SOME EXTREME VALUE RESULTS FOR INDEFINITE HERMITIAN MATRICES[1]

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1. Introduction

Let A be an n-square complex Hermitian matrix, and let $x_1, \cdots, x_k$ be an orthonormal (o.n.) set of vectors in the unitary n-space V_n. In this paper we consider the following two functions:

$$\varphi(x_1, \cdots, x_k) = \prod_{j=1}^{k} (Ax_j, x_j), \tag{1.1}$$

$$\psi(x_1, \cdots, x_k) = E_2((Ax_1, x_1), \cdots, (Ax_k, x_k)), \qquad k \leqq n. \tag{1.2}$$

$E_2(y_1, \cdots, y_k)$ is the second elementary symmetric function of the indicated variables. The problem is to determine the extreme values of the functions φ and ψ as the vectors $x_1, \cdots, x_k$ vary in V_n subject to the restriction

$$(x_i, x_j) = \delta_{ij}.$$

To do this, we examine the structure of extremal sets $x_1, \cdots, x_k$ in terms of invariance under A. We shall consistently use the term "extremal set" to denote a set of extremal vectors, i.e., vectors for which the extreme values of φ and ψ occur. The problem of the minimum for φ when A is nonnegative Hermitian has been solved by K. Fan [4] and later generalized by Amir-Moéz [1]. The maxima for both φ and ψ are contained in [7]. The minimum for ψ, again with A nonnegative Hermitian, has been solved by A. Ostrowski by means of Schur-convex functions [8]. In this paper we will assume that A *has both positive and negative eigenvalues.* The usual techniques do not seem to generalize readily from the case of positive matrices.

2. Invariance results

LEMMA 1. *If A is nonsingular, then an extremal set for φ spans a k-dimensional invariant subspace of A.*

Proof.[2] By the continuity of the inner product it is clear that we may select $y_1, \cdots, y_k$ satisfying $(y_i, y_j) = \delta_{ij}$ such that

$$\min \varphi = \varphi(y_1, \cdots, y_k). \tag{2.1}$$

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[2] This proof is similar to that of Lemma 1 of [7] in which A is assumed positive definite Hermitian.

If the subspace $L(y_1, \cdots, y_k)$ spanned by the y_j is not invariant under A, then we may assume without loss of generality that there exists a unit vector z in the orthogonal complement L^* of L such that

$$(Ay_1, z) = \rho \neq 0. \tag{2.2}$$

Define the set y_j', $j = 1, \cdots, k$, by

$$y_1' = (1 + t^2 |\rho|^2)^{-1/2}(y_1 - t\rho z)$$
$$y_j' = y_j, \qquad j = 2, \cdots, k,$$

where t is a real number. It is clear that y_j' is an o.n. set. Set

$$m(t) = \varphi(y_1', \cdots, y_k').$$

Then

$$\dot{m}(o) = -2|\rho|^2 \prod_{j=2}^{k} (Ay_j, y_j).$$

In view of (2.1) and (2.2), we conclude that

$$\min \varphi = \varphi(y_1, \cdots, y_k) = 0.$$

Let $u_1, \cdots, u_n$ be a set of o.n. eigenvectors of A corresponding respectively to the eigenvalues $\lambda_1, \cdots, \lambda_n$. Then for $1 \leqq i_1 < \cdots < i_{k+1} \leqq n$

$$\varphi(u_{i_1}, u_{i_3}, u_{i_4}, \cdots, u_{i_{k+1}}) = \lambda_{i_1} \prod_{j=3}^{k+1} \lambda_{i_j} > 0,$$
$$\varphi(u_{i_2}, u_{i_3}, u_{i_4}, \cdots, u_{i_{k+1}}) = \lambda_{i_2} \prod_{j=3}^{k+1} \lambda_{i_j} > 0.$$

Consequently $\lambda_{i_1} \lambda_{i_2} > 0$ for any i_1 and i_2. This implies that A is definite, completing the proof for $\min \varphi$. The argument for $\max \varphi$ is the same.

