A FUNDAMENTAL INEQUALITY IN THE THEORY OF EXTENSIONS OF VALUATIONS

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1. Let K be a field and let K^* be a finite algebraic extension of K, of degree n. Let v be a valuation of K, and let $v_1^*, v_2^*, \cdots, v_g^*$ be the distinct extensions of v to K^*. We denote by Γ and Δ respectively the value group and the residue field of v, and by Γ_i^* and Δ_i^* the analogous entities for v_i^*. It is known that (a) the number g of distinct extensions v_i^* of v is finite, that (b) Δ_i is a finite extension of Δ, and that (c) Γ is a subgroup of Γ_i of finite index. The integers $f_i = [\Delta_i:\Delta]$ and e_i = index of Γ in Γ_i are respectively the relative degree and the reduced ramification index[2] of v_i^* with respect to v. The purpose of this note is to prove the following inequality:[3]

$$\sum_{i=1}^{g} e_i f_i \leqq n. \tag{1}$$

At the end of the note we shall give some conditions under which the equality sign in (1) is valid.

2. We shall first introduce some notations and recall a few known facts from valuation theory.

If v and v' are valuations of K and if the valuation ring R_v of v is a proper subring of the valuation ring $R_{v'}$, of v', then we shall write $v' < v$ and we shall say that *v is composite with v'*. In the case of valuations of finite rank the relation $v' < v$ implies that the rank of v' is less than the rank of v.

If v is composite with v' and if Δ' is the residue field of v', then v determines

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[2] If K^* is a separable extension of K, v^* any extension of v to K^* and p^σ the inseparable factor of relative degree f of v (p = characteristic of Δ), then it seems convenient to designate as the *ramification index* of v^* the product ep^σ (e = reduced ramification index of v).

[3] *A Note by O. Zariski.* A proof of (1) for valuations v of finite rank was found by me in 1952 (but remained unpublished). A few weeks before his death (in 1955), Cohen outlined a proof of (1) for valuations of infinite rank in a letter to Oscar Goldman. Although very sketchy, this outline enabled me to reconstruct the proof that Cohen must have had in mind, and the present proof is, to the best of my knowledge, a synthesis of the proofs of both authors. An independent proof of (1) was given by Roquette at about the same time that Cohen communicated the outline of his proof to Goldman. Although priority (for the case of valuation of infinite rank) may readily be conceded to Roquette, the fact that the present note contains, in part, the last piece of research undertaken by I. S. Cohen before his death makes the publication of this note both timely and proper.

uniquely a valuation $\bar{v}$ of Δ', as follows: if ξ' is any element of Δ' and if ξ is a representative element of ξ' in K, then $\bar{v}(\xi') = v(\xi)$. We write $v = v' \circ \bar{v}$. The residue field of $\bar{v}$ is the same as the residue field Δ of v.

Conversely, given any valuation v' of K and a valuation $\bar{v}$ of the residue field Δ' of v', there exists a unique valuation v of K such that $v = v' \circ \bar{v}$.

The valuations v' with which a given valuation v is composite form a linearly ordered set with respect to the relation $<$.

If $v = v' \circ \bar{v}$ and if Γ, Γ', and $\bar{\Gamma}$ are the value groups of v, v', and $\bar{v}$ respectively, then $\bar{\Gamma}$ is a subgroup of Γ, and $\Gamma' \cong \Gamma/\bar{\Gamma}$.

Two valuations v_1, v_2 of K are said to be *independent* if there exists no valuation v such that we have simultaneously $v \leqq v_1$ and $v \leqq v_2$. A finite set of valuations v_i of K is independent if any two of the valuations v_i are independent.

We shall make use of the following well-known facts concerning the extensions of v to K^*:

(A) If v_1^* and v_2^* are two distinct extensions of v to K^*, then $v_1^* \nless v_2^*$ and $v_2^* \nless v_1^*$.

