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A CHARACTERIZATION OF $C(K)$ AMONG FUNCTION ALGEBRAS ON A RIEMANN SURFACE

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Abstract. For a compact subset K of a Riemann surface, necessary and sufficient conditions are given for a function algebra containing $A(K)$ to be all of $C(K)$. Using these results, several conditions are given on a complex-valued function f so that the algebra generated by $A(K)$ and f is all of $C(K)$. In particular, the results are applied to a harmonic function f to give sufficient conditions for the algebra generated by $A(K)$ and f to be all of $C(K)$. Also, sufficient conditions are given for the algebra $A(K)$ to be a maximal subalgebra of $C(K)$.

1. Introduction

Let $\mathcal{R}$ be an open Riemann surface. Throughout this paper, K will denote a compact subset of $\mathcal{R}$ and ∂K will denote the boundary of K. Let $C(K)$ be the algebra of continuous complex-valued functions on K. For a function f that is in $C(K)$ but not in the algebra A, we let $A[f]$ denote the uniformly closed subalgebra of $C(K)$ generated by A and f. Let $A(K)$ be the algebra of functions in $C(K)$ that are holomorphic on $\operatorname{Int}(K)$, the interior of K, and let $M(K)$ consist of the functions in $C(K)$ that can be approximated uniformly by meromorphic functions on $\mathcal{R}$ with poles off K. The containments $M(K) \subset A(K) \subset C(K)$ are apparent. We give necessary and sufficient conditions for an algebra B containing $A(K)$ to satisfy $B = C(K)$.

In Section 2, we state preliminary theorems that will be used throughout the paper.

For $K \subset \mathbb{C}$, let $R(K)$ be the algebra given by the rational functions in $\mathbb{C}$ with poles off K. In the case where $K = D$, the closed unit disc in the complex plane, Wermer [23] found necessary and sufficient conditions for a continuously

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differentiable function $f \in C(D)$ to satisfy $A(D)[f] = C(D)$. In particular, Wermer showed that when f is continuously differentiable on a neighborhood of the closed unit disc D in the complex plane, then $A(D)[f] = C(D)$ if and only if the graph of f is polynomially convex in $\mathbb{C}^2$ and $R(E) = C(E)$, where E is the zero set of $\overline{\partial} f$. Izzo [15] generalized Wermer's result to any compact subset of the complex plane. His approach is based on Wermer's original proof. In Section 3, we generalize Izzo's results to a compact subset of an open Riemann surface. The technique we use follows closely Izzo's approach in [15] while using some ideas from [16]; similar ideas were first used by Freeman in [7].

In 1969, Čirka [6] used Wermer's technique to obtain a generalization of Wermer's result. In particular, Čirka showed the following.

THEOREM 1.1 ([6]). *Let K be a compact set in the complex plane and suppose that every point of ∂K is a peak point for $R(K)$. Let $f \in C(K)$ be harmonic on the interior of K, but nonholomorphic on each component of the interior of K. Then $R(K)[f] = C(K)$.*

In 1987, Axler and Shields [2] used completely different methods to prove the following case where the function to be adjoined is real-valued. Because of the restrictions placed on K, their theorem is actually a special case of Čirka's result.

THEOREM 1.2 ([2]). *Let K be a compact subset of $\mathbb{C}$, and suppose that there is a positive number d such that each component of the complement of K has a diameter greater than d. Let $u \in C(K)$ be real-valued and harmonic in the interior of K but nonconstant on each component of the interior of K. Then $A(K)[u] = C(K)$.*

In 1993, Izzo [14] obtained the following result, which Jiang [17] extended to a compact subset of a Riemann surface in 2003.

THEOREM 1.3 ([14]). *Let K be a compact subset of the complex plane. Let $u \in C(K)$ be real-valued and harmonic on the interior of K, but nonconstant on each component of the interior of K. Then $A(K)[u] = C(K)$.*

Without some restrictions on the compact set K, it is not known whether the analogous result is true for *complex-valued* functions. In 1997, Izzo [15] showed, without any restrictions on K, that if f is in the uniform closure of $\log|A(K)^{-1}|$ and nonholomorphic on each component of the interior of K, then $A(K)[f] = C(K)$. He also gave various conditions on K and f which imply that $A(K)[f] = C(K)$. In Section 4, we generalize Izzo's results to a compact subset of a Riemann surface.

Finally, in Section 5 we apply the results from Sections 3 and 4 to obtain two results about maximal subalgebras.

2. Preliminaries

To fix an atlas on $\mathcal{R}$, we use a result by Gunning and Narasimhan from 1967.

THEOREM 2.1 ([12, Theorem 1.1]). *There exists a globally defined holomorphic function* $\rho : \mathcal{R} \to \mathbb{C}$ *that is locally a homeomorphism.*

Unless stated otherwise, we use such a global parametrization ρ to define all of our local coordinate charts. For a function f defined on $\mathcal{R}$, we denote $\frac{\partial}{\partial \bar{z}}(f \circ \rho^{-1})$ with $\partial f/\partial \overline{\rho}$ and sometimes simply $\overline{\partial} f$.

A parametric disc Δ for ρ on $\mathcal{R}$ is an open connected set on $\mathcal{R}$ on which ρ is one-to-one and such that $\rho(\Delta) = \{z \in \mathbb{C} : |z - z_0| < r\}$ is a disc in $\mathbb{C}$. If $\rho(p) = z_0$, then we call p the center and r the radius of Δ.

Using ρ, Scheinberg [21] and Gauthier [13] constructed a Cauchy kernel F on $\mathcal{R}$ in the following way: If $(p_0, q_0) \in \mathcal{R} \times \mathcal{R}$, let $U(p_0, q_0) = \Delta(p_0) \times \Delta(q_0)$ be a neighborhood of (p_0, q_0), where $\Delta(p_0)$ and $\Delta(q_0)$ are parametric discs centered at p_0 and q_0 respectively. Define the Cousin data $H : \mathcal{R} \times \mathcal{R} \to \mathbb{C}$ by

$$H_{U(p_0,q_0)}(p,q) = \begin{cases} \frac{1}{\rho(p)-\rho(q)} & \text{if } \Delta(p_0) \cap \Delta(q_0) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists a function G meromorphic on $\mathcal{R} \times \mathcal{R}$ such that

$$G|_{U(p_0,q_0)} - H_{U(p_0,q_0)}$$

is holomorphic in U. Define

$$F(p,q) = \frac{1}{2}\big(G(p,q) - G(q,p)\big).$$

Then $F(p,q) = -F(q,p)$ and the only singularities of F are the simple poles with residues ± 1 on the diagonal.

DEFINITION 2.2. If μ is a finite complex Borel measure on $\mathcal{R}$ with compact support, then the *Cauchy transform* $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(q) = \int F(p,q)\, d\mu(p).$$

The Cauchy transform of a measure μ is holomorphic off the closed support of μ. We also use the following results.

THEOREM 2.3 ([5, Theorem 2.1]). *A measure* μ *on* K *is orthogonal to* $M(K)$ *if and only if* $\hat{\mu} = 0$ *on* $\mathcal{R} \setminus K$.

COROLLARY 2.4 ([5, Corollary 2.3]). *If* f *is continuously differentiable in a neighborhood of* K*, and* $\partial f/\partial \overline{\rho} = 0$ *on* K*, then* $f \in M(K)$.

