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Research Article

Stability of Optimal Controls for the Stationary Boussinesq Equations

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The stationary Boussinesq equations describing the heat transfer in the viscous heat-conducting fluid under inhomogeneous Dirichlet boundary conditions for velocity and mixed boundary conditions for temperature are considered. The optimal control problems for these equations with tracking-type functionals are formulated. A local stability of the concrete control problem solutions with respect to some disturbances of both cost functionals and state equation is proved.

1. Introduction

Much attention has been recently given to the optimal control problems for thermal and hydrodynamic processes. In fluid dynamics and thermal convection, such problems are motivated by the search for the most effective mechanisms of the thermal and hydrodynamic fields control [1–4]. A number of papers are devoted to theoretical study of control problems for stationary models of heat and mass transfer (see e.g., [5–19]). A solvability of extremum problems is proved, and optimality systems which describe the necessary conditions of extremum were constructed and studied. Sufficient conditions to the data are established in [16, 18, 19] which provide the uniqueness and stability of solutions of control problems in particular cases.

Along with the optimal control problems, an important role in applications is played by the identification problems for heat and mass transfer models. In these problems, unknown densities of boundary or distributed sources, coefficients of model differential equations, or boundary conditions are recovered from additional information of the original boundary value problem solution. It is significant that the identification problems can be reduced to appropriate extremum problems by choosing a suitable tracking-type cost functional. As a result, both control and identification problems can be studied using an unified approach based on the constrained optimization theory in the Hilbert or Banach spaces (see [1–4]).

The main goal of this paper is to perform an uniqueness and stability analysis of solutions to control problems with tracking-type functionals for the steady-state Boussinesq equations. We shall consider the situation when the boundary or distributed heat sources play roles of controls and the cost functional depends on the velocity. Using some results of [2] we deduce firstly the optimality system for the general control problem which describes the first-order necessary optimality conditions. Then, based on the optimality system analysis, we deduce a special inequality for the difference of solutions to the original and perturbed control problems. The latter is obtained by perturbing both cost functional and one of the functions entering into the state equation. Using this inequality, we shall establish the sufficient conditions for data which provide a local stability and uniqueness of solutions to control problems under consideration in the case of concrete tracking-type cost functionals.

The structure of the paper is as follows. In Section 2, the boundary value problem for the stationary Boussinesq equations is formulated, and some properties of the solution are described. In Section 3, an optimal control problem is stated, and some theorems concerning the problem solvability, validity of the Lagrange principle for it, and regularity of the Lagrange multiplier are given. In addition, some additional properties of solutions to the control problem under consideration will be established. In Section 4, we shall prove the local stability and uniqueness of solutions to control problems with the velocity-tracking cost functionals. Finally, in Section 5, the local uniqueness and stability of optimal controls for the vorticity-tracking cost functional is proved.

2. Statement of Boundary Problem

In this paper we consider the model of heat transfer in a viscous incompressible heat-conducting fluid. The model consists of the Navier-Stokes equation and the convection-diffusion equation for temperature that are nonlinearly related via buoyancy in the Boussinesq approximation and via convective heat transfer. It is described by equations

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} - \widetilde{\beta}\mathbf{G}T, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$-\lambda\Delta T + \mathbf{u}\cdot\nabla T = f \quad \text{in } \Omega, \tag{2.2}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma, \qquad T = \psi \quad \text{on } \Gamma_D, \qquad \lambda\left(\frac{\partial T}{\partial n} + \alpha T\right) = \chi \quad \text{on } \Gamma_N. \tag{2.3}$$

Here Ω is a bounded domain in the space $\mathbb{R}^d$, $d = 2, 3$ with a boundary Γ consisting of two parts Γ_D and Γ_N; $\mathbf{u}$, p, and T denote the velocity and temperature fields, respectively; $p = P/\rho$, where P is the pressure and $\rho = \text{const} > 0$ is the density of the medium; ν is the kinematic viscosity coefficient, $\mathbf{G}$ is the gravitational acceleration vector, $\widetilde{\beta}$ is the volumetric thermal expansion coefficient, λ is the thermal conductivity coefficient, $\mathbf{g}$ is a given vector-function on Γ, ψ is a given function on a part Γ_D of Γ, χ is a function given on another part $\Gamma_N = \Gamma \setminus \Gamma_D$ of Γ, $\mathbf{n}$ is the unit outer normal. We shall refer to problem (2.1)–(2.3) as Problem 1. We note that all quantities in (2.1)–(2.3) are dimensional and their dimensions are defined in terms of SI units.

We assume that the following conditions are satisfied:

(i) Ω is a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with Lipschitz boundary $\Gamma \in C^{0,1}$, consisting of coupled components $\Gamma^{(i)}$, $i = 1, 2, \ldots, N$; $\Gamma = \Gamma_D \cup \Gamma_N$ and meas $\Gamma_D > 0$.

Below we shall use the Sobolev spaces $H^s(D)$ and $L^2(D)$, where $s \in \mathbb{R}$, or $\mathbf{H}^s(D)$ and $\mathbf{L}^2(D)$ for the vector functions where D denotes Ω, its subset Q, Γ or a part Γ_0 of the boundary Γ. In particularly we need the function spaces $H^1(\Omega)$, $L^2(\Omega)$, $\mathbf{H}^1(\Omega)$, $\mathbf{H}^{1/2}(\Gamma)$, $H^{1/2}(\Gamma_D)$ and their subspaces

$$\mathcal{T} = \left\{\theta \in H^1(\Omega) : \theta|_{\Gamma_D} = 0\right\}, \qquad L^2_0(\Omega) = \left\{r \in L^2(\Omega) : \int_\Omega r dx = 0\right\},$$
$$\mathbf{H}^1_{\mathrm{div}}(\Omega) = \left\{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathrm{div}\ \mathbf{v} = 0\right\}, \qquad \mathbf{H}^1_0(\Omega) = \left\{\mathbf{v} \in H^1(\Omega) : \mathbf{v}|_\Gamma = 0\right\},$$
$$\mathbf{V} = \left\{\mathbf{v} \in \mathbf{H}^1_0(\Omega) : \mathrm{div}\ \mathbf{v} = 0\right\}, \qquad \widetilde{\mathbf{H}}^1(\Omega) = \left\{\mathbf{v} \in \mathbf{H}^1(\Omega) : (\mathbf{v}\cdot\mathbf{n}, 1)_{\Gamma^{(i)}} = 0,\ \mathbf{v}\cdot\mathbf{n}|_{\Gamma_N} = 0\right\},$$
$$\widetilde{\mathbf{H}}^{1/2}(\Gamma) = \left\{\mathbf{v}|_\Gamma :\ \mathbf{v} \in \widetilde{\mathbf{H}}^1(\Omega)\right\} \subset \mathbf{H}^{1/2}(\Gamma), \qquad L^2_+(\Gamma_N) = \left\{\phi \in L^2(\Gamma_N) : \phi \geq 0 \text{ a.e. on } \Gamma_N\right\}. \tag{2.4}$$

The inner products and norms in $L^2(\Omega)$, $L^2(Q)$, or $L^2(\Gamma_N)$ are denoted by $(\cdot,\cdot)$, $\|\cdot\|$, $(\cdot,\cdot)_Q$, $\|\cdot\|_Q$, or $(\cdot,\cdot)_{\Gamma_N}$, $\|\cdot\|_{\Gamma_N}$. The inner products, norms and seminorms in $H^1(Q)$ and $\mathbf{H}^1(Q)$ are denoted by $(\cdot,\cdot)_{1,Q}$, $\|\cdot\|_{1,Q}$, and $|\cdot|_{1,Q}$ or $(\cdot,\cdot)_1$, $\|\cdot\|_1$ and $|\cdot|_1$ if $Q = \Omega$. The norms in $\mathbf{H}^{1/2}(\Gamma)$ or $H^{1/2}(\Gamma_D)$ are denoted by $\|\cdot\|_{1/2,\Gamma}$ or $\|\cdot\|_{1/2,\Gamma_D}$; the norm in the dual space $\widetilde{\mathbf{H}}^{1/2}(\Gamma)^*$ is denoted by $\|\cdot\|_{-1/2,\Gamma}$. Set $\mathbf{b} \equiv \widetilde{\beta}\mathbf{G}$. Let in addition to condition (i) the following conditions hold:

(ii) $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, $\mathbf{b} \equiv \widetilde{\beta}\mathbf{G} \in \mathbf{L}^2(\Omega)$, $\alpha \in L^2(\Gamma_N)$.

The following technical lemma holds (see [2, 20]).

Lemma 2.1. *Under conditions (i) there exist constants $\delta_i > 0$, $\gamma_i > 0$, C_d, C_r, and $\beta_1 > 0$ such that*

$$(\nabla\mathbf{v}, \nabla\mathbf{v}) \geq \delta_0\|\mathbf{v}\|_1^2 \quad \forall\mathbf{v} \in \mathbf{H}^1_0(\Omega), \qquad (\nabla T, \nabla T) \geq \delta_1\|T\|_1^2 \quad \forall T \in \mathcal{T}, \tag{2.5}$$
$$|((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w})| \leq \gamma_0\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1, \qquad \left|(\mathbf{u}\cdot\nabla T, \eta)\right| \leq \gamma_1\|\mathbf{u}\|_1\|T\|_1\|\eta\|_1, \tag{2.6}$$
$$|(\mathbf{b}T, \mathbf{v})| \leq \beta_1\|T\|_1\|\mathbf{v}\|_1 \quad \forall T \in H^1(\Omega),\ \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{2.7}$$
$$\begin{gathered}\left|(\chi, T)_{\Gamma_N}\right| \leq \gamma_2\|\chi\|_{\Gamma_N}\|T\|_1, \qquad \left|(\alpha T, \eta)_{\Gamma_N}\right| \leq \gamma_3\|\alpha\|_{\Gamma_N}\|T\|_1\|\eta\|_1, \\ \|T\|_Q \leq \gamma_4\|T\|_1, \qquad \|\mathbf{v}\|_Q \leq \gamma_4\|\mathbf{v}\|_1,\end{gathered} \tag{2.8}$$
$$\|\mathrm{rot}\ \mathbf{v}\| \leq C_r\|\mathbf{v}\|_1, \qquad \|\mathrm{div}\ \mathbf{v}\| \leq C_d\|\mathbf{v}\|_1. \tag{2.9}$$

Bilinear form $-(\mathrm{div}\ \cdot,\cdot)$ *satisfies the* inf-sup *condition*

$$\inf_{\substack{r\in L^2_0(\Omega)\\ r\neq 0}} \sup_{\substack{\mathbf{v}\in\mathbf{H}^1_0(\Omega)\\ \mathbf{v}\neq 0}} \frac{-(\mathrm{div}\ \mathbf{v}, r)}{\|\mathbf{v}\|_1\|r\|} \geq \beta = \mathrm{const} > 0. \tag{2.10}$$

Besides the following identities take place:

$$((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{v})=0 \quad \forall\mathbf{u}\in\mathbf{H}^1_{\mathrm{div}}(\Omega),\ \mathbf{v}\in\mathbf{H}^1_0(\Omega), \tag{2.11}$$

$$(\mathbf{u}\cdot\nabla T,T)=0 \quad \forall\mathbf{u}\in\mathbf{H}^1_{\mathrm{div}}(\Omega)\cap\widetilde{\mathbf{H}}^1(\Omega),\ T\in H^1(\Omega). \tag{2.12}$$

Let $\mathbf{g}\in\widetilde{\mathbf{H}}^{1/2}(\Gamma)$, $\chi\in L^2(\Gamma_N)$, $\psi\in H^{1/2}(\Gamma_D)$, $f\in L^2(\Omega)$ in addition to (i), (ii). We multiply the equations in (2.1), (2.2) by test functions $\mathbf{v}\in\mathbf{H}^1_0(\Omega)$ and $S\in\mathcal{T}$ and integrate the results over Ω with use of Green's formulas to obtain the weak formulation for the model (2.1)–(2.3). It consists of finding a triple $\mathbf{x}\equiv(\mathbf{u},p,T)\in\widetilde{\mathbf{H}}^1(\Omega)\times L^2_0(\Omega)\times H^1(\Omega)$ satisfying the relations

$$\nu(\nabla\mathbf{u},\nabla\mathbf{v})+((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v})-(p,\operatorname{div}\ \mathbf{v})=\langle\mathbf{f},\mathbf{v}\rangle-(\mathbf{b}T,\mathbf{v}) \quad \forall\mathbf{v}\in\mathbf{H}^1_0(\Omega),\ \mathbf{b}\equiv\widetilde{\beta}\mathbf{G}, \tag{2.13}$$

$$\lambda(\nabla T,\nabla S)+\lambda(\alpha T,S)_{\Gamma_N}+(\mathbf{u}\cdot\nabla T,S)=(f,S)+(\chi,S)_{\Gamma_N} \quad \forall S\in\mathcal{T}, \tag{2.14}$$

$$\operatorname{div}\ \mathbf{u}=0 \quad \text{in } \Omega, \qquad \mathbf{u}=\mathbf{g} \quad \text{on } \Gamma, \qquad T=\psi \quad \text{on } \Gamma_D. \tag{2.15}$$

Following theorem (see [2]) establishes the solvability of Problem 1 and gives a priori estimates for its solution.

Theorem 2.2. *Let conditions (i), (ii) be satisfied. Then Problem 1 has for every quadruple* $\mathbf{g}\in\widetilde{\mathbf{H}}^{1/2}(\Gamma)$, $\chi\in L^2(\Gamma_N)$, $f\in L^2(\Omega)$, $\psi\in H^{1/2}(\Gamma_D)$ *a weak solution* $(\mathbf{u},p,T)$ *that satisfies the estimates*

$$\|\mathbf{u}\|_1\le M_{\mathbf{u}}, \qquad \|p\|\le M_p, \qquad \|T\|_1\le M_T. \tag{2.16}$$

Here $M_{\mathbf{u}}$, M_p *and* M_T *are nondecreasing continuous functions of the norms* $\|\mathbf{f}\|_{-1}$, $\|\mathbf{b}\|$, $\|\mathbf{g}\|_{1/2,\Gamma}$, $\|\chi\|_{\Gamma_N}$, $\|f\|$, $\|\psi\|_{1/2,\Gamma_D}$, $\|\alpha\|_{\Gamma_N}$. *If, additionally,* $\mathbf{f},\mathbf{g},\chi,f,\psi,\alpha$ *are small in the sense that*

$$\frac{\gamma_0}{\delta_0\nu}M_{\mathbf{u}}+\frac{\gamma_1}{\delta_0\nu}\frac{\beta_1}{\delta_1\lambda}M_T<1, \tag{2.17}$$

where δ_0, δ_1, γ_0, γ_1 *and* β_1 *are constants entering into* (2.5)–(2.7), *then the weak solution to Problem 1 is unique.*

3. Statement of Control Problems

Our goal is the study of control problems for the model (2.1)–(2.3) with tracking-type functionals. The problems consist in minimization of certain functionals depending on the state and controls. As the cost functionals we choose some of the following ones:

$$I_1(\mathbf{v})=\|\mathbf{v}-\mathbf{v}_d\|^2_Q, \qquad I_2(\mathbf{v})=\|\mathbf{v}-\mathbf{v}_d\|^2_{1,Q}, \qquad I_3(\mathbf{v})=\|\operatorname{rot}\ \mathbf{v}-\zeta_d\|^2_Q. \tag{3.1}$$

Here Q is a subdomain of Ω. The functionals I_1, I_2, and I_3 where functions $\mathbf{u}_d\in\mathbf{L}^2(Q)$ (or $\mathbf{u}_d\in\mathbf{H}^1(Q)$) and $\zeta_d\in\mathbf{L}^2(Q)$ are interpreted as measured velocity or vorticity fields are used to solve the inverse problems for the models in questions [2].

In order to formulate a control problem for the model (2.1)–(2.3) we split the set of all data of Problem 1 into two groups: the group of controls containing the functions $\chi \in L^2(\Gamma_N)$, $\psi \in H^{1/2}(\Gamma_D)$, and $f \in L^2(\Omega)$, which play the role of controls and the group of fixed data comprising the invariable functions $\mathbf{f}, \mathbf{b},$ and α. As to the function $\mathbf{g}$ entering into the boundary condition for the velocity in (2.3), it will play peculiar role since the stability of solutions to control problems under consideration (see below) will be studied with respect to small perturbations, both the cost functional and the function $\mathbf{g}$ in the norm of $\mathbf{H}^{1/2}(\Gamma)$.