(B) If $v = v' \circ \bar{v}$ and v^* is any extension of v, then v^* can be written in one and only one way in the form $v^* = v'^* \circ \bar{v}^*$, where v'^* is an extension of v' to K^* and $\bar{v}^*$ is an extension of the valuation $\bar{v}$ of the residue field Δ' of v' to the residue field Δ'^* of v'^*.

(C) If $v = v' \circ \bar{v}$, v'^* is any extension of v' to K^*, and $\bar{v}^*$ any extension of $\bar{v}$ to the residue field of v'^*, then $v'^* \circ \bar{v}^*$ is an extension of v to K^*.

(D) If v^*, v'^* are valuations of K^* such that v^* is an extension of v and $v'^* < v^*$, then the restriction v' of v'^* to K is such that $v' < v$.

(E) With the same notations as in (B), let e, e', and $\bar{e}$ be the relative ramification indices of v^* (with respect to v), of v'^* (with respect to v'), and of $\bar{v}^*$ (with respect to $\bar{v}$). Then $e = e'\bar{e}$.

We denote by $E(v)$ the set of all extensions of v to K^* and by $L(v)$ the set of all valuations v' of K such that $v' < v$. For given v and v' such that $v' < v$, we have, in view of (B), a mapping of $E(v)$ into $E(v')$. We denote this mapping by $\varphi_{v'}^{v}$:

$$\varphi_{v'}^{v}: E(v) \rightarrow E(v').$$

By construction, the mapping $\varphi_{v'}^{v}$ has the following property:

$$\varphi_{v'}^{v}(v^*) < v^* \qquad \text{for any } v^* \text{ in } E(v). \tag{2}$$

By (C), this mapping is *onto* $E(v')$. If $v'' < v' < v$ then it is clear that

$$\varphi_{v''}^{v'}\varphi_{v'}^{v} = \varphi_{v''}^{v}. \tag{3}$$

For a given v and for a given extension v^* of v to K^* we also have a mapping

$$\psi_{v^*}^{v}: L(v) \rightarrow L(v^*),$$

defined as follows:

$$\psi_{v^*}^{v}(v') = \varphi_{v'}^{v}(v^*), \qquad v' \in L(v). \tag{4}$$

That $\psi^v_{v^*}$ is indeed a mapping into $L(v^*)$ follows from (2). It follows directly from (D) that $\psi^v_{v^*}$ maps $L(v)$ onto $L(v^*)$. It is obvious that $\psi^v_{v^*}$ is *an order preserving mapping and is* (1, 1) (note that $\psi^v_{v^*}(v')$ is always an extension of v').

3. We next prove two preliminary lemmas.

LEMMA 1. *Let* $v = v' \circ \bar{v}$, *let* Δ *be the residue field of* v (*and hence also of* $\bar{v}$), *and let* Δ' *be the residue field of* v'. *Let* $v_1'^*, v_2'^*, \cdots, v_h'^*$ *be the extension of* v' *to* K^* *and let* $\Delta_s'^*$ *be the residue field of* $v_s'^*$ ($s = 1, 2, \cdots, h$). *If the inequality* (1) *holds for* (v', K, K^*) (i.e., *for the valuation* v' *and its extensions* $v_1'^*, v_2'^*, \cdots, v_h'^*$ *to* K^*) *and for each of the h triads* $(\bar{v}, \Delta', \Delta_s'^*)$, *then the inequality* (1) *holds for* (v, K, K^*).

Proof. Let $\bar{v}^*_{s,1}, \bar{v}^*_{s,2}, \cdots, \bar{v}^*_{s,g_s}$ be all the extensions of $\bar{v}$ to $\Delta_s'^*$. Let f_s' and e_s' be respectively the relative degree and the reduced ramification index of $v_s'^*$ with respect to v', and let $\bar{f}_{s,t_s}$ and $\bar{e}_{s,t_s}$ be the corresponding characters of $\bar{v}^*_{s,t_s}$ with respect to $\bar{v}$. By assumption, we have

$$\sum_{s=1}^{h} e_s' f_s' \leqq n$$

$$\sum_{t_s=1}^{g_s} \bar{e}_{s,t_s} \bar{f}_{s,t_s} \leqq f_s', \qquad s = 1, 2, \cdots, h.$$