For ψ it is not true that *any* extremal set spans an invariant subspace of A; however we have

LEMMA 2. *There exists an o.n. set $y_1, \cdots, y_k$ such that*
(i) *$L(y_1, \cdots, y_k)$ is an invariant subspace of A;*
(ii) *$\psi(y_1, \cdots, y_k) = \min \psi$.*

Proof. Let $x_1, \cdots, x_k$ be a minimizing set for ψ, and assume there exists $z \in L^*(x_1, \cdots, x_k)$ such that $\|z\| = 1$ and

$$(Ax_1, z) = \rho \neq 0;$$

then set

$$x_1' = (1 + t^2 |\rho|^2)^{-1/2}(x_1 - t\rho z)$$
$$x_1' = x_j, \qquad j = 2, \cdots, k$$

for t a real number. Then setting

$$m(t) = \psi(x_1', \cdots, x_k'),$$

we have

$$\begin{aligned} m(t) &= (Ax_1', x_k') \textstyle\sum_{j=2}^{k} (Ax_j, x_j) + E_2\big((Ax_2, x_2), \cdots, (Ax_k, x_k)\big) \\ &= (1 + t^2|\rho|^2)^{-1} \big((Ax_1, x_1) - 2t|\rho|^2 + t^2|\rho|^2(Az, z)\big) \textstyle\sum_{j=2}^{k} (Ax_j, x_j) \\ &\qquad + E_2\big((Ax_2, x_2), \cdots, (Ax_k, x_k)\big). \end{aligned}$$

Hence

$$\dot{m}(o) = -2 \mid \rho \mid^2 \textstyle\sum_{j=2}^{k} (Ax_j, x_j).$$

If $\sum_{j=2}^{k} (Ax_j, x_j) \neq 0$, then, since $\min \psi = \psi(x_1, \cdots, x_k)$, we must conclude that $\rho = 0$ and hence that $L(x_1, \cdots, x_k)$ is invariant under A. If

$$\textstyle\sum_{j=2}^{k} (Ax_j, x_j) = 0,$$

then if z is any unit vector in $L^*(x_2, \cdots, x_k)$,

$$\psi(z, x_2, \cdots, x_k) = E_2\big((Ax_2, x_2), \cdots, (Ax_k, x_k)\big).$$

Hence if $L(x_2, \cdots, x_k)$ is invariant under A, we may choose $z \in L^*(x_2, \cdots, x_k)$ to be a unit eigenvector of A, and hence $L(z, x_2, \cdots, x_k)$ is invariant under A. Consequently we assume that

$$(Ax_2, v) = \rho_1 \neq 0,$$

where $v \in L^*(x_2, \cdots, x_k)$ and v is a unit vector. Define

$$x_2' = (1 + t^2 \mid \rho_1 \mid^2)^{-1/2}(x_2 - t\rho_1 v),$$
$$x_2'' = (1 + t^2 \mid \rho_1 \mid^2)^{-1/2}(t\bar{\rho}_1 x_2 + v)$$

and note that x_2', x_2'', x_3, $\cdots$, x_k form an o.n. set. Define

$$K(t) = (1 + t^2 \mid \rho_1 \mid^2)^{-1} \{2t \mid \rho_1 \mid^2 + t^2 \mid \rho_1 \mid^2 \big((Ax_2, x_2) - (Av, v)\big)\},$$

and we may readily verify that

$$\begin{aligned}(Ax_2', x_2') + (Ax_2'', x_2'') &= (1 + t^2 \mid \rho_1 \mid^2)^{-1} \{(A[x_2 - t\rho_1 v], x_2 - t\rho_1 v) \\ &\qquad + (A[t\bar{\rho}_1 x_2 + v], t\bar{\rho}_1 x_2 + v)\} \\ &= (Ax_2, x_2) + (Av, v).\end{aligned}$$

Also,

$$\begin{aligned}(Ax_2', x_2') &= (1 + t^2 \mid \rho_1 \mid^2)^{-1} \{(Ax_2, x_2) - 2t \mid \rho_1 \mid^2 + t^2 \mid \rho_1 \mid^2 (Av, v)\} \\ &= (Ax_2, x_2) - K(t),\end{aligned}$$

and

$$\begin{aligned}(Ax_2'', x_2'') &= (1 + t^2 \mid \rho_1 \mid^2)^{-1} \{t^2 \mid \rho_1 \mid^2 (Ax_2, x_2) + 2t \mid \rho_1 \mid^2 + (Av, v)\} \\ &= (Av, v) + K(t);\end{aligned}$$

thus

$$(Ax_2', x_2')(Ax_2'', x_2'') = (Ax_2, x_2)(Av, v) + K(t)\big((Ax_2, x_2) - (Av, v)\big) - K^2(t).$$