Let $X = \widetilde{\mathbf{H}}^1(\Omega) \times L^2_0(\Omega) \times H^1(\Omega)$, $Y = \mathbf{H}^{-1}(\Omega) \times L^2_0(\Omega) \times \widetilde{\mathbf{H}}^{1/2}(\Gamma) \times \mathcal{T}^* \times H^{1/2}(\Gamma_D)$. Denote by $I : \widetilde{\mathbf{H}}^1(\Omega) \to \mathbb{R}$ a weakly lower semicontinuous functional. We assume that the controls χ, ψ, and f vary in some sets $K_1 \subset L^2(\Gamma_N)$, $K_2 \subset H^{1/2}(\Gamma_D)$, $K_3 \subset L^2(\Omega)$. Setting $K = K_1 \times K_2 \times K_3$, $\mathbf{x} = (\mathbf{u}, p, T)$, $u_0 = (\mathbf{f}, \mathbf{b}, \alpha)$, $u = (\chi, \psi, f)$ we introduce the functional $J : X \times K \to \mathbb{R}$ by the formula

$$J(\mathbf{x}, u) = \frac{\mu_0}{2} I(\mathbf{u}) + \frac{\mu_1}{2} \|\chi\|^2_{\Gamma_N} + \frac{\mu_2}{2} \|\psi\|^2_{1/2,\Gamma_D} + \frac{\mu_3}{2} \|f\|^2. \tag{3.2}$$

Here $\mu_0, \mu_1, \mu_2, \mu_3$ are nonnegative parameters which serve to regulate the relative importance of each of terms in (3.2) and besides to match their dimensions. Another goal of introducing parameters μ_i is to ensure the uniqueness and stability of the solutions to control problems under study (see below).

We assume that following conditions take place:

(iii) $K_1 \subset L^2(\Gamma_N)$, $K_2 \subset H^{1/2}(\Gamma_D)$, $K_3 \subset L^2(\Omega)$ are nonempty closed convex sets;

(iv) $\mu_0 > 0$, $\mu_l > 0$ or $\mu_0 > 0$, $\mu_l \geq 0$ and K_l is a bounded set, $l = 1, 2, 3$.

Considering the functional J at weak solutions to Problem 1 we write the corresponding constraint which has the form of the weak formulation (2.13)–(2.15) of Problem 1 as follows:

$$F(\mathbf{x}, u, \mathbf{g}) = F(\mathbf{u}, p, T, \chi, \psi, f, \mathbf{g}) = 0. \tag{3.3}$$

Here $F = (F_1, F_2, F_3, F_4, F_5) : X \times K \times \widetilde{\mathbf{H}}^{1/2}(\Gamma) \to Y$ is the operator acting by formulas

$$\begin{gathered}
\langle F_1(\mathbf{x}), \mathbf{v} \rangle = \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) - \langle \mathbf{f}, \mathbf{v} \rangle + (\mathbf{b}T, \mathbf{v}), \\
F_2(\mathbf{x}) = \operatorname{div} \mathbf{u}, \qquad F_3(\mathbf{x}, \mathbf{g}) = \mathbf{u}|_\Gamma - \mathbf{g}, \qquad F_5(\mathbf{x}, \psi) = T|_{\Gamma_D} - \psi, \\
\langle F_4(\mathbf{x}, f, \chi), S \rangle = \lambda(\nabla T, \nabla S) + \lambda(\alpha T, S)_{\Gamma_N} + (\mathbf{u} \cdot \nabla T, S) - (f, S) - (\chi, S)_{\Gamma_N}.
\end{gathered} \tag{3.4}$$

The mathematical statement of the optimal control problem is as follows: to seek a pair $(\mathbf{x}, u)$, where $\mathbf{x} = (\mathbf{u}, p, T) \in X$ and $u = (\chi, \psi, f) \in K_1 \times K_2 \times K_3 = K$ such that

$$\begin{gathered}
J(\mathbf{x}, u) \equiv \frac{\mu_0}{2} I(\mathbf{u}) + \frac{\mu_1}{2} \|\chi\|^2_{\Gamma_N} + \frac{\mu_2}{2} \|\psi\|^2_{1/2,\Gamma_D} + \frac{\mu_3}{2} \|f\|^2 \longrightarrow \inf, \\
F(\mathbf{x}, u, \mathbf{g}) = 0, \quad (\mathbf{x}, u) \in X \times K.
\end{gathered} \tag{3.5}$$

Let $X^* \equiv \widetilde{\mathbf{H}}^1(\Omega)^* \times L^2_0(\Omega) \times H^1(\Omega)^*$ and $Y^* \equiv \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega) \times \widetilde{\mathbf{H}}^{1/2}(\Gamma)^* \times \mathcal{T} \times H^{1/2}(\Gamma_D)^*$ be the duals of the spaces X and Y. Let $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{u}, \mathbf{g}) : X \to Y$ denotes the Fréchet derivative of F with respect to $\mathbf{x}$ at the point $(\widehat{\mathbf{x}}, \widehat{u}, \mathbf{g})$. By $F'_{\mathbf{x}}(\widehat{\mathbf{x}}, \widehat{u}, \mathbf{g})^* : Y^* \to X^*$ we denote the adjoint operator of $F'_{\mathbf{x}}(\widehat{\mathbf{x}}, \widehat{u}, \mathbf{g})$ which is determined by the relation

$$\langle F'_{\mathbf{x}}(\widehat{\mathbf{x}}, \widehat{u}, \mathbf{g})^* \mathbf{y}^*, \mathbf{x} \rangle_{X^* \times X} = \langle \mathbf{y}^*, F'_{\mathbf{x}}(\widehat{\mathbf{x}}, \widehat{u}, \mathbf{g}) \mathbf{x} \rangle_{Y^* \times Y} \quad \forall \mathbf{x} \in X, \ \mathbf{y}^* \in Y^*. \tag{3.6}$$

According to the general theory of extremum problems (see [21]) we introduce an element $\mathbf{y}^* = (\xi, \sigma, \zeta, \theta, \zeta^t) \in Y^*$ which is referred to as the adjoint state and define the Lagrangian $\mathcal{L} : X \times K \times \mathbb{R}^+ \times Y^* \times \widetilde{\mathbf{H}}^{1/2}(\Gamma) \to \mathbb{R}$, where $\mathbb{R}^+ = \{x \in \mathbb{R} : \ x \geq 0\}$, by

$$\begin{aligned} \mathcal{L}(\mathbf{x}, u, \lambda_0, \mathbf{y}^*, \mathbf{g}) = \lambda_0 J(\mathbf{x}, u) + \langle \mathbf{y}^*, F(\mathbf{x}, u) \rangle &\equiv \lambda_0 J(\mathbf{x}, u) \\ &+ \langle F_1(\mathbf{x}), \xi \rangle + (F_2(\mathbf{x}), \sigma) + \langle \zeta, F_3(\mathbf{x}, \mathbf{g}) \rangle_\Gamma \\ &+ \kappa \langle F_4(\mathbf{x}, f, \chi), \theta \rangle + \kappa \langle \zeta^t, F_5(\mathbf{x}, \varphi) \rangle_{\Gamma_D}. \end{aligned} \tag{3.7}$$

Here and below $\langle \zeta, \cdot \rangle_\Gamma \equiv \langle \zeta, \cdot \rangle_{\widetilde{\mathbf{H}}^{1/2}(\Gamma)^* \times \widetilde{\mathbf{H}}^{1/2}(\Gamma)}$, $\langle \zeta^t, \cdot \rangle_{\Gamma_D} \equiv \langle \zeta^t, \cdot \rangle_{H^{1/2}(\Gamma_D)^* \times H^{1/2}(\Gamma_D)}$ and κ is an auxiliary dimensional parameter. Its dimension $[\kappa]$ is chosen so that dimensions of ξ, σ, θ at the adjoint state coincide with those at the basic state, that is,

$$[\xi] = [\mathbf{u}] = L_0 T_0^{-1}, \qquad [\theta] = [T] = K_0, \qquad [\sigma] = [p] = L_0^2 T_0^{-2}. \tag{3.8}$$

Here L_0, T_0, M_0, K_0 denote the SI dimensions of the length, time, mass, and temperature units expressed in meters, seconds, kilograms, and degrees Kelvin, respectively. As a result ξ, σ, and θ can be referred to below as the adjoint velocity, pressure, and temperature. Simple analysis shows (see details in [16]) that the necessity for the fulfillment of (3.8) is that $[\kappa]$ is given by $[\kappa] = L_0^2 T_0^{-2} K_0^{-2}$.

The following theorems (see, e.g., [2]) give sufficient conditions for the solvability of control problem (3.5), the validity of the Lagrange principle for it, and a regularity condition for a Lagrange multiplier.

Theorem 3.1. *Let conditions (i)–(iv) hold and* $\mathbf{g} \in \widetilde{\mathbf{H}}^{1/2}(\Gamma)$*. Then there exists at least one solution* $(\widehat{\mathbf{x}}, \widehat{u}) = (\widehat{\mathbf{u}}, \widehat{p}, \widehat{T}, \widehat{\chi}, \widehat{\varphi}, \widehat{f})$ *to problem* (3.5) *for* $I = I_k$, $k = 1, 2, 3$.

Theorem 3.2. *Let under conditions of Theorem 3.1 a pair* $(\widehat{\mathbf{x}}, \widehat{u}) \equiv (\widehat{\mathbf{u}}, \widehat{p}, \widehat{T}, \widehat{\chi}, \widehat{\varphi}, \widehat{f}) \in X \times K$ *be a local minimizer in problem* (3.5) *and let the cost functional I be continuously differentiable with respect to* $\mathbf{u}$ *at the point* $\widehat{\mathbf{x}}$*. Then there exists a nonzero Lagrange multiplier* $(\lambda_0, \mathbf{y}^*) = (\lambda_0, \xi, \sigma, \zeta, \theta, \zeta^t) \in \mathbb{R}^+ \times Y^*$ *such that the Euler-Lagrange equation*

$$F'_{\mathbf{x}}(\widehat{\mathbf{x}}, \widehat{u}, \mathbf{g})^* \mathbf{y}^* = -\lambda_0 J'_{\mathbf{x}}(\widehat{\mathbf{x}}, \widehat{u}) \quad \text{in } X^* \tag{3.9}$$

for the adjoint state $\mathbf{y}^*$ *is satisfied and the minimum principle holds which is equivalent to the inequality*

$$\mathcal{L}(\widehat{\mathbf{x}}, \widehat{u}, \lambda_0, \mathbf{y}^*, \mathbf{g}) \leq \mathcal{L}(\widehat{\mathbf{x}}, u, \lambda_0, \mathbf{y}^*, \mathbf{g}) \quad \forall u \in K. \tag{3.10}$$

Theorem 3.3. *Let the assumptions of Theorem 3.2 be satisfied and condition* (2.17) *holds for all* $u \equiv (\chi, \varphi, f) \in K$. *Then any nontrivial Lagrange multiplier satisfying* (3.9) *is regular, that is, has the form* $(1, \mathbf{y}^*)$ *and is uniquely determined.*

We note that the functional J and Lagrangian $\mathcal{L}$ given by (3.7) are continuously differentiable functions of controls χ, φ, f and its derivatives with respect to χ, φ, and f are given by

$$\begin{gathered}
\left\langle J'_\chi(\widehat{\mathbf{x}}, \widehat{u}), \chi \right\rangle = \mu_1(\widehat{\chi}, \chi)_{\Gamma_N}, \qquad \left\langle J'_\varphi(\widehat{\mathbf{x}}, \widehat{u}), \varphi \right\rangle = \mu_2(\widehat{\varphi}, \varphi)_{1/2,\Gamma_D}, \qquad \left\langle J'_f(\widehat{\mathbf{x}}, \widehat{u}), f \right\rangle = \mu_3\left(\widehat{f}, f\right), \\
\left\langle \mathcal{L}'_\chi(\widehat{\mathbf{x}}, \widehat{u}, \lambda_0, \mathbf{y}^*, \mathbf{g}), \chi \right\rangle = \lambda_0\mu_1(\widehat{\chi}, \chi)_{\Gamma_N} - \kappa(\theta, \chi)_{\Gamma_N} \equiv (\lambda_0\mu_1\widehat{\chi} - \kappa\theta, \chi)_{\Gamma_N} \quad \forall \chi \in K_1, \\
\left\langle \mathcal{L}'_\varphi(\widehat{\mathbf{x}}, \widehat{u}, \lambda_0, \mathbf{y}^*, \mathbf{g}), \varphi \right\rangle = \lambda_0\mu_2(\widehat{\varphi}, \varphi)_{1/2,\Gamma_D} - \kappa\langle \zeta^t, \varphi \rangle_{\Gamma_D} \equiv \langle \lambda_0\mu_2\widehat{\varphi} - \kappa\zeta^t, \varphi \rangle_{\Gamma_D} \quad \forall \varphi \in K_2, \\
\left\langle \mathcal{L}'_f(\widehat{\mathbf{x}}, \widehat{u}, \lambda_0, \mathbf{y}^*, \mathbf{g}), f \right\rangle = \lambda_0\mu_3\left(\widehat{f}, f\right) - \kappa(\theta, f) \equiv \left(\lambda_0\mu_3\widehat{f} - \kappa\theta, f\right) \quad \forall f \in K_3.
\end{gathered} \tag{3.11}$$

Here for example $\mathcal{L}'_\chi(\widehat{\mathbf{x}}, \widehat{u}, \lambda_0, \mathbf{y}^*, \mathbf{g})$ is the Gateaux derivative with respect to χ at the point $(\widehat{\mathbf{x}}, \widehat{u}, \lambda_0, \mathbf{y}^*, \mathbf{g}) \in X \times K \times \mathbb{R}^+ \times Y^* \times \widetilde{\mathbf{H}}^{1/2}(\Gamma)$. Since K_1, K_2, K_3 are convex sets, at the minimum point $\widehat{u} = (\widehat{\chi}, \widehat{\varphi}, \widehat{f})$ of the functional $\mathcal{L}(\widehat{\mathbf{x}}, \cdot, \lambda_0, \mathbf{y}^*, \mathbf{g})$ the following conditions are satisfied (see [22]):

$$\begin{gathered}
\left\langle \mathcal{L}'_\chi(\widehat{\mathbf{x}}, \widehat{u}, \lambda_0, \mathbf{y}^*, \mathbf{g}), \chi - \widehat{\chi} \right\rangle \equiv (\lambda_0\mu_1\widehat{\chi} - \kappa\theta, \chi - \widehat{\chi})_{\Gamma_N} \geq 0 \quad \forall \chi \in K_1, \\
\left\langle \mathcal{L}'_\varphi(\widehat{\mathbf{x}}, \widehat{u}, \lambda_0, \mathbf{y}^*, \mathbf{g}), \varphi - \widehat{\varphi} \right\rangle = \langle \lambda_0\mu_2\widehat{\varphi} - \kappa\zeta^t, \varphi - \widehat{\varphi} \rangle_{1/2,\Gamma_D} \geq 0 \quad \forall \varphi \in K_2, \\
\left\langle \mathcal{L}'_f(\widehat{\mathbf{x}}, \widehat{u}, \lambda_0, \mathbf{y}^*, \mathbf{g}), f - \widehat{f} \right\rangle = \left(\lambda_0\mu_3\widehat{f} - \kappa\theta, f - \widehat{f}\right) \geq 0 \quad \forall f \in K_3.
\end{gathered} \tag{3.12}$$

We also note that the Euler-Lagrange equation (3.9) is equivalent to identities

$$\begin{gathered}
\nu(\nabla \mathbf{w}, \nabla \xi) + ((\widehat{\mathbf{u}} \cdot \nabla)\mathbf{w}, \xi) + ((\mathbf{w} \cdot \nabla)\widehat{\mathbf{u}}, \xi) + \kappa\left(\mathbf{w} \cdot \nabla \widehat{T}, \theta\right) \\
-(\sigma, \operatorname{div} \mathbf{w}) + \langle \zeta, \mathbf{w} \rangle_\Gamma + \lambda_0\langle J'_\mathbf{u}(\widehat{\mathbf{x}}, \widehat{u}), \mathbf{w} \rangle = 0 \quad \forall \mathbf{w} \in \widetilde{\mathbf{H}}^1(\Omega), \\
(r, \operatorname{div} \xi) = 0 \quad \forall r \in L_0^2(\Omega), \\
\kappa\left[\lambda(\nabla \tau, \nabla \theta) + \lambda(\alpha\tau, \theta)_{\Gamma_N} + (\widehat{\mathbf{u}} \cdot \nabla \tau, \theta) + \langle \zeta^t, \tau \rangle_{\Gamma_D}\right] + (\mathbf{b}\tau, \xi) = 0 \quad \forall \tau \in H^1(\Omega).
\end{gathered} \tag{3.13}$$

Relations (3.13), the minimum principle which is equivalent to the inequalities (3.10) or (3.12), and the operator constraint (3.3) which is equivalent to (2.13)–(2.15) constitute the optimality system for control problem (3.5).