COROLLARY 2.5 ([5, Corollary 2.4]). *If* U *is an open subset of* $\mathcal{R}$*, and* μ *is a measure with compact support satisfying* $\hat{\mu} = 0$ *almost everywhere on* U*, then* $|\mu|(U) = 0$.

COROLLARY 2.6. *If $\hat{\mu}=0$ almost everywhere, then $\mu=0$.*

The following theorem is known as the Kodama–Bishop Localization theorem. We say the function f is *locally approximable on K by holomorphic functions* if each point of K is contained in a parametric disc Δ such that f is the uniform limit on $\overline{\Delta}\cap K$ of functions that are holomorphic on $\overline{\Delta}\cap K$.

THEOREM 2.7 ([18]). *Let f be a complex-valued function defined on a compact subset K of an open Riemann surface $\mathcal{R}$. Then f is the uniform limit on K of meromorphic functions on $\mathcal{R}$ each of which has only finitely many poles (all contained in $\mathcal{R}\setminus K$) if and only if f is locally approximable on K by holomorphic functions.*

The following result by Sakai [20] is a generalization of the Bishop splitting lemma to a Rieman surface.

LEMMA 2.8 ([20, Lemma 7]). *Let K be a compact subset of $\mathcal{R}$. Let μ be a measure on K that is orthogonal to $M(K)$. Let $\{U_j\}_{j=1}^{n}$ be a cover of K by coordinate patches. Then there are measures μ_j such that $\mu=\sum_{j=1}^{n}\mu_j$, where μ_j is orthogonal to $M(\overline{U}_j)$ and the closed support of μ_j is contained in U_j.*

In 1949, Behnke and Stein [3] proved the following theorem.

THEOREM 2.9 ([3]). *Let $\mathcal{R}$ be an open Riemann surface, and U an open subset of $\mathcal{R}$ such that $\mathcal{R}\setminus U$ has no compact connected components. Any function holomorphic on U can be approximated uniformly on compact subsets of U by functions holomorphic on all of $\mathcal{R}$.*

As a corollary to Theorem 2.9, we have the following.

COROLLARY 2.10 ([19, Theorem 3.10.13]). *Let $\mathcal{R}$ be an open Riemann surface. The functions holomorphic on $\mathcal{R}$ separate the points of $\mathcal{R}$. In particular, the functions in $M(K)$ separate the points of K.*

In the case where $\mathcal{R}=\mathbb{C}$ and $M(X)=R(X)$, the next result is known as Alexander's theorem. The proof given below follows the proof of Alexander's theorem appearing in [22].

THEOREM 2.11. *Let $\{X_n\}$ be a sequence of compact sets in $\mathcal{R}$ with compact union X. If $M(X_n)=C(X_n)$ for all n, then $M(X)=C(X)$.*

Proof. Suppose, by way of contradiction, that μ is a measure on X that annihilates $M(X)$ and μ is not the zero measure. Let S be the closed support of μ, so S is the minimal closed set of $\mathcal{R}$ with the property that $|\mu|(X\setminus S)=0$. The sets X_n have no interior, so by the Baire category theorem, X has no interior. We claim that $\mu\in M(S)^{\perp}$. To see this, note that $\hat{\mu}$ vanishes on $\mathcal{R}\setminus X$, by Theorem 2.4, and each point of $\mathcal{R}\setminus S$ is in the closure of $\mathcal{R}\setminus X$. Thus, $\hat{\mu}$ vanishes on $\mathcal{R}\setminus S$ and Theorem 2.3 gives that $\mu\in M(S)^{\perp}$.

Now $S = \bigcup(S \cap X_n)$, so by category there is a parametric disc D that meets S and satisfies $S \cap D = (S \cap X_n) \cap D$ for some n. If $D' \subset \overline{D'} \subset D$, where D' is a parametric disc that meets S, then there is a function $f \in C(\mathcal{R})$ with $f|D'$ identically 1 and f identically zero on a neighborhood of $\mathcal{R} \setminus D$. It follows from Theorem 2.7 that the function $f|S$ belongs to $M(S)$ since $M(S \cap X_n) = C(S \cap X_n)$. But then $f\mu \in M(S)^{\perp}$, and this measure is supported in $S \cap X_n$. Since $M(S \cap X_n) = C(S \cap X_n)$, the measure $f\mu$ must be the zero measure. This implies that $|\mu|(D' \cap S) = 0$, which contradicts the minimality of S. □

Following is a generalization of Bishop's peak point criterion to a Riemann surface.

THEOREM 2.12. *Let K be a compact subset of $\mathcal{R}$, and let P_M be the set of peak points of $M(K)$. If $K \setminus P_M$ has measure zero, then $M(K) = C(K)$.*

Proof. Let μ be a measure on K orthogonal to $M(K)$. Suppose p_0 is such that $\int |F(q,p_0)|\, d|\mu|(q) < \infty$, and $\hat{\mu}(p_0) \neq 0$. Then $p_0 \in K$ by Theorem 2.3. If f is a meromorphic function with poles off K, then $p \mapsto F(p,p_0)[f(p) - f(p_0)]$ is also a meromorphic function with poles off K. So

$$\int F(p,p_0)[f(p) - f(p_0)]\, d\mu(p) = 0.$$

Consequently, for all $f \in M(K)$,

$$f(p_0) = \frac{1}{\hat{\mu}(p_0)} \int F(p,p_0) f(p)\, d\mu(p).$$

Hence, $\frac{1}{\hat{\mu}(p_0)} F(p,p_0)\mu$ is a complex representing measure for p_0. Since $\mu\{p_0\} = 0$, this representing measure has no mass at p_0. Then p_0 is not a peak point of $M(K)$ (see [8, Theorem II.11.3], for example). We conclude that $\hat{\mu}$ is nonzero only for points in $K \setminus P_M$. Since $K \setminus P_M$ has zero area, $\hat{\mu}$ vanishes almost everywhere. By Corollary 2.6, then $\mu = 0$. □

We use the following Lemma in the proof of Theorem 2.14 below.

LEMMA 2.13. *Let E be a closed subset of the the open unit disc $\Delta \subset \mathbb{C}$ with empty interior. Let $h_1, \ldots, h_n$ be holomorphic functions on a neighborhood of the closed unit disc $\overline{\Delta}$ with $h_k(0) = 0$ for $k = 1, \ldots, n$. Let $i_{\mathbb{C}}$ denote the identity function on $\mathbb{C}$. Then for some $\varepsilon > 0$ the set*

$$\{\alpha : |\alpha| < \varepsilon \text{ and } \alpha \notin (i_{\mathbb{C}} + \alpha h_1)(E) \cup \cdots \cup (i_{\mathbb{C}} + \alpha h_n)(E)\}$$

is a dense open subset of the disc $\Delta_\varepsilon = \{\alpha \in \mathbb{C} : |\alpha| < \varepsilon\}$.