Combining these results, we have

$$\begin{aligned}\psi(x_2', x_2'', x_3, \cdots, x_k) &= E_2\big((Ax_2', x_2'), (Ax_2'', x_2''), (Ax_3, x_3), \cdots, (Ax_k, x_k)\big) \\ &= (Ax_2', x_2')(Ax_2'', x_2'') + \big((Ax_2', x_2') + (Ax_2'', x_2'')\big) \textstyle\sum_{j=3}^{k} (Ax_j, x_j) \\ &\qquad + E_2\big((Ax_3, x_3), \cdots, (Ax_k, x_k)\big) \\ &= (Ax_2, x_2)(Av, v) + \big((Ax_2, x_2) + (Av, v)\big) \textstyle\sum_{j=3}^{k} (Ax_j, x_j) \\ &\qquad + E_2\big((Ax_3, x_3), \cdots, (Ax_k, x_k)\big) + K(t)\big((Ax_2, x_2) - (Av, v)\big) - K^2(t) \\ &= \min \psi + K(t)\big((Ax_2, x_2) - (Av, v)\big) - K^2(t).\end{aligned}$$

Now

$$\dot{K}(0) = 2 \mid \rho \mid^2, \qquad K(0) = 0,$$

and hence if $(Ax_2, x_2) \neq (Av, v)$, we conclude that $L(x_2, \cdots, x_k)$ is invariant under A. If $(Ax_2, x_2) = (Av, v)$, then

$$\begin{aligned} \psi(x_2', x_2'', x_3, \cdots, x_k) &= \min \psi - \{2t \mid \rho_1 \mid^2/(1 + t^2 \mid \rho_1 \mid^2)\}^2 \\ &< \min \psi \end{aligned}$$

for $t \neq 0$. This completes the proof.

3. The extreme values

Let R_k be the k-dimensional space of k-tuples over the reals. Let $\lambda_1 \geqq \cdots \geqq \lambda_k$ be any set of k real numbers, and let $\lambda = (\lambda_1, \cdots, \lambda_k) \in R_k$. For $y \in R_k$ we define the convex set $M(\lambda)$ as the totality of points y satisfying

$$\sum_{j=1}^{k} y_j = \sum_{j=1}^{k} y_j \tag{3.1}$$

and

$$\sum_{s=1}^{r} y_{i_s} \leqq \sum_{j=1}^{r} \lambda_j, \qquad 1 \leqq i_1 < \cdots < i_r \leqq k, \quad 1 \leqq r \leqq k-1. \tag{3.2}$$

LEMMA 3. *The extreme values of the function*

$$g(y) = \prod_{j=1}^{k} \lambda_j \tag{3.3}$$

defined on $M(\lambda)$ occur in the set of numbers

$$\left\{ \prod_{j=0}^{q-1} \left(\frac{\lambda_{k_j+1} + \cdots + \lambda_{k_{j+1}}}{k_{j+1} - k_j} \right)^{k_{j+1}-k_j}, 0 \right\}, \tag{3.4}$$

where the k_j are any integers satisfying

$$0 = k_0 < k_1 < \cdots < k_q = k.$$

Proof. First note that if y is such a point that

$$\begin{aligned} y_j &= \frac{\lambda_{k_0+1} + \cdots + \lambda_{k_1}}{k_1 - k_0}, & j &= 1, \cdots, k_1, \\ y_j &= \frac{\lambda_{k_1+1} + \cdots + \lambda_{k_2}}{k_2 - k_1} & j &= k_1 + 1, \cdots, k_2, \\ &\vdots \\ y_j &= \frac{\lambda_{k_{q-1}} + \cdots + \lambda_{k_q}}{k_q - k_{q-1}}, & j &= k_{q-1} + 1, \cdots, k_q, \end{aligned}$$

then $y \in M(\lambda)$. Since $g(y)$ is of the form (3.4), we see that these are achievable values on $M(\lambda)$. The remainder of the proof is an induction argument on k. For $k = 2$ the result is clear. Assume the theorem for all integers less than k. If z is an interior point of $M(\lambda)$ such that $g(z)$ is an extreme value of g, then there exists a multiplier μ such that