Theorems 3.1 and 3.2 above are valid without any smallness conditions in relation to the data of Problem 1. The natural smallness condition (2.17) arises only when proving the uniqueness of solution to boundary problem (2.1)–(2.3) and Lagrange multiplier regularity. However, condition (2.17) does not provide the uniqueness of problem (3.5) solution.

Therefore, an investigation of problem (3.5) solution uniqueness is an interesting and complicated problem. Studying of its solution stability with respect to small perturbations of both cost functional I entering into (3.2) and state equation (3.3) is also of interest. In order to investigate these questions we should establish some additional properties of the solution for the optimality system (2.13)–(2.15), (3.12), (3.13). Based on these properties, we shall impose in the next section the sufficient conditions providing the uniqueness and stability of solutions to control problem (3.5) for particular cost functionals introduced in (3.1).

Let us consider problem (3.5). We assume below that the function $\mathbf{g}$ entering into (2.3) can vary in a certain set $G \subset \widetilde{\mathbf{H}}^{1/2}(\Gamma)$. Let $(\mathbf{x}_1, u_1) \equiv (\mathbf{u}_1, p_1, T_1, \chi_1, \varphi_1, f_1) \in X \times K$ be an arbitrary solution to problem (3.5) for a given function $\mathbf{g} = \mathbf{g}_1 \in G$. By $(\mathbf{x}_2, u_2) \equiv (\mathbf{u}_2, p_2, T_2, \chi_2, \varphi_2, f_2) \in X \times K$ we denote a solution to problem

$$\widetilde{J}(\mathbf{x}, u) \equiv \frac{\mu_0}{2}\widetilde{I}(\mathbf{u}) + \frac{\mu_1}{2}\|\chi\|^2_{\Gamma_N} + \frac{\mu_2}{2}\|\varphi\|^2_{1/2,\Gamma_D} + \frac{\mu_3}{2}\|f\|^2 \longrightarrow \inf, \quad F(\mathbf{x}, u, \widetilde{\mathbf{g}}) = 0, \quad (\mathbf{x}, u) \in X \times K. \tag{3.14}$$

It is obtained by replacing the functional I in (3.5) by a close functional $\widetilde{I}$ depending on $\mathbf{u}$ and by replacing a function $\mathbf{g} \in G$ by a close function $\widetilde{\mathbf{g}} \in G$.

By Theorem 3.1 the following estimates hold for triples $(\mathbf{u}_i, p_i, T_i)$:

$$\|\mathbf{u}_i\|_1 \leq M^0_{\mathbf{u}}, \qquad \|p_i\| \leq M^0_p, \qquad \|T_i\|_1 \leq M^0_T. \tag{3.15}$$

Here

$$M^0_{\mathbf{u}} = \sup_{u\in K, \mathbf{g}\in G} M_{\mathbf{u}}(u_0, u, \mathbf{g}), \qquad M^0_p = \sup_{u\in K, \mathbf{g}\in G} M_p(u_0, u, \mathbf{g}), \qquad M^0_T = \sup_{u\in K, \mathbf{g}\in G} M_T(u_0, u, \mathbf{g}), \tag{3.16}$$

where $M_{\mathbf{u}}$, M_p, and M_T are introduced in Theorem 3.1. We introduce "model" Reynolds number $\mathcal{R}e$, Raley number $\mathcal{R}a$, and Prandtl number $\mathcal{P}$ by

$$\mathcal{R}e = \frac{\gamma_0 M^0_{\mathbf{u}}}{\delta_0 \nu}, \qquad \mathcal{R}a = \frac{\gamma_1}{\delta_0 \nu}\frac{\beta_1 M^0_T}{\delta_1 \lambda}, \qquad \mathcal{P} = \frac{\delta_0 \nu}{\delta_1 \lambda}. \tag{3.17}$$

They are analogues of the following dimensionless parameters widely used in fluid dynamics: the Reynolds number Re, the Rayleigh number Ra, and the Prandtl number Pr. We can show that the parameters introduced in (3.17) are also dimensionless if $\|u\|$, $|u|_1$, and $\|u\|_1$ (where u is an arbitrary scalar) are defined as

$$\|u\|^2 = \int_\Omega u^2 dx, \qquad |u|^2_1 = \int_\Omega |\nabla u|^2 dx, \qquad \|u\|^2_1 = l^{-2}\|u\|^2 + |u|^2_1. \tag{3.18}$$

Here l is a dimensional factor of dimension $[l] = L_0$ whose value is equal to 1.

Assume that the following condition takes place:

$$\mathcal{R}e + \mathcal{R}a \equiv \frac{\gamma_0 M^0_{\mathbf{u}}}{\delta_0 \nu} + \frac{\gamma_1}{\delta_0 \nu}\frac{\beta_1 M^0_T}{\delta_1 \lambda} < \frac{1}{2}. \tag{3.19}$$

Let us denote by $(1, \mathbf{y}_i^*)$, where $\mathbf{y}_i^* \equiv (\xi_i, \sigma_i, \zeta_i, \theta_i, \zeta_i^t) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \widetilde{\mathbf{H}}^{1/2}(\Gamma)^* \times \mathcal{T} \times H^{1/2}(\Gamma_D)^*$, $i = 1, 2$, Lagrange multipliers corresponding to solutions $(\mathbf{x}_i, u_i)$. By Theorems 3.2 and 3.3 and (3.12) they satisfy relations

$$\begin{aligned} &\nu(\nabla \mathbf{w}, \nabla \xi_i) + ((\mathbf{u}_i \cdot \nabla)\mathbf{w}, \xi_i) + ((\mathbf{w} \cdot \nabla)\mathbf{u}_i, \xi_i) + \kappa(\mathbf{w} \cdot \nabla T_i, \theta_i) - (\sigma_i, \operatorname{div} \ \mathbf{w}) \\ &\quad + \langle \zeta_i, \mathbf{w} \rangle_\Gamma + \left(\frac{\mu_0}{2}\right)\left\langle (I^i)'(\mathbf{u}_i), \mathbf{w} \right\rangle = 0 \quad \forall \mathbf{w} \in \widetilde{\mathbf{H}}^1(\Omega),\ i = 1, 2, \end{aligned} \tag{3.20}$$

$$(\operatorname{div} \ \xi_i, r) = 0 \quad \forall r \in L_0^2(\Omega), \tag{3.21}$$

$$\kappa\left[\lambda(\nabla \tau, \nabla \theta_i) + \lambda(\alpha\tau, \theta)_{\Gamma_N} + (\mathbf{u}_i \cdot \nabla \tau, \theta_i) + \langle \zeta_i^t, \tau \rangle_{\Gamma_D}\right] + (\mathbf{b}\tau, \xi_i) = 0 \quad \forall \tau \in H^1(\Omega), \tag{3.22}$$

$$(\mu_1 \chi_i - \kappa\theta_i, \chi - \chi_i)_{\Gamma_N} + \langle \mu_2 \psi_i - \kappa \zeta_i^t, \psi - \psi_i \rangle_{\Gamma_D} + (\mu_3 f_i - \kappa\theta_i, f - f_i) \geq 0 \quad \forall (\chi, \psi, f) \in K. \tag{3.23}$$

We renamed $I^1 \equiv I$, $I^2 \equiv \widetilde{I}$ in (3.20). Set $\xi = \xi_1 - \xi_2$, $\sigma = \sigma_1 - \sigma_2$, $\zeta = \zeta_1 - \zeta_2$, $\theta = \theta_1 - \theta_2$, $\zeta^t = \zeta_1^t - \zeta_2^t$, $\mathbf{g} = \mathbf{g}_1 - \mathbf{g}_2$, and

$$\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2, \qquad p = p_1 - p_2, \qquad T = T_1 - T_2, \qquad \chi = \chi_1 - \chi_2, \qquad \psi = \psi_1 - \psi_2, \qquad f = f_1 - f_2. \tag{3.24}$$

Let us subtract (2.13)–(2.15), written for $\mathbf{u}_2, p_2, T_2, u_2, \mathbf{g}_2$ from (2.13)–(2.15) for $\mathbf{u}_1$, p_1, T_1, u_1, $\mathbf{g}_1$. We obtain

$$\nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{u} \cdot \nabla)\mathbf{u}_1 + (\mathbf{u}_2 \cdot \nabla)\mathbf{u}, \mathbf{v}) - (\operatorname{div} \ \mathbf{v}, p) + (\mathbf{b}T, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{3.25}$$

$$\lambda(\nabla T, \nabla S) + \lambda(\alpha T, S)_{\Gamma_N} + (\mathbf{u} \cdot \nabla T_1, S) + (\mathbf{u}_2 \cdot \nabla T, S) = (f, S) + (\chi, S)_{\Gamma_N} \quad \forall S \in \mathcal{T}, \tag{3.26}$$

$$\operatorname{div} \ \mathbf{u} = 0 \quad \text{in } \Omega, \qquad \mathbf{u}|_\Gamma = \mathbf{g}, \qquad T|_{\Gamma_D} = \psi. \tag{3.27}$$

We set $\chi = \chi_1$, $\psi = \psi_1$, $f = f_1$ in the inequality (3.23) under $i = 2$ and $\chi = \chi_2$, $\psi = \psi_2$, $f = f_2$ in the same inequality under $i = 1$ and add. We obtain

$$-\kappa\left[(\chi, \theta)_{\Gamma_N} + \langle \zeta^t, \psi \rangle_{\Gamma_D} + (f, \theta)\right] \leq -\mu_1 \|\chi\|_{\Gamma_N}^2 - \mu_2 \|\psi\|_{1/2, \Gamma_D}^2 - \mu_3 \|f\|^2. \tag{3.28}$$

Subtract the identities (3.20)–(3.22), written for $(\mathbf{x}_2, u_2, \mathbf{y}_2^*, \mathbf{g}_2)$ from the corresponding identities for $(\mathbf{x}_1, u_1, \mathbf{y}_1^*, \mathbf{g}_1)$, set $\mathbf{w} = \mathbf{u}$, $\tau = T$ and add. Using (3.27) we obtain

$$\begin{aligned} &\nu(\nabla \mathbf{u}, \nabla \xi) + ((\mathbf{u}_1 \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}_1, \xi) + 2((\mathbf{u} \cdot \nabla)\mathbf{u}, \xi_2) + \kappa(\mathbf{u} \cdot \nabla T_1, \theta) + \kappa((\mathbf{u} \cdot \nabla)T, \theta_2) \\ &\quad + \langle \zeta, \mathbf{g} \rangle_\Gamma + \kappa\left[\lambda(\nabla T, \nabla \theta) + \lambda(\alpha T, \theta)_{\Gamma_N} + (\mathbf{u}_1 \cdot \nabla T, \theta) + (\mathbf{u} \cdot \nabla T, \theta_2) + \langle \zeta^t, \psi \rangle_{\Gamma_D}\right] + (\mathbf{b}T, \xi) \\ &\quad + \left(\frac{\mu_0}{2}\right)\left\langle I'(\mathbf{u}_1) - \widetilde{I}'(\mathbf{u}_2), \mathbf{u} \right\rangle = 0. \end{aligned} \tag{3.29}$$

Set further $\mathbf{v} = \xi$ in (3.25), $S = \kappa\theta$ in (3.26), and subtract obtained relations from (3.29). Using inequality (3.28) and arguing as in [18], we obtain

$$\begin{aligned}
&((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1+\xi_2) + \kappa(\mathbf{u}\cdot\nabla T, \theta_1+\theta_2) + \left(\frac{\mu_0}{2}\right)\left\langle I'(\mathbf{u}_1) - \widetilde{I}'(\mathbf{u}_2), \mathbf{u}\right\rangle \\
&\quad\le -\langle \zeta, \mathbf{g}\rangle_\Gamma - \mu_1\|\chi\|^2_{\Gamma_N} - \mu_2\|\varphi\|^2_{1/2,\Gamma_D} - \mu_3\|f\|^2.
\end{aligned} \tag{3.30}$$

Thus we have proved the following result.

Theorem 3.4. *Let under conditions of Theorem 3.2 for functionals I and $\widetilde{I}$ and condition* (3.19) *quadruples $(\mathbf{u}_1, p_1, T_1, u_1)$ and $(\mathbf{u}_2, p_2, T_2, u_2)$ be solutions to problem* (3.5) *under $\mathbf{g} = \mathbf{g}_1$ and problem* (3.14) *under $\mathbf{g} = \mathbf{g}_2$, respectively, $\mathbf{y}_i^* = (\xi_i, \sigma_i, \zeta_i, \theta_i, \zeta_i^t)$, $i = 1, 2$ be corresponding Lagrange multipliers. Then the inequality* (3.30) *holds for differences $\mathbf{u}, p, T, \chi, \varphi, f$, defined in* (3.24)*, where $\mathbf{g} = \mathbf{g}_1 - \mathbf{g}_2$, $\zeta = \zeta_1 - \zeta_2$.*

Below we shall need the estimates of differences $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $p = p_1 - p_2$, $T = T_1 - T_2$ entering into (3.25)–(3.27) by differences $\chi = \chi_1 - \chi_2$, $\varphi = \varphi_1 - \varphi_2$, $f = f_1 - f_2$, and $\mathbf{g} = \mathbf{g}_1 - \mathbf{g}_2$. Denote by $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ a vector such that div $\mathbf{u}_0 = 0$ in Ω, $\mathbf{u}_0|_\Gamma = \mathbf{g}$, $\|\mathbf{u}_0\|_1 \le C_0\|\mathbf{g}\|_{1/2,\Gamma}$. Here C_0 is a constant depending on Ω. The existence of $\mathbf{u}_0$ follows from [20, page 24]. We present the difference $\mathbf{u} \equiv \mathbf{u}_1 - \mathbf{u}_2$ as $\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}$, where $\tilde{\mathbf{u}} \in \mathbf{V}$ is a new unknown function. Set $\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}$, $\mathbf{v} = \tilde{\mathbf{u}}$ in (3.25). Taking into account (2.9) we obtain

$$\nu(\nabla\tilde{\mathbf{u}}, \nabla\tilde{\mathbf{u}}) = -\nu(\nabla\mathbf{u}_0, \nabla\tilde{\mathbf{u}}) - ((\mathbf{u}_0\cdot\nabla)\mathbf{u}_1, \tilde{\mathbf{u}}) - ((\tilde{\mathbf{u}}\cdot\nabla)\mathbf{u}_1, \tilde{\mathbf{u}}) - ((\mathbf{u}_2\cdot\nabla)\mathbf{u}_0, \tilde{\mathbf{u}}) - (\mathbf{b}T, \tilde{\mathbf{u}}). \tag{3.31}$$

Using estimates (2.5), (2.6), (2.7), and (3.15), we deduce from (3.31) that

$$\delta_0\nu\|\tilde{\mathbf{u}}\|_1^2 \le \nu\|\mathbf{u}_0\|_1\|\tilde{\mathbf{u}}\|_1 + \gamma_0 M^0_{\mathbf{u}}\|\tilde{\mathbf{u}}\|_1^2 + 2\gamma_0 M^0_{\mathbf{u}}\|\mathbf{u}_0\|_1\|\tilde{\mathbf{u}}\|_1 + \beta_1\|T\|_1\|\tilde{\mathbf{u}}\|_1. \tag{3.32}$$

It follows from (3.19) that

$$\frac{\delta_0\nu}{2} < \delta_0\nu - \gamma_0 M^0_{\mathbf{u}} - \frac{\beta_1\gamma_1}{\delta_1\lambda}M^0_T \le \delta_0\nu - \gamma_0 M^0_{\mathbf{u}}. \tag{3.33}$$

Rewriting the inequality (3.32) by (3.33) as

$$\left(\frac{\delta_0\nu}{2}\right)\|\tilde{\mathbf{u}}\|_1^2 \le \left(\delta_0\nu - \gamma_0 M^0_{\mathbf{u}}\right)\|\tilde{\mathbf{u}}\|_1^2 \le \left(\nu + 2\gamma_0 M^0_{\mathbf{u}}\right)\|\mathbf{u}_0\|_1\|\tilde{\mathbf{u}}\|_1 + \beta_1\|T\|_1\|\tilde{\mathbf{u}}\|_1, \tag{3.34}$$

we obtain that

$$\begin{aligned}
\|\tilde{\mathbf{u}}\|_1 &\le \left(\frac{2}{\delta_0\nu}\right)\left(\nu + 2\gamma_0 M^0_{\mathbf{u}}\right)\|\mathbf{u}_0\|_1 + \left(\frac{2\beta_1}{\delta_0\nu}\right)\|T\|_1 \le \left(2\delta_0^{-1} + 4\mathcal{R}e\right)\|\mathbf{u}_0\|_1 + \left(\frac{2\beta_1}{\delta_0\nu}\right)\|T\|_1 \\
&\le 2\mathcal{R}\|\mathbf{u}_0\|_1 + \left(\frac{2\beta_1}{\delta_0\nu}\right)\|T\|_1, \quad \mathcal{R} \equiv \delta_0^{-1} + 2\mathcal{R}e.
\end{aligned} \tag{3.35}$$