Hence

$$\sum_{s=1}^{h} \sum_{t_s=1}^{g_s} e_s' \bar{e}_{s,t_s} \bar{f}_{s,t_s} \leqq n. \tag{5}$$

By (C), each of the valuations $v_s'^* \circ \bar{v}^*_{s,t_s}$ is an extension of $v (= v' \circ \bar{v})$. By (B), these $g_1 + g_2 + \cdots + g_h$ valuations are distinct and give all the extensions of v to v^*. By (E), each product $e_s' \bar{e}_{s,t_s}$ represents the relative ramification index of the valuation $v_s'^* \circ \bar{v}^*_{s,t_s}$ with respect to v. Since the residue fields of the valuations $\bar{v}$, $\bar{v}^*_{s,t_s}$ coincide respectively with the residue fields of v and $v_s'^* \circ \bar{v}^*_{s,t_s}$, the integer $\bar{f}_{s,t_s}$ is also equal to the relative degree of $v_s'^* \circ \bar{v}^*_{s,t_s}$. It follows that (5) is identical with (1). Q. E. D.

For any valuation v of K we denote by $g(v)$ the number of distinct extensions of v to K^*. From the existence of the mapping $\varphi^v_{v'}$, it follows that $g(v)$ is a monotone increasing function of v. The next lemma is our principal auxiliary result.

LEMMA 2. *If* $L(v)$ *has no last element, then* $g(v) = \max_{v' \epsilon L(v)} \{g(v')\}$.

Proof. Let $g_0 = \max_{v' \epsilon L(v)} \{g(v')\}$ and let us fix in $L(v)$ a valuation v_0' such that $g(v_0') = g_0$. To prove the lemma we have to show that the mapping $\varphi^v_{v_0'}$, of $E(v)$ onto $E(v_0')$, is (1, 1). Let v_1^*, v_2^* be elements of $E(v)$ such that

$$\varphi^v_{v_0'}(v_1^*) = \varphi^v_{v_0'}(v_2^*). \tag{6}$$

Since $L(v)$ has no last element, and since $\psi^v_{v_i^*}$ is a (1, 1) order preserving mapping of $L(v)$ onto $L(v_i^*)$ ($i = 1, 2$), also $L(v_1^*)$ and $L(v_2^*)$ have no last element. Therefore

$$R_{v_i^*} = \bigcap_{v'^* \in L(v_i^*)} R_{v'^*}, \qquad i = 1, 2,$$

where $R_{v_i^*}$ denotes the valuation ring of v_i^*. Thus, in order to show that $v_1^* = v_2^*$, we have only to show that $L(v_1^*) = L(v_2^*)$, and for this it will be sufficient to show that $\psi^v_{v_1^*} = \psi^v_{v_2^*}$.

Let then v' be an arbitrary element of $L(v)$. By the definition (4) of ψ, the equality $\psi^v_{v_1^*}(v') = \psi^v_{v_2^*}(v')$ (which we wish to establish) is equivalent to

$$\varphi^v_{v'}(v_1^*) = \varphi^v_{v'}(v_2^*). \tag{7}$$

We have either (a) $v' \leqq v_0' < v$ or (b) $v_0' < v' < v$. In case (a) we have $\varphi^v_{v'} = \varphi^{v_0'}_{v'} \cdot \varphi^v_{v_0'}$, and in this case (7) follows directly from (6). In case (b) we have $\varphi^v_{v_0'} = \varphi^{v'}_{v_0'} \cdot \varphi^v_{v'}$. We observe that since $v_0' < v'$ we have $g(v_0') \leqq g(v')$, and hence $g(v') = g(v_0')$ since $v' \in L(v)$ (and by our choice of v_0'). Thus $E(v')$ and $E(v_0')$ have the same number of elements, and consequently $\varphi^{v'}_{v_0'}$ is a one-to-one mapping of $E(v')$ onto $E(v_0')$. Thus, again (7) follows from (6) (and from the above relation $\varphi^v_{v_0'} = \varphi^{v'}_{v_0'} \cdot \varphi^v_{v'}$), Q. E. D.