$$\frac{\partial g}{\partial y_j} = \mu, \qquad j = 1, \cdots, k$$

for $y = z$. Thus

$$\prod_{i \neq j} z_i = \mu, \qquad \prod_{i=1}^{k} z_i = \mu z_j . \tag{3.5}$$

Summing on j and using (3.1), we have

$$k \prod_{i=1}^{k} z_i = \mu \sum_{i=1}^{k} \lambda_i ,$$

and substituting in (3.5), we have

$$\prod_{i \neq j} z_i \left(\sum_{i=1}^{k} \lambda_i - k z_j\right) = 0, \qquad j = 1, \cdots , k.$$

Hence $g(z) = 0$ or $g(z) = \{(\sum_{i=1}^{k} \lambda_i)/k\}^k$ and both of these types are included in (3.4). Now suppose z is not an interior point of $M(\lambda)$. Then one of the inequalities (3.2) is an equality. That is, for some $\omega = \{i_1 , \cdots , i_r\}$, $r < k$, we have

$$\sum_{s=1}^{r} y_{i_s} = \sum_{j=1}^{r} \lambda_j \tag{3.6}$$

for $y = z$. We consider the extreme values of g on the set defined by (3.1), (3.2), and (3.6). Set

$$h(y) = \prod_{s=1}^{r} y_{i_s} ,$$

where the indices $i_1 , \cdots , i_r$ are precisely those in ω. For any subset $x_1 , \cdots , x_t$, $1 \leqq t \leqq r$, of $y_{i_1} , \cdots , y_{i_r}$,

$$\sum_{j=1}^{t} x_j \leqq \sum_{j=1}^{t} \lambda_j . \tag{3.7}$$

Hence by (3.6), (3.7), and the induction hypothesis, $h(y)$ has extreme values of the form (3.4) with k replaced by r and involving only $\lambda_1 , \cdots , \lambda_r$. On the other hand, set

$$m(y) = \prod_{j \notin \omega} y_j ,$$

and by (3.1) and (3.6)

$$\sum_{j \notin \omega} y_j = \sum_{j=r+1}^{k} \lambda_j .$$

Let $v_1 , \cdots , v_t$, $1 \leqq t \leqq k - r$, be any subset of $\{y_j\}_{j \notin \omega}$. Then

$$\sum_{s=1}^{r} y_{i_s} + \sum_{j=1}^{t} v_j \leqq \sum_{j=1}^{r+t} \lambda_j ,$$

and hence

$$\sum_{j=1}^{t} v_j \leqq \sum_{j=r+1}^{r+t} \lambda_j .$$

Again, by the induction hypothesis, the extreme values of $m(y)$ are of the form (3.4) using $\lambda_{r+1} , \cdots , \lambda_k$ and $k - r$ in place of k. It follows that numbers of the form (3.4) are bounds on the extreme values of $g(y) = m(y)h(y)$, and since these are achievable values, the induction is complete.

Remark. Lemma 3 has an interesting geometric interpretation. The set $M(\lambda)$ can be described as follows. Consider $H(\lambda)$, the convex hull of the $k!$ points $P\lambda$ as P runs over all k-square permutation matrices. It is known that $H(\lambda) = M(\lambda)$ (see [9]). However, for completeness we recapitulate the brief proof of this fact. For if $y \in M(\lambda)$, there exists a permutation matrix P for which $(Py)_j \geqq (Py)_{j+1}$, $j = 1, \cdots , k - 1$. It is clear that $Py \in M(\lambda)$. By a result in [5; p. 49], $Py = S\lambda$, where S is a k-square doubly stochastic matrix. Hence $y = P^{-1}S\lambda$, and $P^{-1}S$ is clearly doubly stochastic when S is. By a result of G. Birkhoff [2], $P^{-1}S$ is a centroid of permutation matrices. It follows that $y \in H(\lambda)$. Conversely if $y \in H(\lambda)$, then y is a centroid of the points $P\lambda$, and each $P\lambda \in M(\lambda)$. Thus $H(\lambda) = M(\lambda)$.