Taking into account the relation $\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}$, we come to the following estimate $\|\mathbf{u}\|_1$ via $\|\mathbf{g}\|_{1/2,\Gamma}$ and $\|T\|_1$:

$$\|\mathbf{u}\|_1 \leq \|\mathbf{u}_0\|_1 + \|\tilde{\mathbf{u}}\|_1 \leq (2\mathcal{R}+1)\|\mathbf{u}_0\|_1 + \left(\frac{2\beta_1}{\delta_0 \nu}\right)\|T\|_1 \leq C_0(2\mathcal{R}+1)\|\mathbf{g}\|_{1/2,\Gamma} + \left(\frac{2\beta_1}{\delta_0 \nu}\right)\|T\|_1. \tag{3.36}$$

Denote by $T_0 \in H^1(\Omega)$ a function such that $T_0|_{\Gamma_D} = \psi$ and the estimate $\|T_0\|_1 \leq C_1\|\psi\|_{1/2,\Gamma_D}$ holds with a certain constant C_1, which does not depend on ψ. Let us present the difference $T = T_1 - T_2$ as $T = T_0 + \widetilde{T}$, where $\widetilde{T} \in \mathcal{T}$ is a new unknown function. Set $T = T_0 + \widetilde{T}$, $S = \widetilde{T}$ in (3.26). We obtain

$$\begin{aligned} &\lambda\left(\nabla\widetilde{T}, \nabla\widetilde{T}\right) + \lambda\left(\alpha\widetilde{T}, \widetilde{T}\right)_{\Gamma_N} + \left(\mathbf{u}_2 \cdot \nabla\widetilde{T}, \widetilde{T}\right) \\ &\quad = -\lambda\left(\nabla T_0, \nabla\widetilde{T}\right) - \lambda\left(\alpha T_0, \widetilde{T}\right)_{\Gamma_N} \\ &\qquad - \left(\mathbf{u}_2 \cdot \nabla T_0, \widetilde{T}\right) - \left(\mathbf{u} \cdot \nabla T_1, \widetilde{T}\right) + \left(f, \widetilde{T}\right) + \left(\chi, \widetilde{T}\right)_{\Gamma_N}. \end{aligned} \tag{3.37}$$

Using estimates (2.5)–(2.8) and (3.15) we deduce that

$$\begin{aligned} \delta_1\lambda\left\|\widetilde{T}\right\|_1^2 &\leq \lambda\|T_0\|_1\left\|\widetilde{T}\right\|_1 + \gamma_1 M_{\mathbf{u}}^0\|T_0\|_1\left\|\widetilde{T}\right\| + \lambda\gamma_3\|\alpha\|_{\Gamma_N}\|T_0\|_1\left\|\widetilde{T}\right\|_1 \\ &\quad + \gamma_1 M_T^0\|\mathbf{u}\|_1\left\|\widetilde{T}\right\|_1 + \left(\gamma_2\|\chi\|_{\Gamma_N} + \gamma_4\|f\|\right)\left\|\widetilde{T}\right\|_1 \end{aligned} \tag{3.38}$$

or

$$\left\|\widetilde{T}\right\|_1 \leq \frac{1}{\delta_1\lambda}\left(\lambda + \gamma_1 M_{\mathbf{u}}^0 + \lambda\gamma_3\|\alpha\|_{\Gamma_N}\right)\|T_0\|_1 + \frac{\gamma_1 M_T^0}{\delta_1\lambda}\|\mathbf{u}\|_1 + \frac{1}{\delta_1\lambda}\left(\gamma_2\|\chi\|_{\Gamma_N} + \gamma_4\|f\|\right). \tag{3.39}$$

Taking into account the relation $T = T_0 + \widetilde{T}$, we obtain from this estimate that

$$\|T\|_1 \leq C_1(\mathcal{N}+1)\|\psi\|_{1/2,\Gamma_D} + \frac{\gamma_2\|\chi\|_{\Gamma_N} + \gamma_4\|f\|}{\delta_1\lambda} + \frac{\gamma_1 M_T^0}{\delta_1\lambda}\|\mathbf{u}\|_1, \quad \mathcal{N} = \frac{\lambda + \gamma_1 M_{\mathbf{u}}^0 + \lambda\gamma_3\|\alpha\|_{\Gamma_N}}{\delta_1\lambda}. \tag{3.40}$$

Using further the estimate (3.36) for $\mathbf{u}$, we deduce from (3.40) that

$$\begin{aligned} \|T\|_1 &\leq C_1(\mathcal{N}+1)\|\psi\|_{1/2,\Gamma_D} + \frac{\gamma_2\|\chi\|_{\Gamma_N} + \gamma_4\|f\|}{\delta_1\lambda} \\ &\quad + C_0(2\mathcal{R}+1)\frac{\gamma_1 M_T^0}{\delta_1\lambda}\|\mathbf{g}\|_{1/2,\Gamma} + \frac{2\beta_1}{\delta_0\nu}\frac{\gamma_1 M_T^0}{\delta_1\lambda}\|T\|_1. \end{aligned} \tag{3.41}$$

From this inequality and (3.17), (3.19) we come to the following estimate:

$$\|T\|_1 \leq \frac{C_1(\mathcal{N}+1)}{1-2\mathcal{R}a}\|\varphi\|_{1/2,\Gamma_D} + \frac{\gamma_2\|\chi\|_{\Gamma_N}+\gamma_4\|f\|}{\delta_1\lambda(1-2\mathcal{R}a)} + \frac{C_0(2\mathcal{R}+1)}{1-2\mathcal{R}a}\frac{\gamma_1 M_T^0}{\delta_1\lambda}\|\mathbf{g}\|_{1/2,\Gamma}. \tag{3.42}$$

Using (3.42), we deduce from (3.36) that

$$\begin{aligned}\|\mathbf{u}\|_1 \leq{}& C_0(2\mathcal{R}+1)\left(1+\frac{1}{1-2\mathcal{R}a}\frac{2\beta_1}{\delta_0\nu}\frac{\gamma_1 M_T^0}{\delta_1\lambda}\right)\|\mathbf{g}\|_{1/2,\Gamma}\\ &+\frac{2\beta_1}{\delta_0\nu}\left[\frac{C_1(\mathcal{N}+1)}{1-2\mathcal{R}a}\|\varphi\|_{1/2,\Gamma_D}+\frac{\gamma_2\|\chi\|_{\Gamma_N}+\gamma_4\|f\|}{\delta_1\lambda(1-2\mathcal{R}a)}\right].\end{aligned} \tag{3.43}$$

Taking into account (3.17) we come to the following estimate for $\|\mathbf{u}\|_1$:

$$\|\mathbf{u}\|_1 \leq \frac{2\beta_1}{\delta_0\nu}\left[\frac{C_1(\mathcal{N}+1)}{1-2\mathcal{R}a}\|\varphi\|_{1/2,\Gamma_D}+\frac{\gamma_2\|\chi\|_{\Gamma_N}+\gamma_4\|f\|}{\delta_1\lambda(1-2\mathcal{R}a)}\right]+\frac{C_0(2\mathcal{R}+1)\|\mathbf{g}\|_{1/2,\Gamma}}{1-2\mathcal{R}a}. \tag{3.44}$$

An analogous estimate holds and for the pressure difference $p = p_1 - p_2$. In order to establish this estimate we make use of inf-sup condition (2.10). By (2.10) for the function $p = p_1 - p_2$ and any (small) number $\delta > 0$ there exists a function $\mathbf{v}_0 \in \mathbf{H}_0^1(\Omega)$, $\mathbf{v}_0 \neq 0$, such that $-(\operatorname{div}\,\mathbf{v}_0, p) \geq \beta_0\|\mathbf{v}_0\|_1\|p\|$ where $\beta_0 = (\beta-\delta) > 0$. Set $\mathbf{v} = \mathbf{v}_0$ in the identity for $\mathbf{u}$ in (3.25) and make of this estimate and estimates (2.6), (2.7), (3.15). We shall have

$$\beta_0\|\mathbf{v}_0\|_1\|p\| \leq -(\operatorname{div}\,\mathbf{v}_0, p) \leq \left(\nu+2\gamma_0 M_{\mathbf{u}}^0\right)\|\mathbf{v}_0\|_1\|\mathbf{u}\|_1+\beta_1\|T\|_1\|\mathbf{v}_0\|_1. \tag{3.45}$$

Dividing to $\|\mathbf{v}_0\|_1 \neq 0$, we deduce that

$$\|p\| \leq \frac{\nu+2\gamma_0 M_{\mathbf{u}}^0}{\beta_0}\|\mathbf{u}\|_1+\left(\frac{\beta_1}{\beta_0}\right)\|T\|_1 = \frac{\delta_0\nu}{\beta_0}\mathcal{R}\|\mathbf{u}\|_1+\frac{\beta_1}{\beta_0}\|T\|_1. \tag{3.46}$$

Using (3.42) and (3.44), we come to the following final estimate for $\|p\|$:

$$\|p\| \leq \frac{2\mathcal{R}+1}{\beta_0(1-2\mathcal{R}a)}\times\left[\beta_1 C_1(\mathcal{N}+1)\|\varphi\|_{1/2,\Gamma_D}+\frac{\beta_1\left(\gamma_2\|\chi\|_{\Gamma_N}+\gamma_4\|f\|\right)}{\delta_1\lambda}+\delta_0\nu C_0(\mathcal{R}+\mathcal{R}a)\|\mathbf{g}\|_{1/2,\Gamma}\right]. \tag{3.47}$$

Remark 3.5. Along with three-parametric control problem (3.5) we shall consider and one-parametric control problem which corresponds to situation when a function $u = \chi$ is a unique control. This problem can be considered as particular case of the general control problem (3.5), for which the set K_2 consists of one element $\psi_0 \in H^{1/2}(\Gamma_D)$ and the set K_3 consists of one element $f_0 \in L^2(\Omega)$. For this case the conditions $f \equiv f_1 - f_2 = 0$, $\psi \equiv \psi_1 - \psi_2 = 0$ take place, and the estimates (3.42)–(3.47) and inequality (3.30) take the form

$$\|T\|_1 \leq \frac{\gamma_2 \|\chi\|_{\Gamma_N}}{\delta_1 \lambda (1 - 2\mathcal{R}a)} + \frac{C_0(2\mathcal{R} + 1)}{1 - 2\mathcal{R}a} \frac{\gamma_1 M_T^0}{\delta_1 \lambda} \|\mathbf{g}\|_{1/2,\Gamma}, \tag{3.48}$$

$$\|\mathbf{u}\|_1 \leq \frac{2\beta_1 \gamma_2 \|\chi\|_{\Gamma_N}}{\delta_0 \nu \delta_1 \lambda (1 - 2\mathcal{R}a)} + \frac{C_0(2\mathcal{R} + 1)\|\mathbf{g}\|_{1/2,\Gamma}}{1 - 2\mathcal{R}a}, \tag{3.49}$$

$$\|p\| \leq \frac{2\mathcal{R} + 1}{\beta_0 (1 - 2\mathcal{R}a)} \left[\frac{\beta_1 \gamma_2 \|\chi\|_{\Gamma_N}}{\delta_1 \lambda} + \delta_0 \nu C_0 (\mathcal{R} + \mathcal{R}a) \|\mathbf{g}\|_{1/2,\Gamma} \right], \tag{3.50}$$

$$((\mathbf{u} \cdot \nabla)\mathbf{u}, \xi_1 + \xi_2) + \kappa(\mathbf{u} \cdot \nabla T, \theta_1 + \theta_2) + \left(\frac{\mu_0}{2}\right) \left\langle I'(\mathbf{u}_1) - \widetilde{I}'(\mathbf{u}_2), \mathbf{u} \right\rangle \leq -\langle \zeta, \mathbf{g} \rangle_\Gamma - \mu_1 \|\chi\|_{\Gamma_N}^2. \tag{3.51}$$

4. Control Problems for Velocity Tracking-Type Cost Functionals

Based on Theorem 3.4 and estimates (3.42)–(3.47) or (3.48)–(3.50), we study below uniqueness and stability of the solution to problem (3.5) for concrete tracking-type cost functionals. We consider firstly the case mentioned in Remark 3.5 where $I = I_1$ and the heat flux χ on the part Γ_N of Γ is a unique control; that is, we consider one-parametric control problem

$$J(\mathbf{v}, \chi) \equiv \frac{\mu_0}{2} \|\mathbf{v} - \mathbf{v}_d\|_Q^2 + \frac{\mu_1}{2} \|\chi\|_{\Gamma_N}^2 \longrightarrow \inf, \quad F(\mathbf{x}, \chi, \mathbf{g}) = 0, \ \mathbf{x} = (\mathbf{v}, p, T) \in X, \ \chi \in K_1. \tag{4.1}$$

In accordance to Remark 3.5 we can consider problem (4.1) as a particular case of the general control problem (3.5), which corresponds to the situation when every of sets K_2 and K_3 consists of one element.

Let $(\mathbf{x}_1, u_1) \equiv (\mathbf{u}_1, p_1, T_1, \chi_1)$ be a solution to problem (4.1), that corresponds to given functions $\mathbf{v}_d \equiv \mathbf{u}_d^{(1)} \in \mathbf{L}^2(Q)$ and $\mathbf{g} = \mathbf{g}_1 \in G \subset \widetilde{\mathbf{H}}^{1/2}(\Gamma)$, and let $(\mathbf{x}_2, u_2) \equiv (\mathbf{u}_2, p_2, T_2, \chi_2)$ be a solution to problem (4.1), that corresponds to perturbed functions $\widetilde{\mathbf{v}}_d \equiv \mathbf{u}_d^{(2)} \in \mathbf{L}^2(Q)$ and $\widetilde{\mathbf{g}} = \mathbf{g}_2 \in G \subset \widetilde{\mathbf{H}}^{1/2}(\Gamma)$. Setting $\mathbf{u}_d = \mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}$ in addition to (3.24) we note that under conditions of problem (4.1) we have

$$\langle I_1'(\mathbf{u}_i), \mathbf{w} \rangle = 2\left(\mathbf{u}_i - \mathbf{u}_d^{(i)}, \mathbf{w}\right)_Q, \quad \left(I_1'(\mathbf{u}_1) - \widetilde{I}_1'(\mathbf{u}_2), \mathbf{u}\right) = 2\left(\|\mathbf{u}\|_Q^2 - (\mathbf{u}, \mathbf{u}_d)_Q\right). \tag{4.2}$$

Identity (3.22) for problem (4.1) does not change, while identities (3.20), (3.21), and inequality (3.51) take due to (4.2) a form

$$\begin{aligned}\nu(\nabla\mathbf{w},\nabla\xi_i)+((\mathbf{u}_i\cdot\nabla)\mathbf{w},\xi_i)+((\mathbf{w}\cdot\nabla)\mathbf{u}_i,\xi_i)+\kappa(\mathbf{w}\cdot\nabla T_i,\theta_i)-(\sigma_i,\operatorname{div}\,\mathbf{w})\\+\langle\zeta_i,\mathbf{w}\rangle_\Gamma+\mu_0\Big(\mathbf{u}_i-\mathbf{u}_d^{(i)},\mathbf{w}\Big)_Q=0\quad\forall\mathbf{w}\in\widetilde{\mathbf{H}}^1(\Omega),\end{aligned}\tag{4.3}$$

$$(\operatorname{div}\,\xi_i,r)=0\quad\forall r\in L_0^2(\Omega),\tag{4.4}$$

$$((\mathbf{u}\cdot\nabla)\mathbf{u},\xi_1+\xi_2)+\kappa(\mathbf{u}\cdot\nabla T,\theta_1+\theta_2)+\mu_0\Big(\|\mathbf{u}\|_Q^2-(\mathbf{u},\mathbf{u}_d)_Q\Big)\le-\langle\zeta,\mathbf{g}\rangle_\Gamma-\mu_1\|\chi\|_{\Gamma_N}^2.\tag{4.5}$$

Using identities (4.3), (4.4), (3.22) we estimate parameters ξ_i, θ_i, σ_i and ζ_i. Firstly we deduce estimates for norms $\|\xi_i\|_1$ and $\|\theta_i\|_1$. To this end we set $\mathbf{w}=\xi_i$, $\tau=\theta_i$ in (4.3), (3.22). Taking into account (2.11), (2.12), and condition $\xi_i\in\mathbf{V}$, which follows from (4.4), we obtain

$$\nu(\nabla\xi_i,\nabla\xi_i)=-((\xi_i\cdot\nabla)\mathbf{u}_i,\xi_i)-\kappa(\xi_i\cdot\nabla T_i,\theta_i)-\mu_0\Big(\mathbf{u}_i-\mathbf{u}_d^{(i)},\xi_i\Big)_Q,\tag{4.6}$$

$$\kappa\lambda\big[(\nabla\theta_i,\nabla\theta_i)+(\alpha\theta_i,\theta_i)_{\Gamma_N}\big]=-(\mathbf{b}\theta_i,\xi_i),\quad i=1,2.\tag{4.7}$$