Proof. First, we show that for each value of $k = 1, \ldots, n$ we can choose an ε_k so that the set $\{\alpha : |\alpha| < \varepsilon_k \text{ and } \alpha \notin (i_{\mathbb{C}} + \alpha h_k)(E)\}$ is a dense open subset of the disc Δ_{ε_k}. To see this, note that for α small enough, $i_{\mathbb{C}} + \alpha h_k$ is one-to-one on Δ (see [11, Stability Theorem]). Define a function w_k by

$w_k(z) = \dfrac{z}{1-h_k(z)}$. Notice that w_k is defined and holomorphic on a neighborhood of 0, that $w_k(0) = 0$, and that $w_k'(0) = 1$. Restrict the domain and range of w_k so that w_k is a biholomorphic map of a neighborhood of 0 onto another neighborhood of 0. For z and α in the domain and range of w_k, respectively, the following equations are equivalent

$$\begin{aligned} \alpha &= w_k(z), \\ \alpha &= \frac{z}{1-h_k(z)}, \\ z &= \alpha - \alpha h_k(z), \\ \alpha &= (i_{\mathbb{C}} + \alpha h_k)(z). \end{aligned}$$

Choose $\varepsilon_k > 0$ small enough so that $i_{\mathbb{C}} + \alpha h_k$ is one-to-one on Δ whenever $|\alpha| < \varepsilon_k$, and such that the disc Δ_{ε_k} is contained in the range of w_k. For $\alpha \in \Delta_{\varepsilon_k}$, we have $\alpha = w_k(z)$ for some z, and then from above we get that $\alpha = (i_{\mathbb{C}} + \alpha h_k)(z)$. Since $i_{\mathbb{C}} + \alpha h_k$ is one-to-one on Δ, we can conclude that for $\alpha \in \Delta_{\varepsilon_k}$, we have $\alpha \in (i_{\mathbb{C}} + \alpha h_k)(E)$ if and only if $\alpha \in w_k(E)$. Since E is a closed set in Δ with empty interior and w_k is holomorphic, it follows that $\{\alpha : |\alpha| < \varepsilon_k \text{ and } \alpha \notin (i_{\mathbb{C}} + \alpha h)(E)\}$ is a dense open subset of the disc Δ_{ε_k}.

Set $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$. Then for each $k = 1, \dots, n$ the set $\{\alpha : |\alpha| < \varepsilon$ and $\alpha \notin (i_{\mathbb{C}} + \alpha h_k)(E)\}$ is a dense open subset of Δ_ε. Thus, the intersection of these sets, $\{\alpha \notin (i_{\mathbb{C}} + \alpha h_1)(E) \cup \dots \cup (i_{\mathbb{C}} + \alpha h_n)(E)\}$, is also a dense open subset of Δ_ε, and the lemma is proved. □

THEOREM 2.14. *Let K be a compact subset of an open Riemann surface $\mathcal{R}$, and suppose F is a subset of K such that the closure $\overline{F}$ of F has no interior in $\mathcal{R}$. Let $a \in \operatorname{Int}(K) \setminus \overline{F}$. There exists a globally defined holomorphic function $\phi : \mathcal{R} \to \mathbb{C}$ that gives local coordinates on all of K and satisfies $\phi(a) \notin \phi(\overline{F})$. That is, ϕ separates the point a from the closure of the set F.*

Proof. Let ρ be a globally defined holomorphic function that gives local coordinates on $\mathcal{R}$. Without loss of generality, we can assume $\rho(a) = 0$. Since $\overline{F}$ is compact, the set $\rho^{-1}(\rho(a)) \cap \overline{F}$ is a finite set of points. Denote the points of this set by $b_1, \dots, b_n$. Since the holomorphic functions on $\mathcal{R}$ separate points, we can find a holomorphic function h on $\mathcal{R}$ such that $h(a) = 1$ and $h(b_k) = 0$ for $k = 1, \dots, n$.

For each point $a, b_1, \dots, b_n$, choose a parametric disc for ρ centered at that point. Let Δ_0 be the disc centered at a and Δ_k be the disc centered at b_k for $k = 1, \dots, n$. Shrink some of the discs, if necessary, so that they all have the same radius r. Let ψ_k be the inverse of ρ restricted to Δ_k. So ψ_k maps the disc $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$ diffeomorphically onto Δ_k and sends 0 to a for $k = 0$ and to b_k for $k = 1, \dots, n$.

For $\alpha \in \mathbb{C}$, define $\phi_\alpha : \mathcal{R} \to \mathbb{C}$ by $\phi_\alpha = \rho + \alpha h$. Then ϕ_α is holomorphic on all of $\mathcal{R}$, and for all α small enough we have that $\frac{d\phi_\alpha}{d\rho} \neq 0$ on K, so ϕ_α gives local coordinates about each point of K. Thus, the proof will be complete once we show the existence of arbitrarily small values of α satisfying $\phi_\alpha(a) \notin \phi_\alpha(\overline{F})$. Note that $\phi_\alpha(a) = \alpha$, so we want to find an α such that $\alpha \notin \phi_\alpha(\overline{F})$.

Since ρ never takes the value 0 on $\overline{F} \setminus (\Delta_1 \cup \cdots \cup \Delta_n)$ and h is bounded on K, for small enough values of α we have that $\alpha \notin \phi_\alpha(\overline{F} \setminus (\Delta_1 \cup \cdots \cup \Delta_n))$. Thus, it suffices to consider ϕ_α on $\Delta_1, \ldots, \Delta_n$. For $k = 1, \ldots, n$, let $\rho_\alpha^k = \phi_\alpha \circ \psi_k : \Delta_r \to \mathbb{C}$. Observe that $\rho_\alpha^k = i_{\mathbb{C}} + \alpha(h \circ \psi_k)$ and that ρ_α^k is defined on a neighborhood of Δ_r. Now, we can apply the above lemma to conclude that for all $j = 1, \ldots, n$, there are arbitrarily small values of α such that $\alpha \notin \rho_\alpha^j(\bigcup_{j=1}^n \rho(\Delta_j \cap \overline{F}))$. But then $\alpha \notin \phi_\alpha(\Delta_j \cap \overline{F})$ for all j, and the proof is complete. □

The last three lemmas in this section will simplify the proofs of Theorems 3.1 and 3.2.

Lemma 2.15 ([17, Lemma 2.9]). *If μ is a measure on K that annihilates $A(K)$, then $\hat{\mu} = 0$ almost everywhere off* $\mathrm{Int}(K)$.

Lemma 2.16. *If a subset E of K has measure zero, then $\phi^{-1}(\phi(E)) \cap K$ has measure zero for any function ϕ that gives local coordinates on K.*

Proof. Since K is compact, we can cover $\phi(K)$ with finitely many open connected sets $V_j, j = 1, \ldots, n$, where each connected set in $\phi^{-1}(V_j)$ that has nonempty intersection with K is mapped diffeomorphically by ϕ onto V_j. Then for $j = 1, \ldots, n$, $\phi^{-1}(V_j) \cap K$ is a disjoint union of finitely many sets $U_{j_1}, \ldots, U_{j_m}$ in K, each of which is mapped diffeomorphically into V_j.

Fix one set, say V_t. Because $\phi(E) \cap V_t$ has measure zero, and ϕ is a diffeomorphism on each U_{t_i}, we have that $\phi^{-1}(\phi(E)) \cap U_{t_i}$ has measure zero for each $i = 1, \ldots, m$. It follows that $\phi^{-1}(\phi(E) \cap V_t) \cap K = \bigcup_{i=1}^m (\phi^{-1}(\phi(E)) \cap U_{t_i})$ has measure zero.