Lemma 3 asserts that the maximum and minimum signed volumes of the k-dimensional parallelopiped bounded by the planes $x_j = y_j$ and the coordinate planes $x_j = 0$ as y varies over the polyhedron $H(\lambda)$ are of the form (3.4). It seems interesting to ask the same question for a more general elementary symmetric function than (3.3). Lemma 4 answers this for $E_2(y_1, \cdots, y_k)$. Of course, if $\lambda_j > 0$ for $j = 1, \cdots, k$, then $H(\lambda)$ consists of points all of whose coordinates are positive. In this case both Lemma 3 and Lemma 4 follow for the minimum at least by using the concavity of $E_r^{1/r}(y_1, \cdots, y_k)$, $1 \leqq r \leqq k$, where E_r is the r^{th} symmetric function of the indicated variables [6].

LEMMA 4.

$$\min_{y \in M(\lambda)} E_2(y_1, \cdots, y_k) = E_2(\lambda_1, \cdots, \lambda_k) \tag{3.8}$$

$$\max_{y \in M(\lambda)} E_2(y_1, \cdots, y_k) = \binom{k}{2}\left\{\left(\sum_{j=1}^k \lambda_j\right)/k\right\}^2. \tag{3.9}$$

Proof. From (3.1) we see that the right side of (3.9) is an achievable value of $E_2(y_1, \cdots, y_k)$. We need only show that

$$E_2(y_1, \cdots, y_k) \leqq \binom{k}{2}\left\{\left(\sum_{j=1}^k y_j\right)/k\right\}^2. \tag{3.10}$$

This is known for $y_j \geqq 0$, $j = 1, \cdots, k$ [5; p. 52]. Now

$$\left(\sum_{j=1}^k y_j\right)^2 = \sum_{j=1}^k y_j^2 + 2E_2(y_1, \cdots, y_k),$$

and hence (3.10) is equivalent to

$$\left(\sum_{j=1}^k y_j\right)^2 \leqq k\sum_{j=1}^k y_j^2,$$

which follows from the convexity of t^2.

Now if the minimum value of E_2 is achieved at an interior point of $M(\lambda)$, we conclude that

$$\frac{\partial E_2}{\partial y_j} = \mu, \qquad j = 1, \cdots, k \tag{3.11}$$

for y this interior point and μ a constant multiplier. But (3.11) implies that

$$y_1 = y_2 = \cdots = y_k = \left(\sum_{j=1}^k \lambda_j\right)/k.$$

Hence assume that for some $\omega = \{i_1, \cdots, i_r\}$, $r < k$, we have

$$\sum_{s=1}^r y_{i_s} = \sum_{j=1}^r \lambda_j$$

for $y = z$, the minimizing point. The proof now proceeds by induction on k exactly as in Lemma 3. The essential part of the argument is contained in the following sequence:

$$\begin{aligned} E_2(z_1, \cdots, z_k) &= \left(\textstyle\sum_{s=1}^{r} z_{i_s}\right)\left(\textstyle\sum_{j\notin\omega} z_j\right) + E_2(z_{i_1}, \cdots, z_{i_r}) + E_2(z_j\,; j \notin \omega) \\ &= \left(\textstyle\sum_{j=1}^{r} \lambda_j\right)\left(\textstyle\sum_{j=r+1}^{k} \lambda_j\right) + E_2(z_{i_1}, \cdots, z_{i_r}) + E_2(z_j : j \notin \omega) \\ &\geqq \left(\textstyle\sum_{j=1}^{r} \lambda_j\right)\left(\textstyle\sum_{j=r+1}^{k} \lambda_j\right) + E_2(\lambda_1, \cdots, \lambda_r) \\ &\qquad + E_2(\lambda_{r+1}, \cdots, \lambda_k) \\ &= E_2(\lambda_1, \cdots, \lambda_k). \end{aligned}$$

The inequality follows as before from the induction hypothesis.

4. Applications to matrices

THEOREM 1. *For* $1 \leqq k \leqq n$ *the extreme values of* $\varphi(x_1, \cdots, x_k)$ *are of the form*

$$\prod_{j=1}^{q-1} \left(\frac{\lambda_{k_j+1} + \cdots + \lambda_{k_{j+1}}}{k_{j+1} - k_j}\right)^{k_{j+1}-k_j}, \tag{4.1}$$

where the k_j *are integers satisfying* $0 = k_0 < k_1 < \cdots < k_q = k$ *and* $\lambda_1, \cdots, \lambda_k$ *is a choice of* k *eigenvalues of the matrix* A.