Using estimates (2.5)–(2.8) and (3.15) we have

$$(\nabla\xi_i,\nabla\xi_i)\ge\delta_0\|\xi_i\|_1^2,\qquad(\nabla\theta_i,\nabla\theta_i)\ge\delta_1\|\theta_i\|_1^2,\tag{4.8}$$

$$|((\xi_i\cdot\nabla)\mathbf{u}_i,\xi_i)|\le\gamma_0\|\mathbf{u}_i\|_1\|\xi_i\|_1^2\le\gamma_0M_{\mathbf{u}}^0\|\xi_i\|_1^2,\tag{4.9}$$

$$\kappa|(\xi_i\cdot\nabla T_i,\theta_i)|\le\kappa\gamma_1M_T^0\|\xi_i\|_1\|\theta_i\|_1,\qquad|(\mathbf{b}\theta_i,\xi_i)|\le\beta_1\|\theta_i\|_1\|\xi_i\|_1,\tag{4.10}$$

$$\Big\|\mathbf{u}_i-\mathbf{u}_d^{(i)}\Big\|_Q\le\|\mathbf{u}_i\|_Q+\Big\|\mathbf{u}_d^{(i)}\Big\|_Q\le\gamma_4M_{\mathbf{u}}^0+\Big\|\mathbf{u}_d^{(i)}\Big\|_Q\le\delta_0\nu\gamma_4\gamma_0^{-1}\Big(\mathcal{R}e+\mathcal{R}e^0\Big),\tag{4.11}$$

$$\left|\Big(\mathbf{u}_i-\mathbf{u}_d^{(i)},\xi_i\Big)_Q\right|\le\Big\|\mathbf{u}_i-\mathbf{u}_d^{(i)}\Big\|_Q\|\xi_i\|_Q\le\delta_0\nu\gamma\Big(\mathcal{R}e+\mathcal{R}e^0\Big)\|\xi_i\|_1,\tag{4.12}$$

where

$$\gamma=\gamma_4^2\gamma_0^{-1},\qquad\mathcal{R}e^0=\frac{\gamma_0}{\delta_0\nu\gamma_4}\max\Big(\Big\|\mathbf{u}_d^{(1)}\Big\|_Q,\Big\|\mathbf{u}_d^{(2)}\Big\|_Q\Big).\tag{4.13}$$

By virtue of (4.8)–(4.10) and (4.12), we deduce from (4.7) and (4.6) that

$$\kappa\|\theta_i\|_1\le\frac{\beta_1}{\delta_1\lambda}\|\xi_i\|_1,\tag{4.14}$$

$$\delta_0\nu\|\xi_i\|_1^2\le\gamma_0M_{\mathbf{u}}^0\|\xi_i\|_1^2+\kappa\gamma_1M_T^0\|\theta_i\|_1+\mu_0\delta_0\nu\gamma\Big(\mathcal{R}e+\mathcal{R}e^0\Big)\|\xi_i\|_1.\tag{4.15}$$

Taking into account (4.14), we obtain from (4.15) that

$$\left(\delta_0\nu - \gamma_0 M_{\mathbf{u}}^0 - \frac{\beta_1\gamma_1}{\delta_1\lambda}M_T^0\right)\|\xi_i\|_1^2 \le \mu_0\delta_0\nu\gamma\left(\mathcal{R}e + \mathcal{R}e^0\right)\|\xi_i\|_1. \tag{4.16}$$

Using (3.33) we deduce successfully from (4.16), (4.14) that

$$\|\xi_i\|_1 \le 2\mu_0\gamma\left(\mathcal{R}e + \mathcal{R}e^0\right), \qquad \kappa\|\theta_i\|_1 \le 2\mu_0\gamma\frac{\beta_1}{\delta_1\lambda}\left(\mathcal{R}e + \mathcal{R}e^0\right). \tag{4.17}$$

Let us estimate further the norms $\|\sigma_i\|$ and $\|\zeta_i\|_{-1/2,\Gamma}$ from (4.3). In order to estimate $\|\sigma_i\|$ we make use of inf-sup condition (2.10). By (2.10) for a function $\sigma_i \in L_0^2(\Omega)$ and any small number $\delta > 0$ there exists a function $\mathbf{v}_i \in \mathbf{H}_0^1(\Omega)$, $\mathbf{v}_i \neq 0$, such that the inequality

$$-(\operatorname{div}\,\mathbf{v}_i, \sigma_i) \ge \beta_0\|\mathbf{v}_i\|_1\|\sigma_i\|, \quad i = 1,2, \ \beta_0 = \beta - \delta \tag{4.18}$$

holds. Setting in (4.3) $\mathbf{w} = \mathbf{v}_i$ and using this estimate together with estimates (2.6), (3.15), (4.11), we have

$$\begin{aligned}\beta_0\|\mathbf{v}_i\|_1\|\sigma_i\| &\le -(\operatorname{div}\,\mathbf{v}_i, \sigma_i)\\ &\le \nu\|\mathbf{v}_i\|_1\|\xi_i\|_1\\ &\quad + 2\gamma_0 M_{\mathbf{u}}^0\|\mathbf{v}_i\|_1\|\xi_i\|_1 + \kappa\gamma_1 M_T^0\|\mathbf{v}_i\|_1\|\theta_i\|_1 + \mu_0\delta_0\nu\gamma\left(\mathcal{R}e + \mathcal{R}e^0\right)\|\mathbf{v}_i\|_1.\end{aligned} \tag{4.19}$$

From this inequality we deduce by (4.17) that

$$\begin{aligned}\|\sigma_i\| &\le \frac{1}{\beta_0}\left[\left(\nu + 2\gamma_0 M_{\mathbf{u}}^0\right)\|\xi_i\|_1 + \gamma_1 M_T^0\kappa\|\theta_i\|_1 + \mu_0\delta_0\nu\gamma\left(\mathcal{R}e + \mathcal{R}e^0\right)\right]\\ &\le \frac{\delta_0\nu}{\beta_0}\left[\mathcal{R}\|\xi_i\|_1 + \frac{\gamma_1 M_T^0}{\delta_0\nu}\kappa\|\theta_i\|_1 + \mu_0\gamma\left(\mathcal{R}e + \mathcal{R}e^0\right)\right].\end{aligned} \tag{4.20}$$

Taking into account (4.17), we come from (4.20) to the estimate

$$\|\sigma_i\| \le \mu_0\gamma\frac{\delta_0\nu}{\beta_0}\left(\mathcal{R}e + \mathcal{R}e^0\right)(2\mathcal{R} + 2\mathcal{R}a + 1). \tag{4.21}$$

It remains to estimate $\|\zeta\|_{-1/2,\Gamma}$. To this end we make again use of identity (4.3). Using estimates (2.6), (2.9) and (3.15), (4.11), (4.17), (4.21) as well we have

$$\begin{aligned}|\langle\zeta_i, \mathbf{w}\rangle_\Gamma| &\le \left[\left(\nu + 2\gamma_0 M_{\mathbf{u}}^0\right)\|\xi_i\|_1 + \gamma_1 M_T^0\kappa\|\theta_i\|_1 + C_d\|\sigma_i\| + \mu_0\delta_0\nu\gamma\left(\mathcal{R}e + \mathcal{R}e^0\right)\right]\|\mathbf{w}\|_1\\ &\le \mu_0\delta_0\nu\gamma\left(1 + C_d\beta_0^{-1}\right)\left(\mathcal{R}e + \mathcal{R}e^0\right)(2\mathcal{R} + 2\mathcal{R}a + 1)\|\mathbf{w}\|_1 \quad \forall\mathbf{w} \in \widetilde{\mathbf{H}}^1(\Omega).\end{aligned} \tag{4.22}$$

As $\zeta = \zeta_1 - \zeta_2$ we obtain from this inequality that

$$\|\zeta\|_{-1/2,\Gamma} \le \mu_0 a, \quad a = 2\delta_0 \nu\gamma\left(1 + C_d\beta_0^{-1}\right)\left(\mathcal{R}e + \mathcal{R}e^0\right)(2\mathcal{R} + 2\mathcal{R}a + 1). \tag{4.23}$$

Taking into account (2.6), (3.48), (3.49), and estimates (4.17) for ξ_i, θ_i, we have

$$\begin{aligned}
|((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1 + \xi_2)| &\le \gamma_0\|\mathbf{u}\|_1^2(\|\xi_1\|_1 + \|\xi_2\|_1) \\
&\le 4\mu_0\gamma_0\gamma\left(\mathcal{R}e + \mathcal{R}e^0\right)\left[\frac{2\beta_1\gamma_2\|\chi\|_{\Gamma_N}}{\delta_0\nu\delta_1\lambda(1-2\mathcal{R}a)} + \frac{C_0(2\mathcal{R}+1)\|\mathbf{g}\|_{1/2,\Gamma}}{1-2\mathcal{R}a}\right]^2, \\
\kappa|(\mathbf{u}\cdot\nabla T, \theta_1 + \theta_2)| &\le \frac{4\mu_0\gamma_1\gamma\beta_1(\mathcal{R}e + \mathcal{R}e^0)}{\delta_1\lambda}\left[\frac{2\beta_1\gamma_2\|\chi\|_{\Gamma_N}}{\delta_0\nu\delta_1\lambda(1-2\mathcal{R}a)} + \frac{C_0(2\mathcal{R}+1)\|\mathbf{g}\|_{1/2,\Gamma}}{1-2\mathcal{R}a}\right] \\
&\quad\times\left[\frac{\gamma_2\|\chi\|_{\Gamma_N}}{\delta_1\lambda(1-2\mathcal{R}a)} + \frac{C_0(2\mathcal{R}+1)}{1-2\mathcal{R}a}\frac{\gamma_1 M_T^0}{\delta_1\lambda}\|\mathbf{g}\|_{1/2,\Gamma}\right].
\end{aligned} \tag{4.24}$$

It follows from (4.24) that

$$|((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1 + \xi_2) + \kappa(\mathbf{u}\cdot\nabla T, \theta_1 + \theta_2)| \le \mu_0\left(b\|\mathbf{g}\|_{1/2,\Gamma}^2 + c\|\chi\|_{\Gamma_N}^2\right). \tag{4.25}$$

Here constants b and c are given by

$$\begin{aligned}
b &= 4\gamma\gamma_0 C_0^2(2\mathcal{R}+1)^2\frac{(\mathcal{R}e + \mathcal{R}e^0)}{(1-2\mathcal{R}a)^2}\left[3 + \left(\frac{\gamma_1}{\gamma_0}\right)^2 \mathcal{D}^2\mathcal{R}a^2\right], \\
c &= 4\gamma\gamma_0\left(\frac{\beta_1}{\delta_0\nu}\frac{\gamma_2}{\delta_1\lambda}\right)^2\frac{(\mathcal{R}e + \mathcal{R}e^0)}{(1-2\mathcal{R}a)^2}\left[12 + \left(\frac{\gamma_1}{\gamma_0}\right)^2\mathcal{D}^2\right].
\end{aligned} \tag{4.26}$$

Let the data for problem (4.1) and parameters μ_0, μ_1 be such that with a certain constant $\varepsilon > 0$ the following condition takes place:

$$(1-\varepsilon)\mu_1 \ge \mu_0 c, \quad \varepsilon = \text{const} > 0. \tag{4.27}$$

Under condition (4.27) we deduce from (4.25) that

$$|((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1 + \xi_2) + \kappa(\mathbf{u}\cdot\nabla T, \theta_1 + \theta_2)| \le \mu_0 b\|\mathbf{g}\|_{1/2,\Gamma}^2 + (1-\varepsilon)\mu_1\|\chi\|_{\Gamma_N}^2. \tag{4.28}$$

Taking into account (4.28) and the estimate $|\langle\zeta, \mathbf{g}\rangle_\Gamma| \le \|\zeta\|_{-1/2,\Gamma}\|\mathbf{g}\|_{1/2,\Gamma} \le \mu_0 a\|\mathbf{g}\|_{1/2,\Gamma}$ which follows from (4.23), we come from (4.5) to the inequality

$$\begin{aligned}
\mu_0\left(\|\mathbf{u}\|_Q^2 - (\mathbf{u}, \mathbf{u}_d)_Q\right) &\le -((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1 + \xi_2) - \kappa(\mathbf{u}\cdot\nabla T, \theta_1 + \theta_2) - \langle\zeta, \mathbf{g}\rangle_\Gamma - \mu_1\|\chi\|_{\Gamma_N}^2 \\
&\le -\varepsilon\mu_1\|\chi\|_{\Gamma_N}^2 + \mu_0 a\|\mathbf{g}\|_{1/2,\Gamma} + \mu_0 b\|\mathbf{g}\|_{1/2,\Gamma}^2.
\end{aligned} \tag{4.29}$$

It follows from this inequality that

$$\mu_0\|\mathbf{u}\|_Q^2 \le \mu_0(\mathbf{u},\mathbf{u}_d)_Q - \varepsilon\mu_1\left\|\chi\right\|_{\Gamma_N}^2 + \mu_0 a\|\mathbf{g}\|_{1/2,\Gamma} + \mu_0 b\|\mathbf{g}\|_{1/2,\Gamma}^2. \tag{4.30}$$

Excluding nonpositive term $-\varepsilon\mu_1\|\chi\|_{\Gamma_N}^2$ from the right-hand side of (4.30), we deduce from (4.30) that

$$\|\mathbf{u}\|_Q^2 \le \|\mathbf{u}_d\|_Q\|\mathbf{u}\|_Q + a\|\mathbf{g}\|_{1/2,\Gamma} + b\|\mathbf{g}\|_{1/2,\Gamma}^2. \tag{4.31}$$

Equation (4.31) is a quadratic inequality for $\|\mathbf{u}\|_Q$. Solving it we come to the following estimate for $\|\mathbf{u}\|_Q$:

$$\|\mathbf{u}\|_Q \le \|\mathbf{u}_d\|_Q + \left(a\|\mathbf{g}\|_{1/2,\Gamma} + b\|\mathbf{g}\|_{1/2,\Gamma}^2\right)^{1/2}. \tag{4.32}$$

As $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{u}_d = \mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}$, $\mathbf{g} = \mathbf{g}_1 - \mathbf{g}_2$, the estimate (4.32) is equivalent to the following estimate for the velocity difference $\mathbf{u}_1 - \mathbf{u}_2$:

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_Q \le \left\|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\right\|_Q + \left(a\|\mathbf{g}_1 - \mathbf{g}_2\|_{1/2,\Gamma} + b\|\mathbf{g}_1 - \mathbf{g}_2\|_{1/2,\Gamma}^2\right)^{1/2}. \tag{4.33}$$

This estimate under $Q = \Omega$ has the sense of the stability estimate in $\mathbf{L}^2(\Omega)$ of the component $\widehat{\mathbf{u}}$ of the solution $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{T}, \widehat{\chi})$ to problem (4.1) relative to small perturbations of functions $\mathbf{v}_d \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in G$ in the norms of $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^{1/2}(\Gamma)$, respectively. In particular case where $\mathbf{g}_1 = \mathbf{g}_2$ the estimate (4.33) transforms to "exact" a priori estimate $\|\mathbf{u}_1 - \mathbf{u}_2\|_Q \le \|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\|_Q$. It was obtained when studying control problems for Navier-Stokes and in [18] when studying control problems for heat convection equations. If besides $\mathbf{u}_d^{(1)} = \mathbf{u}_d^{(2)}$ it follows from (4.33) that $\mathbf{u}_1 = \mathbf{u}_2$ in Ω, if $Q = \Omega$. This yields together with (4.30), (3.48), (3.50) that $\chi_1 = \chi_2$, $T_1 = T_2$, $p_1 = p_2$. The latter means the uniqueness of the solution to problem (4.1) when $Q = \Omega$ and condition (4.27) holds.

