QQ^T, \tag{5.1}$$

where

$$\operatorname{tr}(W) = kv = \tau. \tag{5.2}$$

Suppose that we perform arbitrary permutations to the rows and to the columns of Q. This is equivalent to multiplying Q on the left by a permutation matrix P_1 and on the right by a permutation matrix P_2. Now if $Q^* = P_1 QP_2$ and $W^* = Q^*Q^{*T}$, then

$$W^* = P_1WP_1^{-1},$$

and

$$\operatorname{tr}(C_r(W^*)) = \operatorname{tr}(C_r(W)), \tag{5.3}$$

$$\operatorname{tr}(P_r(W^*)) = \operatorname{tr}(P_r(W)). \tag{5.4}$$

Thus both $\operatorname{tr}(C_r(W))$ and $\operatorname{tr}(P_r(W))$ are invariant under arbitrary permutations of the rows and of the columns of Q. Such functions of Q are of combinatorial interest because they describe properties of the arrangement of the v elements into the v sets independent of the particular labelling of elements and sets.

By (2.1) and (2.2),

$$C_r(QQ^T) = C_r(Q)C_r(Q^T) = C_r(Q)(C_r(Q))^T.$$

Thus $\operatorname{tr}(C_r(W))$ is equal to the sum of the squares of the r^{th} order minor determinants of Q. Note that

$$\operatorname{tr}(C_1(W)) = \tau, \tag{5.5}$$

$$\operatorname{tr}(C_2(W)) = \Delta, \tag{5.6}$$

$$\operatorname{tr}(C_v(W)) = (\det Q)^2, \tag{5.7}$$

where Δ denotes the number of 2 by 2 nonsingular submatrices of Q. It is clear that

$$\text{tr}\,(C_r(W)) \geqq 0, \tag{5.8}$$

and in particular, $\Delta \geqq 0$. Moreover, $\Delta = 0$ if and only if by permutations of rows and columns, we may write Q in the form

$$Q = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.9}$$

Here S is the matrix of 1's and is of size e by f, where

$$ef = \tau,$$

and the 0's denote zero blocks.

Also,

$$\text{tr}\,(P_1(W)) = kv \tag{5.10}$$

and

$$\text{tr}\,(P_r(W)) \leqq (\lambda_1 + \cdots + \lambda_v)^r = k^r v^r, \tag{5.11}$$

where $\lambda_1, \cdots, \lambda_v$ are the v characteristic roots of $W = QQ^T$. If equality holds in (5.11) for some $r > 1$, then equality holds in (5.11) for every r, and one characteristic root of W must equal kv and the remaining $v - 1$ characteristic roots must equal 0. But then $\Delta = 0$, and by permutations of rows and columns we may write Q in the form (5.9). Conversely, every Q that by permutations of rows and columns may be written in the form (5.9) satisfies

$$\text{tr}\,(P_r(W)) = k^r v^r$$

for every r.

The previous discussion suggests the study of the arrangement of the τ 1's in Q for tr $(C_r(W))$ maximal, and the related problem for tr $(P_r(W))$ minimal. Such a study will lead us to matrices of considerable combinatorial importance. Moreover, their structure is diametrically unlike those of (5.9). Let tr $(W) = kv = \tau$ and let S be the v by v matrix of 1's. Let $SWS = \mu S$, where

$$\mu = (k + (v - 1)\lambda(Q))v. \tag{5.12}$$

Here $\lambda(Q)$ is a rational number determined by the arrangement of the 0's and 1's within Q. Indeed, if c_i denotes the sum of column i of Q, then