Proof. By a standard continuity argument we may assume A is nonsingular. By Lemma 1 an extremal set spans an invariant subspace L under A. By a result of K. Fan [3: Theorem 1] and the invariance of L, we conclude that

$$\big((Ax_1, x_1), \cdots, (Ax_k, x_k)\big) \in M(\lambda),$$

where $\lambda = (\lambda_1, \cdots, \lambda_k)$ and $\lambda_1, \cdots, \lambda_k$ is some choice of k eigenvalues of A. The theorem of Fan that we are applying here states that if B is any Hermitian n-square complex matrix and $x_1, \cdots, x_k$ is an o.n. set ($k \leqq n$), then

$$\textstyle\sum_{j=1}^{k} \beta_{n-j+1} \leqq \sum_{j=1}^{k} (Bx_j, x_j) \leqq \sum_{j=1}^{k} \beta_j,$$

where $\beta_1 \geqq \cdots \geqq \beta_n$ are the eigenvalues of B. By Lemma 3 the extreme values of φ are bounded above and below by expressions of the form (4.1) or 0. However, the argument used in Lemma 1 excludes 0. Now a typical value (4.1) can be obtained by choosing

$$x_t = \sum_{i=k_j+1}^{i=k_{j+1}} \frac{\theta_j^{i(t-k_j)} u_i}{(k_{j+1} - k_j)^{1/2}}, \qquad t = k_j + 1, \cdots, k_{j+1} \tag{4.2}$$

for $j = 0, \cdots, q - 1$, where θ_j is a primitive $(k_{j+1} - k_j)$ root of unity and $u_1, \cdots, u_k$ are o.n. eigenvectors of A corresponding to $\lambda_1, \cdots, \lambda_k$ respectively. It is a straightforward calculation to verify that the vectors x_t are o.n. and have the required property. For example, if $j = 0$ then (4.2) becomes

$$x_t = \sum_{i=1}^{k_1} \frac{\theta_0^{it} u_i}{(k_1)^{1/2}}, \qquad t = 1, \cdots, k_1$$

and

$$(x_t, x_s) = (k_1)^{-1} \textstyle\sum_{i=1}^{k_1} \theta_0^{i(t-s)} = \delta_{ts},$$

where s and t are less than or equal to k_1, and θ_0 is a primitive k^{th} root of unity. This completes the proof.

THEOREM 2. *For* $1 \leqq k \leqq n$

$$\min \psi(x_1, \cdots, x_k) = E_2(\lambda_1, \cdots, \lambda_k) \tag{4.3}$$

$$\max \psi(x_1, \cdots, x_k) = k^{-2}\binom{k}{2}\left\{\max\left(\left|\sum_{j=1}^k \alpha_j\right|, \left|\sum_{j=1}^k \alpha_{n-j+1}\right|\right)\right\}^2, \tag{4.4}$$

where $\alpha_1 \geqq \cdots \geqq \alpha_n$ *are the eigenvalues of* A *and* $\lambda_1, \cdots, \lambda_k$ *is some choice of* k *of the* α_j, $j = 1, \cdots, n$.

Proof. The fact that ψ is bounded above by the right side of (4.4) follows immediately from (3.10) and Fan's result [3]. This value is clearly achieved by making a choice of vectors x_t as in (4.2) with $q = 2$. To establish (4.3), we use Lemma 2 to conclude that there exists a minimizing set for ψ, $x_1, \cdots, x_k$, that spans an invariant subspace of A. As in Theorem 1

$$((Ax_1, x_1), \cdots, (Ax_k, x_k)) \in M(\lambda),$$

where $\lambda_1, \cdots, \lambda_k$ is a choice of k of the α_j. Hence by Lemma 3

$$\psi(x_1, \cdots, x_k) \geqq E_2(\lambda_1, \cdots, \lambda_k). \tag{4.5}$$

The right side of (4.5) is clearly achievable by an appropriate choice of k o.n. eigenvectors of A.

Remark. It would be of interest to determine the extreme values of

$$E_r((Ax_1, x_1), \cdots, (Ax_k, x_k))$$

for $1 \leqq r < k$, $r \geqq 3$. The methods used here do not seem to generalize readily except when A is nonnegative Hermitian.

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