4. We now proceed to the proof of the inequality (1). This inequality is known for valuations v which are discrete, of rank 1. The proof of this inequality in that case uses only the fact that the extensions of v to K^* are in that case independent. Therefore our inequality (1) can be regarded as known whenever the extensions of v are independent, in particular, if $g(v) = 1$. We shall therefore use induction with respect to $g(v)$.

The case in which the extensions of v to K^* are independent is characterized by the condition that for any v' in $L(v)$ the mapping $\varphi^v_{v'}$ is (1, 1), or, equivalently, that $g(v') = g(v)$ for any v' in $L(v)$. We may therefore assume that there exist valuations v' in $L(v)$ such that $g(v') < g(v)$. Let $L_0(v)$ denote the set of all such valuations v'. We assert *that* $L_0(v)$ *has a last element.* For, assume the contrary. Since $L_0(v)$ is a linearly ordered set of valuations, the intersection of the valuation rings $R_{v'}$, $v' \in L_0(v)$, is the valuation ring of some valuation v_0', and we have $v_0' \leqq v$, $v_0' \notin L_0(v)$. It is clear that $L_0(v) = L(v_0')$ (since $L_0(v)$ contains with any valuation v' also all the valuations v'' such that $v'' < v'$). By Lemma 2 we have therefore $g(v_0') = \max_{v' \in L_0(v)} \{g(v')\}$, whence $g(v_0') < g(v)$, $v_0' \in L_0(v)$, a contradiction.

Let then v' be the last element of $L_0(v)$ and let $v = v' \circ \bar{v}$. We use the notations of the proof of Lemma 1. Since $g(v') < g(v)$, our induction hypothesis implies that the inequality (1) holds for (v', K, K^*). We now have to consider two cases, according as $h = g(v') > 1$, or $h = g(v') = 1$.

If $h > 1$, then each of the integers $g_1, g_2, \cdots, g_h$ is *less* than $g(v)$ (since $g_1 + g_2 + \cdots + g_h = g = g(v)$). Hence, inequality (1) holds for each of the

h triads $(\bar{v}, \Delta', \Delta_s'^*)$. Therefore, inequality (1) holds also for (v, K, K^*), by Lemma 1.

Assume now that $h = 1$. In that case, v' has only one extension to K^*, but any valuation v'' such that $v' < v'' \leqq v$ has exactly g extensions to K^* (in view of the fact that v' is the *last* element of $L_0(v)$). When interpreted as a property of $\bar{v}$, this property of v' has the following meaning: the valuation $\bar{v}$ of Δ', and *every valuation* $\bar{v}'$ *of* Δ' *such that* $\bar{v}' < \bar{v}$, has exactly g extensions [to the residue field Δ'^* of the (unique) extension v'^* of v' to K^*]. This means that the g extensions of $\bar{v}$ are independent. Hence inequality (1) holds for $(\bar{v}, \Delta', \Delta'^*)$, and therefore it follows again from Lemma 1 that our inequality (1) holds for (v, K, K^*).

5. The following is a sufficient condition for the equality sign to hold in (1): *if R^* is the integral closure of R_v in K^*, then R^* is a finite R_v-module.* The proof runs along lines which are familiar in the case of discrete valuations of rank 1, and it will be sufficient to give an outline of the proof.

If $v_1^*, v_2^*, \cdots, v_g^*$ are the extensions of v to K^*, R_i^* the valuation ring of v_i^*, and $\mathfrak{M}_i^*$ the maximal ideal of R_i^*, we set $\mathfrak{P}_i^* = \mathfrak{M}_i^* \cap R^*$. Then $\mathfrak{P}_1^*$, $\mathfrak{P}_2^*, \cdots, \mathfrak{P}_g^*$ are the only maximal ideals in R^*, and the quotient ring of R^* with respect to $\mathfrak{P}_i^*$ is the valuation ring of v_i^*. If $\mathfrak{P}$ denotes the maximal ideal of the valuation ring $R = R_v$ of v, then $\mathfrak{Q}_i^* = R_i^* \cdot \mathfrak{P} \cap R^*$ is a primary ideal whose radical is $\mathfrak{P}_i^*$, and the extended ideal $\mathfrak{P}^* = R^*\mathfrak{P}$ is the intersection of the g primary ideal $\mathfrak{Q}_i^*$. It follows that the vector space $R^*/\mathfrak{P}^*$ (over the residue field $\Delta = R/\mathfrak{P}$ of v) is a direct sum of the g vector spaces $\mathfrak{H}_i^*/\mathfrak{P}^*$, where