It is important to note that the uniqueness and stability of the solution to problem (4.1) under condition (4.27) take place and in the case where $Q \subset \Omega$; that is, Q is only a part of domain Ω. In order to prove this fact let us consider the inequality (4.30). Using (4.32) we deduce from (4.30) that

$$\begin{aligned}
\varepsilon\mu_1\left\|\chi\right\|_{\Gamma_N}^2 &\le -\mu_0\|\mathbf{u}\|_Q^2 + \mu_0\|\mathbf{u}_d\|_Q\|\mathbf{u}\|_Q + \mu_0 a\|\mathbf{g}\|_{1/2,\Gamma} + \mu_0 b\|\mathbf{g}\|_{1/2,\Gamma}^2 \\
&\le \mu_0\|\mathbf{u}_d\|_Q^2 + \mu_0\|\mathbf{u}_d\|_Q\left(a\|\mathbf{g}\|_{1/2,\Gamma} + b\|\mathbf{g}\|_{1/2,\Gamma}^2\right)^{1/2} + \mu_0 a\|\mathbf{g}\|_{1/2,\Gamma} + \mu_0 b\|\mathbf{g}\|_{1/2,\Gamma}^2 \\
&\le \mu_0\left[\|\mathbf{u}_d\|_Q + \left(a\|\mathbf{g}\|_{1/2,\Gamma} + b\|\mathbf{g}\|_{1/2,\Gamma}^2\right)^{1/2}\right]^2.
\end{aligned} \tag{4.34}$$

From (4.34) and (3.48)–(3.50) we come to the following stability estimates:

$$\|\chi_1 - \chi_2\|_{\Gamma_N} \leq \sqrt{\frac{\mu_0}{\varepsilon\mu_1}}\Delta, \tag{4.35}$$

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_1 \leq \frac{2\beta_1\gamma_2}{\delta_0\nu\delta_1\lambda(1-2\mathcal{R}a)}\sqrt{\frac{\mu_0}{\varepsilon\mu_1}}\Delta + \frac{C_0(2\mathcal{R}+1)\|\mathbf{g}_1 - \mathbf{g}_2\|_{1/2,\Gamma}}{1-2\mathcal{R}a}, \tag{4.36}$$

$$\|T_1 - T_2\|_1 \leq \frac{\gamma_2}{\delta_1\lambda(1-2\mathcal{R}a)}\sqrt{\frac{\mu_0}{\varepsilon\mu_1}}\Delta + \frac{C_0(2\mathcal{R}+1)}{1-2\mathcal{R}a}\frac{\gamma_1 M_T^0}{\delta_1\lambda}\|\mathbf{g}_1 - \mathbf{g}_2\|_{1/2,\Gamma}, \tag{4.37}$$

$$\|p_1 - p_2\| \leq \frac{(2\mathcal{R}+1)\beta_1\gamma_2}{\beta_0\delta_1\lambda(1-2\mathcal{R}a)}\sqrt{\frac{\mu_0}{\varepsilon\mu_1}}\Delta + \frac{C_0(2\mathcal{R}+1)(\mathcal{R}+\mathcal{R}a)}{1-2\mathcal{R}a}\frac{\delta_0\nu}{\beta_0}\|\mathbf{g}_1 - \mathbf{g}_2\|_{1/2,\Gamma}, \tag{4.38}$$

where

$$\Delta = \left\|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\right\|_Q + \left(a\|\mathbf{g}_1 - \mathbf{g}_2\|_{1/2,\Gamma} + b\|\mathbf{g}_1 - \mathbf{g}_2\|_{1/2,\Gamma}^2\right)^{1/2}. \tag{4.39}$$

Thus we have proved the theorem.

Theorem 4.1. *Let, under conditions (i), (ii), (iii) for* K_1 *and* (3.19), *the quadruple* $(\mathbf{u}_i, p_i, T_i, \chi_i)$ *be a solution to problem* (4.1) *corresponding to given functions* $\mathbf{v}_d = \mathbf{u}_d^{(i)} \in \mathbf{L}^2(Q)$ *and* $\mathbf{g}_i \in G \subset \tilde{\mathbf{H}}^{1/2}(\Gamma)$, $i = 1, 2$, *where* $Q \subset \Omega$ *is an arbitrary open subset, and let the parameters* a *and* b, c *are defined in* (4.23) *and* (4.26) *in which parameters* γ *and* $\mathcal{R}e^0$ *are given by* (4.13). *Suppose that condition* (4.27) *is satisfied. Then stability estimates* (4.33) *and* (4.35)–(4.38) *hold true where* Δ *is defined in* (4.39).

Now we consider three-parametric control problem

$$\begin{gathered} J(\mathbf{v}, \chi, \varphi, f) \equiv \frac{\mu_0}{2}\|\mathbf{v} - \mathbf{v}_d\|_Q^2 + \frac{\mu_1}{2}\|\chi\|_{\Gamma_N}^2 + \frac{\mu_2}{2}\|\varphi\|_{1/2,\Gamma_D}^2 + \frac{\mu_3}{2}\|f\|^2 \longrightarrow \inf, \\ F(\mathbf{x}, u, \mathbf{g}) = 0, \quad \mathbf{x} = (\mathbf{v}, p, T) \in X, \; u = (\chi, \varphi, f) \in K \end{gathered} \tag{4.40}$$

corresponding to the cost functional $I_1(\mathbf{v}) = \|\mathbf{v} - \mathbf{v}_d\|_Q^2$. Let $(\mathbf{x}_1, u_1) \equiv (\mathbf{u}_1, p_1, T_1, \chi_1, \varphi_1, f_1)$ be a solution to problem (4.40) corresponding to given functions $\mathbf{v}_d \equiv \mathbf{u}_d^{(1)} \in \mathbf{L}^2(Q)$ and $\mathbf{g} = \mathbf{g}_1 \in G$, and let $(\mathbf{x}_2, u_2) \equiv (\mathbf{u}_2, p_2, T_2, \chi_2, \varphi_2, f_2)$ be a solution to problem (4.40) corresponding to perturbed functions $\tilde{\mathbf{v}}_d \equiv \mathbf{u}_d^{(2)} \in \mathbf{L}^2(Q)$ and $\tilde{\mathbf{g}} = \mathbf{g}_2 \in G$. Setting $\mathbf{u}_d = \mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}$ in addition to (3.24), we note that under conditions of problem (4.40) identities (3.20) and (3.21) transform to identities (4.3), (4.4), identity (3.22) does not change, while inequality (3.30) takes by (4.2) a form

$$\begin{gathered} ((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1 + \xi_2) + \kappa(\mathbf{u}\cdot\nabla T, \theta_1 + \theta_2) + \mu_0\left(\|\mathbf{u}\|_Q^2 - (\mathbf{u}, \mathbf{u}_d)_Q\right) \\ \leq -\langle\zeta, \mathbf{g}\rangle_\Gamma - \mu_1\|\chi\|_{\Gamma_N}^2 - \mu_2\|\varphi\|_{1/2,\Gamma_D}^2 - \mu_3\|f\|^2. \end{gathered} \tag{4.41}$$

From (4.3), (4.4), and (3.22) we come to the same estimates (4.17), (4.21), and (4.23) for norms $\|\xi_i\|_1$, $\|\theta_i\|_1$, $\|\sigma_i\|$ and $\|\zeta\|_{-1/2,\Gamma}$. Taking into account these estimates and estimates (3.42), (3.44) for $\|T\|_1$, $\|\mathbf{u}\|_1$, we deduce that

$$
\begin{aligned}
&|((\mathbf{u}\cdot\nabla)\mathbf{u},\xi_1+\xi_2)| \\
&\quad\le \gamma_0\|\mathbf{u}\|_1^2(\|\xi_1\|_1+\|\xi_2\|_1)\le 4\mu_0\gamma_0\gamma\left(\mathcal{R}e+\mathcal{R}e^0\right) \\
&\qquad\times\left[\frac{2\beta_1C_1(\mathcal{N}+1)\|\varphi\|_{1/2,\Gamma_D}}{\delta_0\nu(1-2\mathcal{R}a)}+\frac{2\beta_1\left(\gamma_2\|\chi\|_{\Gamma_N}+\gamma_4\|f\|\right)}{\delta_0\nu\delta_1\lambda(1-2\mathcal{R}a)}+\frac{C_0(2\mathcal{R}+1)\|\mathbf{g}\|_{1/2,\Gamma}}{1-2\mathcal{R}a}\right]^2, \\
&|(\mathbf{u}\cdot\nabla T,\theta_1+\theta_2)| \\
&\quad\le \frac{4\mu_0\gamma_1\gamma\beta_1(\mathcal{R}e+\mathcal{R}e^0)}{\delta_1\lambda} \\
&\qquad\times\left[\frac{2\beta_1C_1(\mathcal{N}+1)\|\varphi\|_{1/2,\Gamma_D}}{\delta_0\nu(1-2\mathcal{R}a)}+\frac{2\beta_1\left(\gamma_2\|\chi\|_{\Gamma_N}+\gamma_4\|f\|\right)}{\delta_0\nu\delta_1\lambda(1-2\mathcal{R}a)}+\frac{C_0(2\mathcal{R}+1)\|\mathbf{g}\|_{1/2,\Gamma}}{1-2\mathcal{R}a}\right] \\
&\qquad\times\left[\frac{C_1(\mathcal{N}+1)}{1-2\mathcal{R}a}\|\varphi\|_{1/2,\Gamma_D}+\frac{\gamma_2\|\chi\|_{\Gamma_N}+\gamma_4\|f\|}{\delta_1\lambda(1-2\mathcal{R}a)}+\frac{C_0(2\mathcal{R}+1)}{1-2\mathcal{R}a}\frac{\gamma_1M_T^0}{\delta_1\lambda}\|\mathbf{g}\|_{1/2,\Gamma}\right].
\end{aligned}
\tag{4.42}
$$

Here parameters γ and $\mathcal{R}e^0$ are given by (4.13). From (4.42) we obtain that

$$
|((\mathbf{u}\cdot\nabla)\mathbf{u},\xi_1+\xi_2)+\kappa(\mathbf{u}\cdot\nabla T,\theta_1+\theta_2)|\le\mu_0\left(b\|\mathbf{g}\|_{1/2,\Gamma}^2+c_1\|\chi\|_{\Gamma_N}^2+c_2\|\varphi\|_{1/2,\Gamma_D}^2+c_3\|f\|^2\right).
\tag{4.43}
$$

Here constants b, c_1, c_2, and c_3 are given by relations

$$
\begin{aligned}
b&=8\gamma_0\gamma C_0^2(2\mathcal{R}+1)^2\frac{(\mathcal{R}e+\mathcal{R}e^0)}{(1-2\mathcal{R}a)^2}\left[3+\left(\frac{\gamma_1}{\gamma_0}\right)^2\mathcal{P}^2\mathcal{R}a^2\right],\\
c_1&=8\gamma_0\gamma\left(\frac{\beta_1}{\delta_0\nu}\frac{\gamma_2}{\delta_1\lambda}\right)^2\frac{(\mathcal{R}e+\mathcal{R}e^0)}{(1-2\mathcal{R}a)^2}\left[12+\left(\frac{\gamma_1}{\gamma_0}\right)^2\mathcal{P}^2\right],\\
c_2&=8\gamma_0\gamma C_1^2(\mathcal{N}+1)^2\left(\frac{\beta_1}{\delta_0\nu}\right)^2\frac{(\mathcal{R}e+\mathcal{R}e^0)}{(1-2\mathcal{R}a)^2}\left[12+\left(\frac{\gamma_1}{\gamma_0}\right)^2\mathcal{P}^2\right],\\
c_3&=8\gamma_0\gamma\left(\frac{\beta_1}{\delta_0\nu}\frac{\gamma_4}{\delta_1\lambda}\right)^2\frac{(\mathcal{R}e+\mathcal{R}e^0)}{(1-2\mathcal{R}a)^2}\left[12+\left(\frac{\gamma_1}{\gamma_0}\right)^2\mathcal{P}^2\right].
\end{aligned}
\tag{4.44}
$$

Let the data for problem (4.40) and parameters μ_0, μ_1, μ_2, and μ_3 be such that

$$\mu_1(1-\varepsilon) \geq \mu_0 c_1, \quad \mu_2(1-\varepsilon) \geq \mu_0 c_2, \quad \mu_3(1-\varepsilon) \geq \mu_0 c_3, \quad \varepsilon = \text{const} > 0. \tag{4.45}$$

Under condition (4.45) we deduce from (4.43) that

$$\begin{aligned} &|((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1+\xi_2) + \kappa(\mathbf{u}\cdot\nabla T, \theta_1+\theta_2)| \\ &\quad\leq (1-\varepsilon)\mu_1\|\chi\|_{\Gamma_N}^2 + (1-\varepsilon)\mu_2\|\varphi\|_{1/2,\Gamma_D}^2 \\ &\qquad + (1-\varepsilon)\mu_3\|f\|^2 + \mu_0 b\|\mathbf{g}\|_{1/2,\Gamma}^2. \end{aligned} \tag{4.46}$$

Taking into account (4.46) and (4.23), we come from (4.41) to the inequality

$$\begin{aligned} \mu_0\left(\|\mathbf{u}\|_Q^2 - (\mathbf{u},\mathbf{u}_d)_Q\right) &\leq -((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1+\xi_2) - \kappa(\mathbf{u}\cdot\nabla T, \theta_1+\theta_2) \\ &\quad - \langle\zeta,\mathbf{g}\rangle_\Gamma - \mu_1\|\chi\|_{\Gamma_N}^2 - \mu_2\|\varphi\|_{1/2,\Gamma_D}^2 - \mu_3\|f\|^2 \\ &\leq -\varepsilon\mu_1\|\chi\|_{\Gamma_N}^2 - \varepsilon\mu_2\|\varphi\|_{1/2,\Gamma_D}^2 - \varepsilon\mu_3\|f\|^2 + \mu_0 a\|\mathbf{g}\|_{1/2,\Gamma} + \mu_0 b\|\mathbf{g}\|_{1/2,\Gamma}^2. \end{aligned} \tag{4.47}$$

It follows from this inequality that

$$\begin{aligned} \mu_0\|\mathbf{u}\|_Q^2 &\leq \mu_0(\mathbf{u},\mathbf{u}_d)_Q - \varepsilon\mu_1\|\chi\|_{\Gamma_N}^2 - \varepsilon\mu_2\|\varphi\|_{1/2,\Gamma_D}^2 \\ &\quad - \varepsilon\mu_3\|f\|^2 + \mu_0 a\|\mathbf{g}\|_{1/2,\Gamma} + \mu_0 b\|\mathbf{g}\|_{1/2,\Gamma}^2. \end{aligned} \tag{4.48}$$

Excluding nonpositive terms from the right-hand side of (4.48), we come to the inequality (4.31) where constants a and b are defined in (4.23) and (4.44). From (4.31) we deduce the estimate (4.32) for $\|\mathbf{u}\|_Q$ with mentioned constants a and b given by (4.23) and (4.44). As in the case of problem (4.1), stability in the norm $\mathbf{L}^2(\Omega)$ of the component $\widehat{\mathbf{u}}$ of the solution to problem (4.40) relative to small perturbations of functions $\mathbf{v}_d \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in G$ in the norms of $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^{1/2}(\Gamma)$, respectively, and uniqueness of the solution to problem (4.40) follow from (4.32) in the case when $Q = \Omega$ and (4.45) holds.

We note again that the uniqueness and stability of the solution to problem (4.40) under condition (4.45) take place and in the case $Q \subset \Omega$ where Q is only a part of the domain Ω. In order to establish this fact we consider inequality (4.48) which we rewrite taking into account (4.32) as

$$\begin{aligned} \varepsilon\mu_1\|\chi\|_{\Gamma_N}^2 + \varepsilon\mu_2\|\varphi\|_{1/2,\Gamma_D}^2 + \varepsilon\mu_3\|f\|^2 &\leq -\mu_0\|\mathbf{u}\|_Q^2 + \mu_0\|\mathbf{u}\|_Q\|\mathbf{u}_d\|_Q + \mu_0 a\|\mathbf{g}\|_{1/2,\Gamma} + \mu_0 b\|\mathbf{g}\|_{1/2,\Gamma}^2 \\ &\leq \mu_0\left[\|\mathbf{u}_d\|_Q + \left(a\|\mathbf{g}\|_{1/2,\Gamma} + b\|\mathbf{g}\|_{1/2,\Gamma}^2\right)^{1/2}\right]^2. \end{aligned} \tag{4.49}$$

From this inequality and from (3.42)–(3.47) we come to the following stability estimates:

$$\|\chi_1-\chi_2\|_{\Gamma_N}\le\sqrt{\frac{\mu_0}{\varepsilon\mu_1}}\Delta,\qquad \|\varphi_1-\varphi_2\|_{1/2,\Gamma_D}\le\sqrt{\frac{\mu_0}{\varepsilon\mu_2}}\Delta,\qquad \|f_1-f_2\|\le\sqrt{\frac{\mu_0}{\varepsilon\mu_3}}\Delta, \tag{4.50}$$

$$\|\mathbf{u}_1-\mathbf{u}_2\|_1\le\frac{2\beta_1 d}{\delta_0\nu(1-2\mathcal{R}a)}\sqrt{\frac{\mu_0}{\varepsilon}}\Delta+\frac{C_0(2\mathcal{R}+1)\|\mathbf{g}_1-\mathbf{g}_2\|_{1/2,\Gamma}}{1-2\mathcal{R}a}, \tag{4.51}$$

$$\|T_1-T_2\|_1\le\frac{d}{1-2\mathcal{R}a}\sqrt{\frac{\mu_0}{\varepsilon}}\Delta+\frac{C_0(2\mathcal{R}+1)}{1-2\mathcal{R}a}\frac{\gamma_1 M_T^0}{\delta_1\lambda}\|\mathbf{g}_1-\mathbf{g}_2\|_{1/2,\Gamma}, \tag{4.52}$$

$$\|p_1-p_2\|\le\frac{(2\mathcal{R}+1)}{\beta_0(1-2\mathcal{R}a)}\left[\beta_1 d\sqrt{\frac{\mu_0}{\varepsilon}}\Delta+\delta_0\nu C_0(\mathcal{R}+\mathcal{R}a)\|\mathbf{g}_1-\mathbf{g}_2\|_{1/2,\Gamma}\right]. \tag{4.53}$$

Here a constant d depending on μ_1, μ_2, and μ_3 is given by

$$d=\frac{\gamma_2}{\delta_1\lambda\sqrt{\mu_1}}+\frac{C_1(\mathcal{N}+1)}{\sqrt{\mu_2}}+\frac{\gamma_4}{\delta_1\lambda\sqrt{\mu_3}}, \tag{4.54}$$

and a quantity Δ is defined in (4.39). Thus the following theorem is proved.