Since V_t was arbitrary, $\phi^{-1}(\phi(E) \cap V_j) \cap K$ has measure zero for each $j = 1, \ldots, n$. Thus,

$$\phi^{-1}(\phi(E)) \cap K = \phi^{-1}\left(\bigcup_{j=1}^n \phi(E) \cap V_j\right) \cap K$$
$$= \bigcup_{j=1}^n \left(\phi^{-1}(\phi(E) \cap V_j) \cap K\right)$$

has measure zero. □

Lemma 2.17. *Let E be a subset of* $\mathrm{Int}(K)$ *such that for each compact subset E' of $\overline{E}$ we have $M(E') = C(E')$. Then for every point q in $\rho^{-1}(\rho(E)) \cap$* $\mathrm{Int}(K)$, *there is a parametric disc Δ_q centered at q whose closure is contained in* $\mathrm{Int}(K)$ *and for which $M(\rho^{-1}(\rho(\overline{E})) \cap \overline{\Delta}_q) = C(\rho^{-1}(\rho(\overline{E})) \cap \overline{\Delta}_q)$.*

Proof. Fix a point p_0 in $\rho^{-1}(\rho(\overline{E})) \cap \mathrm{Int}(K)$. Since $\overline{E}$ is compact, there are finitely many points $p_0, p_1, \ldots, p_n$ in $\rho^{-1}(\rho(p_0)) \cap \overline{E} \cap \mathrm{Int}(K)$. Choose parametric discs $\Delta_0, \Delta_1, \ldots, \Delta_n$ centered at $p_0, p_1, \ldots, p_n$, respectively, so that each disc is contained in $\mathrm{Int}(K)$ and mapped diffeomorphically by ρ onto $\rho(\Delta_0)$. Shrink the radius of the discs further to obtain discs $\Delta_0^*, \ldots, \Delta_n^*$, that are each mapped diffeomorphically onto $\rho(\Delta_0^*)$ and satisfy $\overline{\Delta}_j^* \subset \Delta_j$ for all $j = 0, \ldots, n$.

For each $j = 0, 1, \ldots, n$, let ϕ_j be the diffeomorphism of Δ_j onto Δ_0 given by ρ. More specifically, $\phi_j = \psi \circ (\rho|_{\Delta_j})$, where $\psi : \rho(\Delta_0) \to \Delta_0$ is the inverse of ρ restricted to Δ_0. Denote the set $\phi_j(\overline{E} \cap \overline{\Delta}_j^*)$ by $\tilde{E}_j$. So $\tilde{E}_j$ is a diffeomorphic copy of $\overline{E} \cap \overline{\Delta}_j^*$ inside $\overline{\Delta}_0^*$. Note furthermore, that $\rho^{-1}(\rho(\overline{E})) \cap \overline{\Delta}_0^* = \bigcup_{j=0}^n \tilde{E}_j$.

Let $f \in C(\tilde{E}_j)$. Then $f \circ \phi_j^{-1}$ is in $C(\overline{E} \cap \overline{\Delta}_j^*) = M(\overline{E} \cap \overline{\Delta}_j^*)$. It follows that $f \circ \phi_j^{-1}$ can be uniformly approximated on $\overline{E} \cap \overline{\Delta}_j^*$ by functions holomorphic in a neighborhood of $\overline{E} \cap \overline{\Delta}_j^*$. That is, $f \circ \phi_j^{-1} = \lim_{m\to\infty} g_m$ on $\overline{E} \cap \overline{\Delta}_j^*$, where each g_m is holomorphic in a neighborhood of $\overline{E} \cap \overline{\Delta}_j^*$ that is contained in Δ_j.

Now on $\tilde{E}_j$, we have $f = f \circ \phi_j^{-1} \circ \phi_j = \lim(g_m \circ \phi_j)$, where, for each m, the function $g_m \circ \phi_j$ is holomorphic in a neighborhood of $\tilde{E}_j$ that is contained in Δ_0. Thus, Theorem 2.7 gives that $f \in M(\tilde{E}_j)$. It follows that $M(\tilde{E}_j) = C(\tilde{E}_j)$, and this is true for all $j = 0, \ldots, n$. Then by Theorem 2.11, we have

$$
\begin{aligned}
M\big(\rho^{-1}(\rho(\overline{E})) \cap \overline{\Delta}_0^*\big) &= M\left(\bigcup_{j=0}^n \tilde{E}_j\right) \\
&= C\left(\bigcup_{j=0}^n \tilde{E}_j\right) \\
&= C\big(\rho^{-1}(\rho(\overline{E})) \cap \overline{\Delta}_0^*\big). \quad \square
\end{aligned}
$$

3. Main theorems

Theorems 3.1 and 3.2 below are the main results of this paper. They generalize results due to Izzo [15] to a compact subset of a Riemann surface. (A function algebra on a set K is a uniformly closed subalgebra of $C(K)$ that contains the constants and separates the points of K.)

THEOREM 3.1. *Let K be a compact subset of an open Riemann surface $\mathcal{R}$. Suppose B is a function algebra on K that contains $A(K)$. Then $B = C(K)$ if and only if both of the following conditions hold:*

(i) *the maximal ideal space of B is K, and*
(ii) *for almost every point a in $\mathrm{Int}(K)$ there is a function f in B that is differentiable at a and such that $(\partial f/\partial\overline{\rho})(a) \neq 0$.*

THEOREM 3.2. *Let K be a compact subset of an open Riemann surface $\mathcal{R}$. Suppose B is a function algebra on K that contains $A(K)$. Let $E = \{\zeta \in \mathrm{Int}(K)$: if $f \in B$, then either $(\partial f/\partial\overline{\rho})(\zeta) = 0$ or f is not differentiable at $\zeta\}$. Then $B = C(K)$ if and only if both of the following conditions hold:*

(i) *the maximal ideal space of B is K, and*
(ii) *for each compact subset E' of $\overline{E} \cap \mathrm{Int}(K)$ we have $M(E') = C(E')$.*

The major portion of the proofs of Theorems 3.1 and 3.2 will be accomplished with the following lemma. Notice that if a is any point in K, and ϕ is a function that gives local coordinates on K, then since K is compact, $\phi^{-1}(\phi(a)) \cap K$ is a finite set of points. Also, if a is any point in $\mathrm{Int}(K)$, and ϕ gives local coordinates on K and separates a from the boundary of K, then each of the points in the finite set $\phi^{-1}(\phi(a)) \cap K$ is in $\mathrm{Int}(K)$.

LEMMA 3.3. *Suppose B is a function algebra on K with maximal ideal space K and such that $M(K) \subset B$. Let μ be a measure on K that annihilates B, and let a be a point in $\mathrm{Int}(K)$. Let ϕ be a globally defined holomorphic function that gives local coordinates on K and separates the point a from the boundary of K, as given by Theorem 2.14. Let $a_1, \dots, a_d$ denote the points in the finite set $\phi^{-1}(\phi(a)) \cap K$. Suppose that $\int |F(p, a_j)|\, d|\mu|(p) < \infty$ for each $j = 1, \dots, d$.*

If there are functions $f_1, \dots, f_d$ in B such that f_j is differentiable at a_j and $(\partial f_j/\partial\overline{\phi})(a_j) \neq 0$ for each $j = 1, \dots, d$, then $\hat{\mu}(a) = 0$.

Proof. Since the proof is long, we divide it into steps.

Step 1: Show there exist finitely many functions $f_0, f_1, \dots, f_m$ in B, a neighborhood Ω of $\sigma(\phi, f_0, f_1, \dots, f_m)$ (the joint spectrum of $\phi, f_0, f_1, \dots, f_m$) in $\mathbb{C}^{m+2}$, and holomorphic functions h and h_1 on Ω such that:

(1) $h = (z_1 - \phi(a))h_1$ where z_1 is the first complex coordinate function on $\mathbb{C}^{m+2}$,
(2) the only zeros of h on $\sigma(\phi, f_0, f_1, \dots, f_m)$ are at the points $(\phi(a_j), f_0(a_j), f_1(a_j), \dots, f_m(a_j))$, $j = 1, \dots, d$,
(3) for some $\varepsilon > 0$ the circular sector $T = \{z \in \mathbb{C} : -\frac{\pi}{4} \leq \arg z \leq \frac{\pi}{4}, |z| < \varepsilon\}$ satisfies $h(\sigma(\phi, f_0, f_1, \dots, f_m)) \cap T = \{0\}$.