$$\mathfrak{H}_i^* = \bigcap_{j \neq i} \mathfrak{Q}_j^* .$$

We shall prove below the inequality

$$\dim \mathfrak{H}_i^*/\mathfrak{P}^* \leqq e_i f_i , \qquad i = 1, 2, \cdots, g. \tag{8}$$

Assuming for a moment this inequality, we find then that

$$\dim R^*/\mathfrak{P}^* \leqq \sum_{i=1}^{g} e_i f_i . \tag{9}$$

Now assume that R^* is a finite R-module. Let $\{w_1, w_2, \cdots, w_m\}$ be a minimal R-basis of R^*. Since R is a valuation ring, a well-known consideration shows that the elements w_ν are linearly independent over K. On the other hand, these elements obviously span K^* over K. Hence $m = n = [K^*:K]$. It is also immediate (and is well known) that if $\bar{w}_\nu$ denotes the $\mathfrak{P}^*$-residue of w_ν, then the n vectors $\bar{w}_\nu$ are linearly independent (over $R/\mathfrak{P}$), and, of course, they span the space $R^*/\mathfrak{P}^*$. Consequently, by (9), we have $n \leqq \sum_{i=1}^{g} e_i f_i$, and combining this with inequality (1) we find $n = \sum_{i=1}^{g} e_i f_i$, thus proving our assertion.

It remains to prove (8). We shall give the full details of the proof, for the details involved differ to some extent from the consideration used in the discrete case.

Let us prove (8), for instance, for $i = 1$. We first make some straightforward considerations about the value groups Γ_1^* and Γ of v_1^* and v respectively. Let G_1 denote the set of all non-negative elements α^* of Γ_1^* such that $\alpha^* < \beta$ for every positive element β of Γ. If α_1^*, α_2^* are distinct elements of G_1, and if, say, $\alpha_1^* < \alpha_2^*$, then $0 < \alpha_2^* - \alpha_1^* < \alpha_2^*$, and hence $\alpha_2^* - \alpha_1^* \notin \Gamma$. Thus, *distinct elements of* G_1 *belong to distinct* Γ*-cosets, and hence* G_1 *is a finite set, consisting of at most* e_1 *elements.*

The significance of the set G_1 is the following: *if* x^* *is an element of* $\mathfrak{H}_1^*$ *then* $x^* \notin \mathfrak{P}^*$ *if and only if* $v_1^*(x^*) \in G_1$. For, if $x^* \in \mathfrak{P}^* = R^*\mathfrak{P}$, then it is clear that $v_1^*(x^*) \geqq v_1^*(y) = v(y)$ for some y in $\mathfrak{P}$, and hence $v_1^*(x^*) \notin G_1$ since $0 < v(y) \in \Gamma$. Conversely, if $v_1^*(x^*) \notin G_1$, then $v_1^*(x^*) \geqq v(y)$ for some $y \in \mathfrak{P}$ and hence $x^* = (x^*/y)\, y \in \mathfrak{H}_1^* \cap R_1^*\,\mathfrak{P} = R^*\,\mathfrak{P}$.

It follows from these remarks that if $\mathfrak{A}^*$ is any R-submodule of $\mathfrak{H}_1^*$ which contains $\mathfrak{P}^*$ as a proper subset, then $\mathfrak{A}^*$ contains elements of least value, and this minimum value is an element of G_1. We denote this minimum value by $v_1^*(\mathfrak{A}^*)$.