$$\lambda(Q) = \frac{\sum c_i^2 - kv}{v(v - 1)}. \tag{5.13}$$

Now define

$$\lambda = \frac{k(k - 1)}{v - 1}. \tag{5.14}$$

Every 0, 1 matrix Q of order v containing $\tau = kv$ 1's must satisfy

$$\lambda(Q) \geqq \lambda.$$

For we have

$$\sum c_i = kv$$

and

$$k^2v^2 = (\sum c_i)^2 \leqq v \sum c_i^2,$$

whence by (5.13),

$$\lambda(Q) = \frac{\sum c_i^2 - kv}{v(v-1)} \geqq \frac{k(k-1)}{v-1}.$$

Also,

$$\sum c_i^2 \leqq v \sum c_i = kv^2,$$

and by (5.13),

$$\lambda(Q) \leqq \frac{kv^2 - kv}{v(v-1)} = k.$$

Hence it follows that

$$\lambda \leqq \lambda(Q) \leqq k. \tag{5.15}$$

We describe now some special 0, 1 matrices A of order v, called incidence matrices of v, k, λ configurations. Let v elements $x_1, \cdots, x_v$ be arranged into v sets $S_1, \cdots, S_v$ such that every set contains exactly k distinct elements and such that every pair of sets has exactly λ elements in common, $0 < \lambda < k < v$. Such an arrangement is called a v, k, λ *configuration*. Every v, k, λ configuration must satisfy (5.14) [11]. For such a configuration, let $a_{ij} = 1$ if x_j is an element of S_i, and let $a_{ij} = 0$ if x_j is not an element of S_i. The v by v matrix $A = [a_{ij}]$ of 0's and 1's is called the *incidence matrix of the* v, k, λ *configuration*. One verifies easily that if $0 < \lambda < k < v$, then a v, k, λ configuration exists if and only if there exists a 0, 1 matrix A of order v such that

$$AA^T = B = (k - \lambda)I + \lambda S. \tag{5.16}$$

The v, k, λ configurations and their related incidence matrices have been studied very extensively in recent years. The central problem concerns the determination of the precise range of values of v, k, and λ for which configurations exist. Certain nonexistence theorems are established in [2] and [4], and a general survey of the literature is available in [5; 10; 11]. The following theorems show that an incidence matrix A of a v, k, λ configuration has the 0, 1 arrangement with tr $(C_r(AA^T))$ maximal and tr $(P_r(AA^T))$ minimal.

THEOREM 5.1. *Let Q be a 0, 1 matrix of order v, containing exactly $\tau = kv$ 1's. Let $\lambda = k(k-1)/(v-1)$ and $B = (k-\lambda)I + \lambda S$, where $0 < \lambda < k < v$. Then*

$$\text{tr}\,(C_r(QQ^T)) \leqq \text{tr}\,(C_r(B)).$$

Equality holds for $r = 1$. If equality holds for an $r > 1$, then Q is the incidence matrix of a v, k, λ configuration.

By Theorem 4.3,

$$\text{tr}\,(C_r(QQ^T)) \leqq \text{tr}\,(C_r(B^*)),$$

where $B^* = (k - \lambda(Q))I + (\lambda(Q))S$. But by (5.15), $\lambda \leqq \lambda(Q) \leqq k$, and hence

$$\operatorname{tr}(C_r(B^*)) \leqq \operatorname{tr}(C_r(B)).$$

If equality holds in the theorem for an $r > 1$, then

$$QQ^T = B^* = B,$$

and Q is the incidence matrix of a v, k, λ configuration.

Note that Theorem 5.1 implies that

$$(\det Q)^2 \leqq k^2(k - \lambda)^{v-1}, \tag{5.17}$$

where equality holds if and only if Q is the incidence matrix of a v, k, λ configuration [12].

THEOREM 5.2. *Under the hypothesis of Theorem* 5.1,

$$\operatorname{tr}(P_r(QQ^T)) \geqq \operatorname{tr}(P_r(B)).$$

Equality holds for $r = 1$. *If equality holds for an* $r > 1$, *then* Q *is the incidence matrix of a* v, k, λ *configuration.*

By Theorem 4.4,

$$\operatorname{tr}(P_r(QQ^T)) \geqq \operatorname{tr}(P_r(B^*)),$$

and $\lambda \leqq \lambda(Q) \leqq k$ implies $\operatorname{tr}(P_r(B^*)) \geqq \operatorname{tr}(P_r(B))$.

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