Theorem 4.2. *Let, under conditions (i), (ii), (iii), and* (3.19), *an element* $(\mathbf{u}_i,p_i,T_i,\chi_i,\varphi_i,f_i)$ *be a solution to problem* (4.40) *corresponding to given functions* $\mathbf{v}_d=\mathbf{u}_d^{(i)}\in\mathbf{L}^2(Q)$ *and* $\mathbf{g}_i\in G$, $i=1,\ 2$, *where* Q *is an arbitrary open subset, and let parameters* a *and* b,c_1,c_2,c_3 *be defined in* (4.23) *and* (4.44), *where* γ *and* $\mathcal{R}e^0$*are given by* (4.13). *Suppose that conditions* (4.45) *are satisfied. Then stability estimates* (4.33) *and* (4.50)–(4.53) *hold where* Δ *and* d *are defined in* (4.39) *and* (4.54).

In the same manner one can study control problem

$$J(\mathbf{v},\chi)\equiv\frac{\mu_0}{2}\|\mathbf{v}-\mathbf{v}_d\|_{1,Q}^2+\frac{\mu_1}{2}\|\chi\|_{\Gamma_N}^2\longrightarrow\inf,\qquad F(\mathbf{x},\chi,\mathbf{g})=0,\ \mathbf{x}\in X,\ \chi\in K_1 \tag{4.55}$$

corresponding to the cost functional $I_2(\mathbf{v})=\|\mathbf{v}-\mathbf{v}_d\|_{1,Q}^2$. Let us denote by $(\mathbf{x}_1,u_1)\equiv(\mathbf{u}_1,p_1,T_1,\chi_1)$ a solution to problem (4.55) which corresponds to given functions $\mathbf{v}_d\equiv\mathbf{u}_d^{(1)}\in\mathbf{L}^2(Q)$ and $\mathbf{g}=\mathbf{g}_1\in G$; by $(\mathbf{x}_2,u_2)\equiv(\mathbf{u}_2,p_2,T_2,\chi_2)$ we denote a solution to problem (4.1) which corresponds to perturbed functions $\tilde{\mathbf{v}}_d\equiv\mathbf{u}_d^{(2)}\in\mathbf{L}^2(Q)$ and $\tilde{\mathbf{g}}=\mathbf{g}_2\in G$. Setting $\mathbf{u}_d=\mathbf{u}_d^{(1)}-\mathbf{u}_d^{(2)}$ in addition to (3.24) we note that under conditions of problem (4.55) we have

$$\langle I_2'(\mathbf{u}_i),\mathbf{w}\rangle=2\left(\mathbf{u}_i-\mathbf{u}_d^{(i)},\mathbf{w}\right)_{1,Q},\qquad \left(I_2'(\mathbf{u}_1)-\tilde{I}_2'(\mathbf{u}_2),\mathbf{u}\right)=2\left(\|\mathbf{u}\|_{1,Q}^2-(\mathbf{u},\mathbf{u}_d)_{1,Q}\right). \tag{4.56}$$

Identity (3.22) for problem (4.1) does not change while identities (3.20), (3.21) and inequality (3.51) transform by (4.56) to (4.4) and relations

$$\begin{aligned} &\nu(\nabla \mathbf{w}, \nabla \xi_i) + ((\mathbf{u}_i \cdot \nabla)\mathbf{w}, \xi_i) + ((\mathbf{w} \cdot \nabla)\mathbf{u}_i, \xi_i) + \kappa(\mathbf{w} \cdot \nabla T_i, \theta_i) - (\sigma_i, \operatorname{div} \mathbf{w}) + \langle \zeta_i, \mathbf{w} \rangle_\Gamma \\ &\quad = -\mu_0 \left(\mathbf{u}_i - \mathbf{u}_d^{(i)}, \mathbf{w} \right)_{1,Q} \quad \forall \mathbf{w} \in \widetilde{\mathbf{H}}^1(\Omega), \ i = 1, 2, \end{aligned} \tag{4.57}$$

$$((\mathbf{u} \cdot \nabla)\mathbf{u}, \xi_1 + \xi_2) + \kappa(\mathbf{u} \cdot \nabla T, \theta_1 + \theta_2) + \mu_0 \left(\|\mathbf{u}\|_{1,Q}^2 - (\mathbf{u}, \mathbf{u}_d)_{1,Q} \right) \le -\langle \zeta, \mathbf{g} \rangle_\Gamma - \mu_1 \|\chi\|_{\Gamma_N}^2. \tag{4.58}$$

Using identities (4.57), (4.4), and (3.22) we estimate parameters ξ_i, θ_i, σ_i and ζ_i. To this end we set $\mathbf{w} = \xi_i$, $\tau = \theta_i$ in (4.57), (3.22). Taking into account (2.11), (2.12) and condition $\xi_i \in \mathbf{V}$ which follows from (4.4) we obtain (4.7) and relation

$$\nu(\nabla \xi_i, \nabla \xi_i) = -((\xi_i \cdot \nabla)\mathbf{u}_i, \xi_i) - \kappa(\xi_i \cdot \nabla T_i, \theta_i) - \mu_0 \left(\mathbf{u}_i - \mathbf{u}_d^{(i)}, \xi_i \right)_{1,Q}. \tag{4.59}$$

Using estimates (3.15) we deduce in addition to (4.8)–(4.10) that

$$\begin{aligned} &\left\| \mathbf{u}_i - \mathbf{u}_d^{(i)} \right\|_{1,Q} \le \|\mathbf{u}_i\|_{1,Q} + \left\| \mathbf{u}_d^{(i)} \right\|_{1,Q} \le M_{\mathbf{u}}^0 + \left\| \mathbf{u}_d^{(i)} \right\|_Q \le \delta_0 \nu \gamma_0^{-1} \left(\mathcal{R}e + \mathcal{R}e^0 \right), \\ &\left| \left(\mathbf{u}_i - \mathbf{u}_d^{(i)}, \xi_i \right)_{1,Q} \right| \le \left\| \mathbf{u}_i - \mathbf{u}_d^{(i)} \right\|_{1,Q} \|\xi_i\|_{1,Q} \le \left(M_{\mathbf{u}}^0 + \left\| \mathbf{u}_d^{(i)} \right\|_{1,Q} \right) \|\xi_i\|_1 \le \delta_0 \nu \gamma \left(\mathcal{R}e + \mathcal{R}e^0 \right) \|\xi_i\|_1, \end{aligned} \tag{4.60}$$

where

$$\gamma = \gamma_0^{-1}, \qquad \mathcal{R}e^0 = \frac{\gamma_0}{\delta_0 \nu} \max\left(\left\| \mathbf{u}_d^{(1)} \right\|_{1,Q}, \left\| \mathbf{u}_d^{(2)} \right\|_{1,Q} \right). \tag{4.61}$$

Proceeding further as above in study of problem (4.1) we come to the estimates for ξ_i, θ_i, σ_i and $\zeta = \zeta_1 - \zeta_2$. They have a form (4.17), (4.21), and (4.23), where parameters γ and $\mathcal{R}e^0$ are given by (4.61).

Let us assume that the condition (4.27) takes place where parameter c is defined in (4.26), (4.61). Using (4.27) and estimates (4.17), (4.21), (4.23) we deduce inequality (4.28) where parameter b is given by relations (4.26), (4.61). Taking into account (4.28) and (4.23), we come from (4.58) to the inequality

$$\begin{aligned} \mu_0 \left(\|\mathbf{u}\|_{1,Q}^2 - (\mathbf{u}, \mathbf{u}_d)_{1,Q} \right) &\le -((\mathbf{u} \cdot \nabla)\mathbf{u}, \xi_1 + \xi_2) - \kappa(\mathbf{u} \cdot \nabla T, \theta_1 + \theta_2) - \|\zeta\|_{-1/2,\Gamma} \|\mathbf{g}\|_{1/2,\Gamma} - \mu_1 \|\chi\|_{\Gamma_N}^2 \\ &\le -\varepsilon \mu_1 \|\chi\|_{\Gamma_N}^2 + \mu_0 a \|\mathbf{g}\|_{1/2,\Gamma} + \mu_0 b \|\mathbf{g}\|_{1/2,\Gamma}^2. \end{aligned} \tag{4.62}$$

It follows from this inequality that

$$\mu_0 \|\mathbf{u}\|_{1,Q}^2 \le \mu_0 (\mathbf{u}, \mathbf{u}_d)_{1,Q} - \varepsilon \mu_1 \|\chi\|_{\Gamma_N}^2 + \mu_0 a \|\mathbf{g}\|_{1/2,\Gamma} + \mu_0 b \|\mathbf{g}\|_{1/2,\Gamma}^2. \tag{4.63}$$

Excluding nonpositive term $-\varepsilon\mu_1\|\chi\|^2_{\Gamma_N}$, we deduce from (4.63) that

$$\|\mathbf{u}\|^2_{1,Q} \le \|\mathbf{u}_d\|_{1,Q}\|\mathbf{u}\|_{1,Q} + a\|\mathbf{g}\|_{1/2,\Gamma} + b\|\mathbf{g}\|^2_{1/2,\Gamma}. \tag{4.64}$$

Equation (4.31) is a quadratic inequality relative to $\|\mathbf{u}\|_{1,Q}$. By solving it we come to the estimate

$$\|\mathbf{u}\|_{1,Q} \le \|\mathbf{u}_d\|_{1,Q} + \left(a\|\mathbf{g}\|_{1/2,\Gamma} + b\|\mathbf{g}\|^2_{1/2,\Gamma}\right)^{1/2} \tag{4.65}$$

which is equivalent to the following estimate for $\mathbf{u}_1 - \mathbf{u}_2$:

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{1,Q} \le \left\|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\right\|_{1,Q} + \left(a\|\mathbf{g}_1 - \mathbf{g}_2\|_{1/2,\Gamma} + b\|\mathbf{g}_1 - \mathbf{g}_2\|^2_{1/2,\Gamma}\right)^{1/2}. \tag{4.66}$$

The estimate (4.66) under $Q = \Omega$ has the sense of the stability estimate in the norm $\mathbf{H}^1(\Omega)$ of the component $\widehat{\mathbf{u}}$ of the solution $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{T}, \widehat{\chi})$ to problem (4.55) relative to small perturbations of functions $\mathbf{v}_d \in \mathbf{H}^1(\Omega)$ and $\mathbf{g} \in G$ in the norms of $\mathbf{H}^1(\Omega)$ and $\mathbf{H}^{1/2}(\Gamma)$ respectively. In the case where $\mathbf{u}_d^{(1)} = \mathbf{u}_d^{(2)}$ and $\mathbf{g}_1 = \mathbf{g}_2$ it follows from (4.66) that $\mathbf{u}_1 = \mathbf{u}_2$ in Ω, if $Q = \Omega$. This yields together with (4.63), (3.48), (3.50) that $\chi_1 = \chi_2$, $T_1 = T_2$, $p_1 = p_2$. The latter means the uniqueness of the solution to problem (4.55) when $Q = \Omega$ and (4.27) holds.

We note again that using (4.63), (4.65) we can deduce rougher stability estimates of the solution to problem (4.55) which take place even in the case where $Q \ne \Omega$. In fact we deduce from (4.63) (4.65) that

$$\begin{aligned}
\varepsilon\mu_1\|\chi\|^2_{\Gamma_N} &\le \mu_0\|\mathbf{u}\|_{1,Q}\|\mathbf{u}_d\|_{1,Q} + \mu_0 a\|\mathbf{g}\|_{1/2,\Gamma} + \mu_0 b\|\mathbf{g}\|^2_{1/2,\Gamma} \\
&\le \mu_0\left[\|\mathbf{u}_d\|_{1,Q} + \left(a\|\mathbf{g}\|_{1/2,\Gamma} + b\|\mathbf{g}\|^2_{1/2,\Gamma}\right)^{1/2}\right]^2.
\end{aligned} \tag{4.67}$$

From (4.67) and (3.48)–(3.50) we come to the estimates (4.35)–(4.38) where one should set

$$\Delta = \left\|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\right\|_{1,Q} + \left(a\|\mathbf{g}_1 - \mathbf{g}_2\|_{1/2,\Gamma} + b\|\mathbf{g}_1 - \mathbf{g}_2\|^2_{1/2,\Gamma}\right)^{1/2}. \tag{4.68}$$

Thus we have proved the following theorem.

Theorem 4.3. *Let, under conditions (i), (ii), (iii) for K_1 and* (3.19)*, the quadruple $(\mathbf{u}_i, p_i, T, \chi_i)$ be a solution to problem* (4.55) *corresponding to given functions $\mathbf{v}_d = \mathbf{u}_d^{(i)} \in \mathbf{H}^1(Q)$ and $\mathbf{g}_i \in G$, $i = 1, 2$, where $Q \subset \Omega$ is an arbitrary open subset, and let parameters a, b, c be defined in* (4.23) *and* (4.26)*, in which γ and $\mathcal{R}e^0$ are given by* (4.61)*. Suppose that condition* (4.27) *is satisfied. Then the stability estimates* (4.66) *and* (4.35)–(4.38) *hold where Δ is defined in* (4.68).

In the similar way one can study three-parametric control problem

$$J(\mathbf{v},\chi,\varphi,f) \equiv \frac{\mu_0}{2}\|\mathbf{v}-\mathbf{v}_d\|_{1,Q}^2 + \frac{\mu_1}{2}\|\chi\|_{\Gamma_N}^2 + \frac{\mu_2}{2}\|\varphi\|_{1/2,\Gamma_D}^2 + \frac{\mu_3}{2}\|f\|^2 \longrightarrow \inf,$$
$$F(\mathbf{x},\chi,\varphi,f,\mathbf{g}) = 0, \quad (\mathbf{x},\chi,\varphi,f) \in X \times K_1 \times K_2 \times K_3. \tag{4.69}$$

It is obtained from (4.40) by replacing of the cost functional $I_1(\mathbf{v})$ by $I_2(\mathbf{v})$. Analogous analysis shows that the following theorem holds.

Theorem 4.4. *Let, under conditions (i), (ii), (iii) and* (3.19)*, an element* $(\mathbf{u}_i, p_i, T_i, \chi_i, \varphi_i, f_i)$ *be a solution to problem* (4.69) *corresponding to given functions* $\mathbf{u}_d = \mathbf{u}_d^{(i)} \in \mathbf{H}^1(Q)$ *and* $\mathbf{g}_i \in G$, $i = 1, 2$, *where* $Q \subset \Omega$ *is an arbitrary open subset and let parameters* a *and* b, c_1, c_2, c_3 *are defined in* (4.23) *and* (4.26)*, in which* γ *and* $\mathcal{R}e^0$ *are given by* (4.61)*. Suppose that conditions* (4.45) *are satisfied. Then the stability estimates* (4.66) *and* (4.50)–(4.53) *hold where* Δ *is defined in* (4.68).