It follows from Corollary 2.10 that there is a function f_0 in B such that $f_0(a_j) = j$ for $j = 1, \dots, d$. Also there is a $\eta > 0$ such that $\{z \in K : |\phi(z) - \phi(a)| < \eta\}$ is a disjoint union $N_1 \cup \dots \cup N_d$ with ϕ forming a local coordinate system on each N_j and

$$N_j = \{z \in K : |\phi(z) - \phi(a)| < \eta\} \cap \{z \in K : |f_0(z) - j| < 1/3\}.$$

For each a_j, choose a function $f_j \in B$ such that $(\partial f_j/\partial\overline{\phi})(a_j) \neq 0$. Now for z in K, we have

$$f_j(z) = f_j(a_j) + \frac{\partial f_j}{\partial \phi}(a_j)\big(\phi(z) - \phi(a)\big) + \frac{\partial f_j}{\partial \overline{\phi}}(a_j)\big(\overline{\phi(z)} - \overline{\phi(a)}\big) + r(z),$$

where $r(z)$ satisfies $r(z)/|\phi(z) - \phi(a)| \to 0$ as $z \to a_j$, or equivalently

$$(4)\quad \frac{(\phi(z)-\phi(a))(f_j(z)-f_j(a_j)-\frac{\partial f_j}{\partial \phi}(a_j)(\phi(z)-\phi(a)))}{\frac{\partial f_j}{\partial \overline{\phi}}(a_j)} = |\phi(z)-\phi(a)|^2+s(z),$$

where $s(z)=r(z)(\phi(z)-\phi(a))/\frac{\partial f_j}{\partial \overline{\phi}}(a_j)$ satisfies $s(z)/|\phi(z)-\phi(a)|^2 \to 0$ as $z \to a_j$. Let g_j be the function defined on $\mathbb{C}^{d+2}$ by

$$g_j(z_1,\dots,z_{d+2}) = \frac{-(z_1-\phi(a))(z_{j+2}-f_j(a_j)-\frac{\partial f_j}{\partial \phi}(a_j)(z_1-\phi(a)))}{\frac{\partial f_j}{\partial \overline{\phi}}}(a_j).$$

Then for $z \in K$ we have by (4) that

$$g_j(\phi(z), f_0(z), f_1(z), \dots, f_d(z)) = -|\phi(z)-\phi(a)|^2 - s(z).$$

Thus, for each $j=1,\dots,d$, there is a $\delta_j>0$ with $\delta_j<\eta$ such that if $U_j' = \{(z_1,\dots,z_{d+2}) \in \mathbb{C}^{d+2} : |z_1-\phi(a)|<\delta_j \text{ and } |z_2-j|<1/3\}$, then the real part $\operatorname{Re} g_j(x)$ of $g_j(x)$ satisfies $\operatorname{Re} g_j(x)<0$ for

$$x \in \left\{(\phi(z), f_0(z), f_1(z), \dots, f_d(z)) : z \in K \setminus \{a_j\}_{j=1}^d\right\} \cap U_j',$$

while $g_j(\phi(a_j), f_0(a_j), f_1(a_j), \dots, f_d(a_j)) = 0$ for $j=1,\dots,d$. Choose a number δ such that $0<\delta<\min\{\delta_1,\dots,\delta_d\}$ and let $U_j = \{(z_1,\dots,z_{d+2}) \in \mathbb{C}^{d+2} : |z_1-\phi(a)|<\delta \text{ and } |z_2-j|<1/3\}$. Let $U = U_1 \cup \dots \cup U_d$ and let

$$V = \{(z_1,\dots,z_{d+2}) \in \mathbb{C}^{d+2} : |z_1-\phi(a)|>\delta\} \cup \left(\bigcup_{j=1}^d (\{\operatorname{Re} g_j<0\} \cap \{|z_2-j|<1/3\})\right).$$

Notice that $U \cap V = (U_1 \cap V) \cup \dots \cup (U_d \cap V)$ and that on $U_j \cap V$ we have $\operatorname{Re} g_j<0$. Thus, if we define g on U by setting $g=g_j$ on U_j, then $\operatorname{Re} g<0$ on $U \cap V$. Since the maximal ideal space of B is K, we have that

$$U \cup V \supset \{(\phi(z), f_0(z), f_1(z), \dots, f_d(z)) : z \in K\} = \sigma(\phi, f_0, f_1, \dots, f_d).$$

Hence, [8, Lemma III.5.2] (the Arens–Calderón lemma) shows that there exist functions $f_{d+1},\dots,f_m \in B$ such that

$$\pi(\hat{\sigma}(\phi, f_0, f_1, \dots, f_m)) \subset U \cup V,$$

where $\hat{\sigma}(\phi, f_0, f_1, \dots, f_m)$ is the polynomially convex hull of the joint spectrum $\sigma(\phi, f_0, f_1, \dots, f_m)$ and $\pi : \mathbb{C}^{m+2} \to \mathbb{C}^{d+2}$ is the projection onto the first $d+2$ coordinates. Extend g to $\pi^{-1}(U)$ by making it independent of the last $m-d$ variables. The open sets $\pi^{-1}(U)$ and $\pi^{-1}(V)$ cover $\hat{\sigma}(\phi, f_0, f_1, \dots, f_m)$ and $\operatorname{Re} g<0$ on $\pi^{-1}(U) \cap \pi^{-1}(V)$. By [1, Theorem 9.4], there exist a neighborhood W of $\hat{\sigma}(\phi, f_0, f_1, \dots, f_m)$ and holomorphic functions φ and ψ on $\pi^{-1}(U) \cap W$ and $\pi^{-1}(V) \cap W$, respectively, with

$$\log(g) = \psi - \varphi \quad \text{on } \pi^{-1}(U) \cap \pi^{-1}(V) \cap W.$$

Then $ge^{\varphi} = e^{\psi}$ on $\pi^{-1}(U) \cap \pi^{-1}(V) \cap W$. The left-hand side is holomorphic on $\pi^{-1}(U) \cap W$, and the right-hand side is holomorphic on $\pi^{-1}(V) \cap W$. Hence, the function h defined by

$$h = \begin{cases} ge^{\varphi} & \text{on } \pi^{-1}(U) \cap W, \\ e^{\psi} & \text{on } \pi^{-1}(V) \cap W, \end{cases}$$

is holomorphic on $(\pi^{-1}(U) \cup \pi^{-1}(V)) \cap W$.

Let

$$h_1 = \frac{h}{z_1 - \phi(a)}.$$

Since $z_1 - \phi(a)$ never vanishes on V, h_1 is holomorphic on $\pi^{-1}(V) \cap W$. Moreover, $g/(z_1 - \phi(a))$ is a polynomial on each U_j, so h_1 is also holomorphic on $\pi^{-1}(U) \cap W$. Thus, h_1 is holomorphic on $(\pi^{-1}(U) \cup \pi^{-1}(U)) \cap W$.

Letting $\Omega = (\pi^{-1}(U) \cup \pi^{-1}(U)) \cap W$, we can see that (1) and (2) hold.