If for a given element α^* of G_1 there exists an element x^* in $\mathfrak{H}_1^*$ such that $v_1^*(x^*) = \alpha^*$, then the set of all elements y^* of $\mathfrak{H}_1^*$ such that $v_1^*(y^*) \geqq \alpha^*$ is an R-submodule $\mathfrak{A}^*$ of $\mathfrak{H}_1^*$ which contains $\mathfrak{P}^*$ as a proper subset and is such that $v_1^*(\mathfrak{A}^*) = \alpha^*$. If we denote by $\alpha_1^*, \alpha_2^*, \cdots, \alpha_s^*$ ($s \leqq e_1$) those elements of G_1 which are v_1^*-values of elements of $\mathfrak{H}_1^*$, where we assume that $0 < \alpha_1^* < \alpha_2^* < \cdots < \alpha_s^*$, then we obtain in this fashion a strictly descending chain of R-submodules of $\mathfrak{H}_1^*$

$$\mathfrak{H}_1^* = \mathfrak{A}_1^* > \mathfrak{A}_2^* > \cdots > \mathfrak{A}_s^* > \mathfrak{A}_{s+1}^* = \mathfrak{P}^*, \tag{10}$$

where $\mathfrak{A}_i^*$ is the set of all y^* in $\mathfrak{H}_1^*$ such that $v_1^*(y^*) \geqq \alpha_i^*$. It is clear that for $i = 2, 3, \cdots, s+1$, the module $\mathfrak{A}_i^*$ consists of all the elements y^* in $\mathfrak{H}_1^*$ such that $v_1(y^*) > \alpha_{i-1}^*$.

The subspaces of $\mathfrak{H}_1^*/\mathfrak{P}^*$ correspond in (1, 1) fashion to the R-submodules of $\mathfrak{H}_1^*$ which contain $\mathfrak{P}^*$. Since $s \leqq e_1$, the inequality $\dim \mathfrak{H}_1^*/\mathfrak{P}^* \leqq e_1 f_1$ will be established if we prove that $\dim \mathfrak{A}_{i-1}^*/\mathfrak{A}_i^* \leqq f_1$ for $i = 2, 3, \cdots, s+1$. Here $\mathfrak{A}_{i-1}^*/\mathfrak{A}_i^*$ $(= (\mathfrak{A}_{i-1}^*/\mathfrak{P}^*)/(\mathfrak{A}_i^*/\mathfrak{P}^*))$ is regarded as a vector space over $R/\mathfrak{P}$. Let then $x_1^*, x_2^*, \cdots, x_{f_1+1}^*$ be any $f_1 + 1$ elements of $\mathfrak{A}_{i-1}^*$. We fix in $\mathfrak{A}_{i-1}^*$ an element y^* of least value: $v_1^*(y^*) = \alpha_{i-1}^*$, and we set $z_j^* = x_j^*/y^*$ $(j = 1, 2, \cdots, f_1 + 1)$. Then the z_j^* belong to the valuation ring of v_1^*, and since f_1 is the relative degree of v_1^*, it follows that we can find elements $u_1, u_2, \cdots, u_{f_1+1}$ in R, *not all in* $\mathfrak{P}$, such that $v_1^*(u_1 z_1^* + u_2 z_2^* + \cdots + u_{f_1+1} z_{f_1+1}^*) > 0$. We have then

$$v_1^*(u_1 x_1^* + u_2 x_2^* + \cdots + u_{f_1+1} \cdot x_{f_1+1}^*) > v_1^*(y^*) = \alpha_{i-1}^*,$$

and therefore $u_1 x_1^* + u_2 x_2^* + \cdots + u_{f_1+1} x_{f_1+1}^* \in \mathfrak{A}_i^*$. This establishes the inequality $\dim \mathfrak{A}_{i-1}^*/\mathfrak{A}_i^* \leqq f_1$ and completes the proof.