5. Control Problem for Vorticity Tracking-Type Cost Functional

Consider now one-parametric control problem

$$J(\mathbf{v},\chi) \equiv \frac{\mu_0}{2}\|\text{rot } \mathbf{v} - \zeta_d\|_Q^2 + \frac{\mu_1}{2}\|\chi\|_{\Gamma_N}^2 \longrightarrow \inf, \qquad F(\mathbf{x},\chi,\mathbf{g}) = 0, \quad \mathbf{x} \in X, \ \chi \in K_1, \tag{5.1}$$

which corresponds to the cost functional $I_3(\mathbf{v}) = \|\text{ rot } \mathbf{v} - \zeta_d\|_Q^2$. Let $(\mathbf{x}_1, u_1) \equiv (\mathbf{u}_1, p_1, T_1, \chi_1)$ be a solution to problem (5.1) corresponding to given functions $\zeta_d \equiv \zeta_d^{(1)} \in \mathbf{L}^2(Q)$ and $\mathbf{g} = \mathbf{g}_1 \in G$, and let $(\mathbf{x}_2, u_2) \equiv (\mathbf{u}_2, p_2, T_2, \chi_2)$ be a solution to problem (4.1) corresponding to perturbed functions $\widetilde{\zeta}_d \equiv \zeta_d^{(2)} \in \mathbf{L}^2(Q)$ and $\widetilde{\mathbf{g}} = \mathbf{g}_2 \in G$. Setting $\mathbf{u}_d = \mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}$ in addition to (3.24), we have under conditions of problem (4.1)

$$\langle I_3'(\mathbf{u}_i), \mathbf{w}\rangle = 2\Big(\text{rot } \mathbf{u}_i - \zeta_d^{(i)}, \text{rot } \mathbf{w}\Big)_Q, \qquad \Big(I_3'(\mathbf{u}_1) - \widetilde{I}_3'(\mathbf{u}_2), \mathbf{u}\Big) = 2\Big(\|\text{rot } \mathbf{u}\|_Q^2 - (\text{rot } \mathbf{u}, \zeta_d)_Q\Big). \tag{5.2}$$

Identity (3.22) for problem (5.1) does not change, while identities (3.20), (3.21) and inequality (3.51) transform due to (5.2) to (4.4) and relations

$$\nu(\nabla \mathbf{w}, \nabla \xi_i) + ((\mathbf{u}_i \cdot \nabla)\mathbf{w}, \xi_i) + ((\mathbf{w} \cdot \nabla)\mathbf{u}_i, \xi_i) + \kappa(\mathbf{w} \cdot \nabla T_i, \theta_i) - (\sigma_i, \text{div } \mathbf{w}) + \langle \zeta_i, \mathbf{w}\rangle_\Gamma$$
$$= -\mu_0\Big(\text{rot } \mathbf{u}_i - \zeta_d^{(i)}, \mathbf{w}\Big)_Q \quad \forall \mathbf{w} \in \widetilde{\mathbf{H}}^1(\Omega), \tag{5.3}$$

$$((\mathbf{u} \cdot \nabla)\mathbf{u}, \xi_1 + \xi_2) + \kappa(\mathbf{u} \cdot \nabla T, \theta_1 + \theta_2) + \mu_0\Big(\|\text{rot } \mathbf{u}\|_Q^2 - (\text{rot } \mathbf{u}, \zeta_d)_Q\Big) \le -\langle \zeta, \mathbf{g}\rangle_\Gamma - \mu_1\|\chi\|_{\Gamma_N}^2. \tag{5.4}$$

Using identities (5.3), (3.22), (4.4) we estimate parameters ξ_i, θ_i, σ_i, and ζ_i. Firstly we deduce estimates of norms $\|\xi_i\|_1$ and $\|\theta_i\|_1$. To this end we set $\mathbf{w} = \xi_i$, $\tau = \theta_i$ in (5.3), (3.22). Taking into account (2.11), (2.12) and condition $\xi_i \in \mathbf{V}$, which follows from (4.4), we obtain (4.7) and relation

$$\nu(\nabla \xi_i, \nabla \xi_i) = -((\xi_i \cdot \nabla)\mathbf{u}_i, \xi_i) - \kappa(\xi_i \cdot \nabla T_i, \theta_i) - \mu_0\Big(\text{rot } \mathbf{u}_i - \zeta_d^{(i)}, \xi_i\Big)_Q. \tag{5.5}$$

Using (2.9), (3.15) we deduce in addition to (4.8)–(4.10) that

$$\begin{aligned}
&\left\|\text{rot }\mathbf{u}_i-\zeta_d^{(i)}\right\|_Q\le\|\text{rot }\mathbf{u}_i\|_Q+\left\|\zeta_d^{(i)}\right\|_Q\le C_rM_{\mathbf{u}}^0+\left\|\zeta_d^{(i)}\right\|_Q\le\frac{\delta_0\nu C_r^2}{\gamma_0}\left(\mathcal{R}e+\mathcal{R}e^0\right),\\
&\left|\left(\text{rot }\mathbf{u}_i-\zeta_d^{(i)},\text{rot }\xi_i\right)_Q\right|\le\left\|\text{rot }\mathbf{u}_i-\zeta_d^{(i)}\right\|_Q\|\xi_i\|_Q\le C_r\left(C_rM_{\mathbf{u}}^0+\left\|\zeta_d^{(i)}\right\|_Q\right)\|\xi_i\|_1\\
&\qquad\le\delta_0\nu\gamma\left(\mathcal{R}e+\mathcal{R}e^0\right)\|\xi_i\|_1,
\end{aligned}\tag{5.6}$$

where

$$\gamma=C_r^2,\qquad\mathcal{R}e^0=\frac{\gamma_0}{\delta_0\nu C_r}\max\left(\left\|\zeta_d^{(1)}\right\|_Q,\left\|\zeta_d^{(2)}\right\|_Q\right).\tag{5.7}$$

Arguing as above in analysis of problem (4.1) we come to the same estimates (4.17), (4.21), and (4.23) for $\|\xi_i\|_1$, $\|\theta_i\|_1$, $\|\sigma_i\|$, and $\|\zeta\|_{-1/2,\Gamma}$ in which parameters γ and $\mathcal{R}e^0$ are given by (5.7).

Let us assume that the condition (4.27) takes place where parameter c is defined in (4.26), (5.7). Using (4.27) and (4.17), (4.21), (4.23) we deduce inequality (4.28) where parameter b is given by (4.26), (5.7). Taking into account (4.28) and (4.23) with parameter a defined in (4.23), (5.7) we come from (5.4) to the inequality

$$\begin{aligned}
&\mu_0\left(\|\text{rot }\mathbf{u}\|_Q^2-(\text{rot }\mathbf{u},\zeta_d)_Q\right)\\
&\quad\le-((\mathbf{u}\cdot\nabla)\mathbf{u},\xi_1+\xi_2)-\kappa(\mathbf{u}\cdot\nabla T,\theta_1+\theta_2)+\|\zeta\|_{-1/2,\Gamma}\|\mathbf{g}\|_{1/2,\Gamma}-\mu_1\|\chi\|_{\Gamma_N}^2\\
&\quad\le-\varepsilon\mu_1\|\chi\|_{\Gamma_N}^2+\mu_0a\|\mathbf{g}\|_{1/2,\Gamma}+\mu_0b\|\mathbf{g}\|_{1/2,\Gamma}^2.
\end{aligned}\tag{5.8}$$

It follows from this inequality that

$$\mu_0\|\text{rot }\mathbf{u}\|_Q^2\le\mu_0(\text{rot }\mathbf{u},\zeta_d)_Q-\varepsilon\mu_1\|\chi\|_{\Gamma_N}^2+\mu_0a\|\mathbf{g}\|_{1/2,\Gamma}+\mu_0b\|\mathbf{g}\|_{1/2,\Gamma}^2.\tag{5.9}$$

Excluding nonpositive term $-\varepsilon\mu_1\|\chi\|_{\Gamma_N}^2$, we deduce from (5.9) that

$$\|\text{rot }\mathbf{u}\|_Q^2\le\|\zeta_d\|_Q\|\text{rot }\mathbf{u}\|_Q+a\|\mathbf{g}\|_{1/2,\Gamma}+b\|\mathbf{g}\|_{1/2,\Gamma}^2.\tag{5.10}$$

Equation (5.10) is a quadratic inequality relative to $\|\text{rot }\mathbf{u}\|_Q$. Solving it we come to the estimate

$$\|\text{rot }\mathbf{u}\|_Q\le\|\zeta_d\|_Q+\left(a\|\mathbf{g}\|_{1/2,\Gamma}+b\|\mathbf{g}\|_{1/2,\Gamma}^2\right)^{1/2},\tag{5.11}$$

which is equivalent to the following estimate for the difference rot $\mathbf{u}_1$ − rot $\mathbf{u}_2$:

$$\|\text{rot }\mathbf{u}_1-\text{rot }\mathbf{u}_2\|_Q\le\left\|\zeta_d^{(1)}-\zeta_d^{(2)}\right\|_Q+\left(a\|\mathbf{g}_1-\mathbf{g}_2\|_{1/2,\Gamma}+b\|\mathbf{g}_1-\mathbf{g}_2\|_{1/2,\Gamma}^2\right)^{1/2}.\tag{5.12}$$

The estimate (5.12) under $Q = \Omega$ has the sense of the stability estimate in the norm $\mathbf{L}^2(\Omega)$ of the vorticity curl $\hat{\mathbf{u}}$ of the component $\hat{\mathbf{u}}$ of the solution $(\hat{\mathbf{u}}, \hat{p}, \widehat{T}, \hat{\chi})$ to problem (5.1) relative to small perturbations of functions $\zeta_d \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in \widetilde{\mathbf{H}}^{1/2}(\Gamma)$ in the norms of $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^{1/2}(\Gamma)$, respectively. In particular case where $\zeta_d^{(1)} = \zeta_d^{(2)}$ and $\mathbf{g}_1 = \mathbf{g}_2$ it follows from (5.11) that rot $\mathbf{u}_1 =$ rot $\mathbf{u}_2$ in Ω, if $Q = \Omega$. From this relation and from (4.30), (3.48), (3.50) it follows that $\chi_1 = \chi_2, T_1 = T_2, p_1 = p_2$. The latter means the uniqueness of the solution to problem (4.1) when $Q = \Omega$ and condition (4.27) holds.

If $Q \neq \Omega$ we can deduce from (5.11) and (5.9) rougher stability estimates of the solution to problem (5.1), which are analogous to estimates (4.35)–(4.38). In fact using (5.11) we deduce from (5.9) that

$$\begin{aligned}
\varepsilon\mu_1 \|\chi\|_{\Gamma_N}^2 &\leq -\mu_0 \|\text{rot }\mathbf{u}\|_Q^2 + \mu_0 \|\zeta_d\|_Q \|\text{rot }\mathbf{u}\|_Q + \mu_0 a \|\mathbf{g}\|_{1/2,\Gamma} + \mu_0 b \|\mathbf{g}\|_{1/2,\Gamma}^2 \\
&\leq \mu_0 \|\zeta_d\|_Q^2 + \mu_0 \|\zeta_d\|_Q \left(a\|\mathbf{g}\|_{1/2,\Gamma} + b\|\mathbf{g}\|_{1/2,\Gamma}^2\right)^{1/2} + \mu_0 a \|\mathbf{g}\|_{1/2,\Gamma} + \mu_0 b \|\mathbf{g}\|_{1/2,\Gamma}^2 \\
&\leq \mu_0 \left[\|\zeta_d\|_Q + \left(a\|\mathbf{g}\|_{1/2,\Gamma} + b\|\mathbf{g}\|_{1/2,\Gamma}^2\right)^{1/2}\right]^2.
\end{aligned} \tag{5.13}$$

From (5.13) and (3.48)–(3.50) we come to the estimates (4.35)–(4.38) where

$$\Delta = \left\|\zeta_d^{(1)} - \zeta_d^{(2)}\right\|_Q + \left(a\|\mathbf{g}_1 - \mathbf{g}_2\|_{1/2,\Gamma} + b\|\mathbf{g}_1 - \mathbf{g}_2\|_{1/2,\Gamma}^2\right)^{1/2}. \tag{5.14}$$

Thus the following theorem is proved.

Theorem 5.1. *Let, under conditions (i), (ii), (iii) for K_1 and* (3.19)*, the quadruple $(\mathbf{u}_i, p_i, T_i, \chi_i)$ be a solution to problem* (5.1) *corresponding to given functions $\zeta_d^{(i)} \in \mathbf{L}^2(Q)$ and $\mathbf{g}_i \in G$, $i = 1, 2$, where $Q \subset \Omega$ is an arbitrary open subset, and let parameters a and b, c be defined in relations* (4.23) *and* (4.26)*, in which γ and $\mathcal{R}e^0$ are given by* (5.7)*. Suppose that condition* (4.27) *is satisfied. Then the stability estimates* (5.12) *and* (4.35)–(4.38) *hold true where Δ is defined in* (5.14)*.*

In the similar way one can study three-parametric control problem

$$\begin{gathered}
J(\mathbf{v}, \chi, \varphi, f) \equiv \frac{\mu_0}{2}\|\text{rot }\mathbf{v} - \zeta_d\|_Q^2 + \frac{\mu_1}{2}\|\chi\|_{\Gamma_N}^2 + \frac{\mu_2}{2}\|\varphi\|_{1/2,\Gamma_D}^2 + \frac{\mu_3}{2}\|f\|^2 \longrightarrow \inf, \\
F(\mathbf{x}, \chi, \varphi, f, \mathbf{g}) = 0, \quad (\mathbf{x}, \chi, \varphi, f) \in X \times K_1 \times K_2 \times K_3.
\end{gathered} \tag{5.15}$$

It is obtained from (4.40) by replacing the cost functional $I_1(\mathbf{v})$ by $I_3(\mathbf{v})$. The following theorem holds.

Theorem 5.2. *Let, under conditions (i), (ii), (iii), and* (3.19)*, an element $(\mathbf{u}_i, p_i, T_i, \chi_i, \varphi_i, f_i)$ be a solution to problem* (5.15) *corresponding to given functions $\zeta_d = \zeta_d^{(i)} \in \mathbf{L}^2(Q)$ and $\mathbf{g}_i \in G$, $i = 1, 2$, where $Q \subset \Omega$ is an arbitrary open subset, and let parameters a and b, c_1, c_2, c_3 be given by relations* (4.23) *and* (4.44)*, in which* γ *and* $\mathcal{R}e^0$ *be defined in* (5.7)*. Suppose that conditions* (4.45) *are satisfied. Then the stability estimates* (5.12) *and* (4.50)–(4.53) *hold where* Δ *and* d *are defined in* (5.14) *and* (4.54)*.*

6. Conclusion

In this paper we studied control problems for the steady-state Boussinesq equations describing the heat transfer in viscous heat-conducting fluid under inhomogeneous Dirichlet boundary conditions for velocity and mixed boundary conditions for temperature. These problems were formulated as constrained minimization problems with tracking-type cost functionals. We studied the optimality system which describes the first-order necessary optimality conditions for the general control problem and established some properties of its solution. In particular we deduced a special inequality for the difference of solutions to the original and perturbed control problem. The latter is obtained by perturbing both the cost functional and the boundary function entering into the Dirichlet boundary condition for the velocity. Using this inequality we found the group of sufficient conditions for the data which provide a local stability and uniqueness of concrete control problems with velocity-tracking or vorticity-tracking cost functionals. This group consists of two conditions: the first is the same for all control problems and has the form of the standard condition (3.19) which ensures the uniqueness of the solution to the original boundary value problem for the Boussinesq equations. The second one depends on the form of control problem under study. In particular for the one-parametric problem (4.1) corresponding to velocity-tracking functional $I_1(\mathbf{v})$ it has the form of estimates (4.27) of the parameters μ_0 and μ_1 included in (4.1), while for the three-parametric problem (4.40) it has the form of estimates (4.45) of the parameters μ_0, μ_1, μ_2, and μ_3 included in (4.40). Similar conditions take place for another tracking-type functionals.

On the one hand, conditions (4.27) and (4.45) are similar to the uniqueness and stability conditions for the solution to the coefficient identification problems for the linear convection-diffusion-reaction equation. On the other hand, these conditions contain compressed information on the Boussinesq heat transfer model (2.1), (2.2) in the form of the constant c defined in (4.26) for problem (4.1) or in the form of three constants c_1, c_2, c_3 defined in (4.44) for problem (4.40). An analysis of the expressions for c or c_1, c_2, c_3 shows that for fixed values of the parameters μ_l inequality (4.27) or inequalities (4.45) represent additional constraints on the Reynolds number $\mathcal{R}e$, Rayleigh number $\mathcal{R}a$, and Prandtl number $\mathcal{P}$ which together with (3.19) ensure the uniqueness and stability of the solution to problem (4.1) or (4.40). We also note that for fixed values of $\mathcal{R}e$, $\mathcal{R}a$, and $\mathcal{P}$ inequalities (4.27) and (4.45) imply that to ensure the uniqueness and stability of the solution to problem (4.1) or (4.40) the values of the parameters μ_1, μ_2, and μ_3 should be positive and exceed the constants on the right-hand sides of inequalities (4.27) and (4.45). This means that the term $(\mu_1/2)\|\chi\|^2_{\Gamma_N}$ in the expression for minimized functional in (4.1) or the terms $(\mu_1/2)\|\chi\|^2_{\Gamma_N}$, $(\mu_2/2)\|\psi\|^2_{1/2,\Gamma_D}$ and $(\mu_3/2)\|f\|^2$ in the expression for minimized functional in (4.40) have a regularizing effect on the control problem under consideration. The same conclusions hold true and for another control problems studied in this paper.

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