Let $y_j = (\phi(a_j), f_0(a_j), f_1(a_j), \ldots, f_m(a_j))$ and let $s = e^{\varphi}$. Then $h = sg$ on $\pi^{-1}(U) \cap W$. Since we can replace h by the product of h with any entire function on $\mathbb{C}^{m+2}$ having no zeros, we may assume that $s(y_j) = 1$ for $j = 1, \ldots, d$. Choose a neighborhood U' of $\{y_1, \ldots, y_d\}$ with U' contained in $\pi^{-1}(U)$ and $|s - 1| < 1/\sqrt{2}$ on U'. Suppose x is a point in $\sigma(\phi, f_0, f_1, \ldots, f_m) \cap U'$ with $x \neq y_j$ for $j = 1, \ldots, d$. Then

$$|h(x) - g(x)| = |s(x) - 1||g(x)| < \frac{1}{\sqrt{2}}|g(x)|.$$

Since $\operatorname{Re} g(x) < 0$, this implies that $\arg h(x)$ lies outside $[-\frac{\pi}{4}, \frac{\pi}{4}]$, and hence $h(x)$ is outside the sector T.

On the other hand, $\sigma(\phi, f_0, f_1, \ldots, f_m) \setminus U'$ is a compact subset of the joint spectrum $\sigma(\phi, f_0, f_1, \ldots, f_m)$ that does not intersect $\{y_1, \ldots, y_d\}$, and by (2) the only zeros of h on $\sigma(\phi, f_0, f_1, \ldots, f_m)$ are at $\{y_1, \ldots, y_d\}$. Hence, the modulus of h is bounded away from zero on $\sigma(\phi, f_0, f_1, \ldots, f_m) \setminus U'$. Therefore, for some $\varepsilon > 0$, we have that everywhere on $\sigma(\phi, f_0, f_1, \ldots, f_m) \setminus \{y_1, \ldots, y_d\}$ the value of h lies outside the sector T. So (3) holds.

Step 2: Show there exists a sequence of functions $\{\alpha_n\}$ in B and a positive constant c such that

(5) $\displaystyle\lim_{n\to\infty} \alpha_n(z) = \frac{1}{\phi(z) - \phi(a)}$ for $z \in \mathrm{K} \setminus \{a_1, \ldots, a_d\}$, and

(6) $\displaystyle|\alpha_n(z)| \leq \frac{c}{|\phi(z) - \phi(a)|}$ for all $z \in K$ and all n large.

With h and h_1 as in Step 1, let

$$\psi_n(x) = \frac{h_1(x)}{h(x) - 1/n}.$$

By (3), for each n large, there is a neighborhood of $\sigma(\phi, f_0, f_1, \ldots, f_m)$ on which h never takes the value $1/n$. Then since h and h_1 are holomorphic on a neighborhood of $\sigma(\phi, f_0, f_1, \ldots, f_m)$, we see that ψ_n is holomorphic on a neighborhood of $\sigma(\phi, f_0, f_1, \ldots, f_m)$ for n large. Let $G: K \to \mathbb{C}^{m+2}$ be given by $G(z) = (\phi(z), f_0(z), f_1(z), \ldots, f_m(z))$. The functional calculus (see [8], Chapter III, for further information) shows that $\psi_n \circ G$ is in B. Let $\alpha_n = \psi_n \circ G$. For $z \in K \setminus \{a_1, \ldots, a_d\}$, we have

$$\begin{aligned}\lim_{n\to\infty} \alpha_n(z) &= \lim_{n\to\infty} \frac{h_1(G(z))}{h(G(z)) - 1/n} \\ &= \frac{h_1(G(z))}{h(G(z))} \\ &= \frac{1}{\phi(z) - \phi(a)}.\end{aligned}$$

So (5) holds.

There is a positive constant c_1 such that for all n large and all w outside the sector T we have

$$\left|1 - \frac{1}{nw}\right| \geq c_1,$$

or equivalently,

$$\left|w - \frac{1}{n}\right| \geq c_1|w|.$$

Thus, by (3), we have for all $z \in K$ and n large,

$$\left|h(G(z)) - \frac{1}{n}\right| \geq c_1|h(G(z))|,$$

or equivalently,

$$\left|h(G(z)) - \frac{1}{n}\right| \geq c_1|\phi(z) - \phi(a)||h_1(G(z))|.$$

Rearranging the last inequality and using the definition of α_n gives

$$|\alpha_n(z)| \leq \frac{1}{c_1|\phi(z) - \phi(a)|},$$

so (6) holds with $c = 1/c_1$.

Step 3: Since by hypothesis $\int |F(p, a_j)|\, d|\mu|(p) < \infty$, $j = 1, \ldots, d$, we can see that $|\mu|$ has no mass at a_j for $j = 1, \ldots, d$. Also, by (6), the functions $a_n(z)(\phi(z) - \phi(a))F(z, a)$ are dominated by the L^1 function $c|F(z, a)|$.

Notice that $F(z, a)$ has a simple pole at a, and $(\phi(z) - \phi(a))F(z, a)$ has a removable singularity at a. Thus, $(\phi(z) - \phi(a))F(z, a)$ is in B. Now, since the α_n are also in B, and μ annihilates B, Lebesgue's dominated convergence theorem gives that

$$\begin{aligned}\hat{\mu}(a) &= \int F(z,a)\,d\mu(z)\\ &= \lim_{n\to\infty}\int \alpha_n(z)\big(\phi(z)-\phi(a)\big)F(z,a)\,d\mu(z)\\ &= 0. \quad \square\end{aligned}$$

We are now ready to give the proofs of the two main theorems.

Proof of Theorem 3.1. Necessity is clear. To prove sufficiency, suppose μ is a measure on K that annihilates B. It suffices to show that $\hat{\mu}=0$ almost everywhere since this implies that $\mu=0$. Since $B\supset A(K)$, Lemma 2.15 shows that $\hat{\mu}=0$ almost everywhere off $\mathrm{Int}(K)$.

Let ρ be a globally defined holomorphic function that gives local coordinates on K, and let $E=\{p\in\mathrm{Int}(K):$ if $f\in B$, then either $(\partial f/\partial\overline{\rho})(p)=0$ or f is not differentiable at $p\}$. Condition (ii) in Theorem 3.1 implies that E has measure zero. Note that if $\overline{\partial} f(p)=0$ with respect to ρ at a point p, then $\overline{\partial} f(p)=0$ with respect to any analytic local coordinate defined in a neighborhood of p.

By Theorem 2.14, we know that for every point $a\in\mathrm{Int}(K)$ there is a globally defined holomorphic function ϕ_a that gives local coordinates on all of K and satisfies $\phi_a(a)\notin\phi_a(\partial K)$. Since ∂K is closed, we can find a neighborhood U_a around each point $a\in\mathrm{Int}(K)$ such that $\phi_a(b)\notin\phi_a(\partial K)$ for every $b\in U_a$. Then there is a countable cover $\{U_{a_i}\}_{i=1}^{\infty}$ of $\mathrm{Int}(K)$, such that for each $i=1,2,\dots$, we have $\phi_{a_i}(b)\notin\phi_{a_i}(\partial K)$ for every $b\in U_{a_i}$.