If R^* is not a finite R-module, then it may very well happen that we have the strict inequality in (1). Thus, in an example of F. K. Schmidt (discussed in O. Zariski, *The concept of a simple point of an abstract algebraic variety*, Trans. Amer. Math. Soc., vol. 62 (1947), p. 24), one has the case in which $n = p =$ characteristic of K ($p > 0$), K^* is a purely inseparable extension of K, $g = 1$, $e_1 = f_1 = 1$.

It is well-known that R^* is always a finite R-module in the following special case: *v is a discrete valuation of rank 1 and K^* is a separable extension of K.* Hence in this case we always have the equality sign in (1).

It may be observed that our preceding result concerning the finiteness of the R-module R^* can be inverted if v is a discrete valuation of rank 1. We have namely the following result: *if v is a discrete valuation of rank 1 and if the equality sign holds in (1), then R^* is a finite R-module.* This follows easily from the above proof of the inequalities (8). Namely, since Γ now is the group of integers, it follows easily that $s = e_1$ and that the elements $\alpha_1^*, \alpha_2^*, \cdots, \alpha_s^*$ are now equal to $0, 1/e_1, \cdots, (e_1 - 1/)e_1$ respectively and that

$$\dim \mathfrak{A}_{i-1}^*/\mathfrak{A}_i^* = f_1$$

(for the proof of this last equality one uses the approximation theorem for the set of extensions $v_1^*, v_2^*, \cdots, v_g^*$ of v). Hence

$$\dim R^*/\mathfrak{P}^* = \textstyle\sum_{i=1}^{g} e_i f_i = n.$$

We can fix a set of n elements $u_{s_i,t_i}^{*(i)}$ in R^* ($i = 1, 2, \cdots, g$; $s_i = 0, 1, \cdots, e_i - 1$; $t_i = 1, 2, \cdots, f_i$) such that

$$\text{(11)} \qquad \begin{aligned} v_i^*(u_{s_i,t_i}^{*(i)}) &= s_i/e_i \\ v_j^*(u_{s_i,t_i}^{*(i)}) &= 1, \qquad \text{if } j \neq i \end{aligned}$$

and such that the $\mathfrak{P}^*$-residues of these n-elements form a basis of the vector space $R^*/\mathfrak{P}^*$ (over $R/\mathfrak{P}$). It is clear that these n elements also form a basis of K^*/K. *We show that they form also an R-basis of R^*.*

Let z^* be any element of R^* and let

$$z^* = \sum_{i,s_i,t_i} a_{s_i,t_i}^{(i)} u_{s_i,t_i}^{*(i)}, \qquad a_{s_i,t_i}^{(i)} \in K.$$

Upon factoring out a coefficient a of least value in v we can write z^* in the form $z^* = by^*$, where $b \in K$,

$$y^* = \sum_{i,s_i,t_i} b_{s_i,t_i}^{(i)} u_{s_i,t_i}^{*(i)}, \qquad b_{s_i,t_i}^{(i)} \in R,$$

and where not all the coefficients b are in $\mathfrak{P}$. Let, say, $b_{s,t}^{(1)} \notin \mathfrak{P}$. Then it follows from (11) that $v_1^*(y^*)$ is one of the numbers $0, 1/e_1, \cdots, (e_1 - 1)/e_1$, hence is less than 1. On the other hand we have $v_1^*(y^*) + v(b) = v_1^*(z^*) \geqq 0$. Hence $v(b)$ is necessarily a nonnegative integer, and thus $b \in R$. Consequently $a_{s_i,t_i}^{(i)} = b \cdot b_{s_i,t_i}^{(i)} \in R$, showing that the elements $u_{s_i,t_i}^{*(i)}$ form an R-basis of R^*.

If K^* is a Galoisian extension of K (and v is an arbitrary valuation of K), then the following results can be established:

(1) The quotient $n/\Sigma e_i f_i$ is a power p^δ of the characteristic p of K ($p \neq 0$, $\delta \geqq 0$). This integer δ may be referred to as the *ramification deficiency* of v.

(2) If the residue field Δ of v is of characteristic zero, then the equality sign holds in (1).

For the proofs of (1) and (2) we refer the reader to the forthcoming book *Commutative Algebra* (vol. 2) of Zariski and Samuel.

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