Since ϕ_{a_i} gives local coordinates on K, it follows from Lemma 2.16 that $\phi_{a_i}^{-1}(\phi_{a_i}(E))\cap K$ has measure zero for each $i=1,2,\dots$, and consequently $\bigcup_{i=1}^{\infty}\phi_{a_i}^{-1}(\phi_{a_i}(E))\cap K$ has measure zero. Then for almost every point $b\in\mathrm{Int}(K)$ there is a function ϕ_{a_j} with $\phi_{a_j}(b)\notin\phi_{a_j}(\partial K)$ and $\phi_{a_j}(b)\notin\phi_{a_j}(E)$. So for almost every point $b\in\mathrm{Int}(K)$, there is a local coordinate ϕ_{a_j} that separates the point b from the boundary of K and satisfies the following: if we let $b_1,\dots,b_d$ denote the points in the finite set $\phi_{a_j}^{-1}(\phi_{a_j}(b))\cap K$, then there are functions $f_1,\dots,f_d$ in B, such that f_k is differentiable at b_k and $(\partial f_k/\partial\overline{\rho})(b_k)\neq 0$ for each $k=1,\dots,d$. In addition, for almost every $b\in\mathrm{Int}(K)$, letting $b_1,\dots,b_d$ denote the same points as above, we have that $\int|F(p,b_k)|\,d|\mu|(p)<\infty$ for each $k=1,\dots,d$. Then by Lemma 3.3, $\hat{\mu}=0$ almost everywhere on $\mathrm{Int}(K)$. Thus, $\hat{\mu}=0$ almost everywhere, and so $\mu=0$. $\square$

Proof of Theorem 3.2. Necessity is clear. To prove sufficiency, suppose μ is a measure on K that annihilates B. It suffices to show that $\hat{\mu}=0$ almost everywhere since this implies that $\mu=0$. Since $B\supset A(K)$, Lemma 2.15 shows that $\hat{\mu}=0$ almost everywhere off $\mathrm{Int}(K)$. Moreover, Lemmas 2.16 and 3.3

show that $\hat{\mu}=0$ almost everywhere on $\mathrm{Int}(K)\setminus\rho^{-1}(\rho(E))$. Thus, we need only show that $\hat{\mu}=0$ almost everywhere on $\rho^{-1}(\rho(E))\cap\mathrm{Int}(K)$.

Since we have already noted that $\hat{\mu}=0$ almost everywhere off $\rho^{-1}(\rho(E))$, the measure μ is supported on $\rho^{-1}(\rho(\overline{E}))$ (by Corollary 2.5), and since the Cauchy transform of a measure is holomorphic off its closed support, $\hat{\mu}=0$ everywhere off $\rho^{-1}(\rho(\overline{E}))$ also. For each $q\in\rho^{-1}(\rho(E))\cap\mathrm{Int}(K)$, choose a parametric disc Δ_q centered at q whose closure is contained in $\mathrm{Int}(K)$ and such that $M(\rho^{-1}(\rho(\overline{E}))\cap\overline{\Delta}_q)=C(\rho^{-1}(\rho(\overline{E}))\cap\overline{\Delta}_q)$ as given by Lemma 2.17. Let Δ_q' be the disc centered at q with radius half that of Δ_q.

Now fix $w\in\rho^{-1}(\rho(\overline{E}))\cap\mathrm{Int}(K)$. Let $U_1=\rho^{-1}(\rho(\overline{E}))\cap\Delta_w$ and let $\{U_j\}_{j=2}^n$ be a cover of $\rho^{-1}(\rho(\overline{E}))\cap(K\setminus\Delta_w)$ by coordinate patches with $\overline{U}_j\cap\Delta_w'=\emptyset$ for all $j=2,\dots,n$. Since $\hat{\mu}=0$ off $\rho^{-1}(\rho(\overline{E}))$, Theorem 2.3 gives that $\mu\perp M(\rho^{-1}(\rho(\overline{E})))$. Hence, by Lemma 2.8, there exist measures $\mu_1,\dots,\mu_n$ such that $\mu=\sum_{i=1}^n\mu_i$ with $\mu_i\perp M(\overline{U}_i)$ and the closed support of μ_i is contained in $\overline{U_i}$ $(i=1,\dots,n)$. Now $\mu_1\perp M(\overline{U}_1)=M(\rho^{-1}(\rho(\overline{E}))\cap\overline{\Delta}_w)=C(\rho^{-1}(\rho(\overline{E}))\cap\overline{\Delta}_w)$, so $\mu_1=0$. Moreover, $\hat{\mu}_j=0$ off $\overline{U}_j$ for all $j=2,\dots,n$ (since $\mu_j\perp M(\overline{U}_j)$), so $\hat{\mu}_j=0$ on Δ_w' for all j. Thus, $\hat{\mu}=\sum_{j=2}^n\hat{\mu}_j=0$ on Δ_w'. We conclude that $\hat{\mu}=0$ on $\rho^{-1}(\rho(\overline{E}))\cap\mathrm{Int}(K)$. □

The next theorem is a consequence of Theorem 3.2. This theorem is a characterization of the continuously differentiable complex-valued functions f such that $A(K)[f]=C(K)$.

THEOREM 3.4. *Let K be a compact subset of an open Riemann surface $\mathcal{R}$. Suppose $f\in C(K)$ is continuously differentiable on* $\mathrm{Int}(K)$. *Then $A(K)[f]=C(K)$ if and only if:*

- (i) *the maximal ideal space of $A(K)[f]$ is K, and*
- (ii) *for each compact subset E' of the set $\{\zeta\in\mathrm{Int}(K):(\partial f/\partial\overline{\rho})(\zeta)=0\}$ we have $M(E')=C(E')$.*

Proof. The "if" part is an immediate consequence of Theorem 3.2. Conversely, if $A(K)[f]=C(K)$, then the maximal ideal space of $A(K)[f]$ is K. Moreover, if there were a compact set E' contained in $\mathrm{Int}(K)$ on which $\partial f/\partial\overline{\rho}$ were identically zero for which $M(E')\neq C(E')$, then the restriction of every member of the set $A(K)\cup\{f\}$ to E' would be in $M(E')$ by Corollary 2.4, and hence the same would be true of every member of $A(K)[f]$. Thus, we would have $A(K)[f]\neq C(K)$. □

The following is a consequence of Theorem 3.1 and also a special case of Theorem 3.4. This corollary will be used in the next section to obtain results about harmonic functions.

COROLLARY 3.5. *Suppose $f\in C(K)$ is continuously differentiable on* $\mathrm{Int}(K)$ *and such that:*

(i) *the maximal ideal space of* $A(K)[f]$ *is* K, *and*
(ii) $\partial f/\partial\overline{\rho}$ *is nonzero almost everywhere on* $\mathrm{Int}(K)$.

Then $A(K)[f]=C(K)$.

4. Harmonic functions

In this section, we generalize some of Izzo's results in [15] for harmonic functions to an open Riemann surface. Following Izzo's approach, we use the notion of subharmonicity with respect to a function algebra as defined by Gamelin and Sibony in [9]. Let A be a function algebra with maximal ideal space M_A, and let u be an upper semicontinuous function on M_A. The function u is said to be *subharmonic with respect to* A if $u(x)\le\int u\,d\sigma$ for every $x\in M_A$ and every Jensen measure σ for x. A real-valued function u on M_A is called *harmonic with respect to* A if both u and $-u$ are subharmonic with respect to A. A complex-valued function on M_A is called *harmonic with respect to* A if its real and imaginary parts are harmonic with respect to A. Notice that a continuous complex-valued function f on M_A is harmonic with respect to A if and only if $\int f d\sigma = f(x)$ for every $x\in M_A$ and every Jensen measure σ for x. Lemma 4.2 shows that harmonicity with respect to an algebra is related to ordinary harmonicity.

The harmonic measure for a point $p\in\mathrm{Int}(K)$ is the unique representing measure for p on ∂K with respect to the functions continuous on K and harmonic on $\mathrm{Int}(K)$.

LEMMA 4.1 ([10, Lemma 7.3]). *Let* K *be a compact subset of an open Riemann surface* $\mathcal{R}$. *If* $p\in\mathrm{Int}(K)$, *then harmonic measure is a Jensen measure for* p *with respect to* $A(K)$.

LEMMA 4.2. *If* h *is harmonic with respect to* $A(K)$, *then* h *is harmonic on* $\mathrm{Int}(K)$.

Proof. Suppose h is harmonic with respect to $A(K)$. Notice that since h and $-h$ are upper semicontinuous on M_A, the function h is continuous on K. Let ρ be a globally defined holomorphic function that gives local coordinates on K. Let $\Delta\subset\mathrm{Int}(K)$ be a parametric disc for ρ with center p_0 and radius r, and set $z_0=\rho(p_0)$. For any function u harmonic in Δ and continuous on $\overline{\Delta}$, we have

$$(*)\qquad u(p_0)=\frac{1}{2\pi}\int_0^{2\pi}u\big(\rho^{-1}(z_0+re^{i\theta})\big)\,d\theta.$$

Then the measure μ_{p_0} defined by $\int f\,d\mu_{p_0}=\frac{1}{2\pi}\int_0^{2\pi}f(\rho^{-1}(z_0+re^{i\theta}))\,d\theta$ is the unique harmonic measure for $A(\overline{\Delta})$. By Lemma 4.1, μ_{p_0} is a Jensen measure for p_0 with respect to $A(\overline{\Delta})$. Then since $A(K)$ is contained in $A(\overline{\Delta})$, μ_{p_0} is a Jensen measure for p_0 with respect to $A(K)$. Therefore, equation $(*)$ also holds with u replaced by the function h. Since Δ was an arbitrary parametric disc in K, we get that h is a harmonic function on $\mathrm{Int}(K)$. □

We also need the following lemma to apply the results from the last section.

LEMMA 4.3 ([15, Lemma 2.1]). *If A is a function algebra on its maximal ideal space X, and f is a complex-valued function on X that is harmonic with respect to A, then the maximal ideal space of $A[f]$ is also X.*

THEOREM 4.4. *Let K be a compact subset of an open Riemann surface $\mathcal{R}$. If $f \in C(K)$ is harmonic with respect to $A(K)$, and f is nonholomorphic on each component of $\mathrm{Int}(K)$, then $A(K)[f] = C(K)$.*

Proof. By Lemma 4.2, the functions harmonic with respect to $A(K)$ are harmonic on $\mathrm{Int}(K)$ in the ordinary sense. Thus, f is continuously differentiable on $\mathrm{Int}(K)$ and $\partial f/\partial\overline{\rho}$ has at most countably many zeros on $\mathrm{Int}(K)$. Moreover, the preceding lemma shows that the maximal ideal space of $A(K)[f]$ is K. Therefore, Corollary 3.5 shows that $A(K)[f] = C(K)$. □

Every function that is in the uniform closure of the complex-linear span of $\log|A(K)^{-1}|$ is harmonic with respect to $A(K)$, so the following is a consequence of Theorem 4.4.

THEOREM 4.5. *If f is in the uniform closure of the complex-linear span of $\log|A(K)^{-1}|$ and f is nonholomorphic on each component of $\mathrm{Int}(K)$, then $A(K)[f] = C(K)$.*

The following theorem was proved by Izzo [15, Theorem 2.6] in the case where K is a compact subset of the complex plane. Since Izzo's proof also holds on a Riemann surface, we simply state the theorem here and refer the reader to [15] for a proof. A *Jensen boundary point* for a function algebra A on X is a point of X for which the only Jensen measure is the point mass.

THEOREM 4.6. *Suppose K is such that every point of ∂K is a Jensen boundary point for $A(K)$. If $f \in C(K)$ is harmonic on $\mathrm{Int}(K)$ and nonholomorphic on each component of $\mathrm{Int}(K)$, then $A(K)[f] = C(K)$.*

Since every peak point is a Jensen boundary point, the following is a special case of Theorem 4.6.

COROLLARY 4.7. *Suppose K is such that every point of ∂K is a peak point for $A(K)$. If $f \in C(K)$ is harmonic on $\mathrm{Int}(K)$ and nonholomorphic on each component of $\mathrm{Int}(K)$, then $A(K)[f] = C(K)$.*

Although in this section we considered only the algebra generated by $A(K)$ and a *single* harmonic function, we could just as easily have considered the algebra generated by $A(K)$ and a whole *family* of harmonic functions. To see this, first observe that, as noted in [15], Lemma 4.3 remains valid if the function f is replaced by a family of complex-valued functions on X each harmonic with respect to A. From this and Theorem 3.1, we obtain the following generalization of Theorem 4.4.

THEOREM 4.8. *If* $\{f_\alpha\}$ *is a family of functions in* $C(K)$ *that are harmonic with respect to* $A(K)$ *and for each component* U *of* $\mathrm{Int}(K)$ *there is some* f_α *that is nonholomorphic on* U*, then the function algebra generated by* $A(K)$ *and* $\{f_\alpha\}$ *is* $C(K)$*.*

Analogs for the remaining results of this section also hold.

5. Maximal subalgebras

By a theorem of Wermer [24], the disc algebra on the circle, $A_{\partial D}$, is a maximal subalgebra of $C(\partial D)$. An open question is whether $A(K)|\partial K$ is a maximal subalgebra of $C(\partial K)$ whenever K is a compact set in the plane with connected interior. Various results, which put restrictions on the set K, can be found in [4], [10], and [15]. In this section, we state two of the maximality results from [15] that also hold on a Riemann surface.

THEOREM 5.1. *Let* K *be a compact subset of an open Riemann surface* $\mathcal{R}$*. Suppose* K *is regular for the Dirichlet problem and is such that every function that is continuous on* K *and harmonic on* $\mathrm{Int}(K)$ *is harmonic with respect to* $A(K)$*. If* $\mathrm{Int}(K)$ *is connected, then* $A(K)|\partial K$ *is a maximal subalgebra of* $C(\partial K)$*.*

Izzo [15] has given two different proofs of the above theorem in the case where K is a compact subset of the complex plane. The first proof given in [15, Theorem 3.1], with Theorem 2.2 in [15] replaced by Theorem 4.4 in this paper, also holds on a Riemann surface. Similarly, for the theorem below, Izzo's first proof [15, Theorem 3.2], with Lemma 1.3 in [15] replaced by Lemmas 2.16 and 3.3 in this paper, also holds for a Riemann surface.

THEOREM 5.2. *Let* K *be a compact subset of an open Riemann surface* $\mathcal{R}$*. Suppose* K *is regular for the Dirichlet problem and is such that every function that is continuous on* K *and harmonic on* $\mathrm{Int}(K)$ *is harmonic with respect to* $A(K)$*. A function algebra* B *on* ∂K *that contains* $A(K)|\partial K$ *is maximal in* $C(\partial K)$ *if and only if there is a component* U *of* $\mathrm{Int}(K)$ *such that* B *consists of all the continuous functions on* ∂K *whose harmonic extension to* K *is holomorphic on* U*.*

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