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Research Article

Periodic and Solitary-Wave Solutions for a Variant of the $K(3,2)$ Equation

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We employ the bifurcation method of planar dynamical systems and qualitative theory of polynomial differential systems to derive new bounded traveling-wave solutions for a variant of the $K(3,2)$ equation. For the focusing branch, we obtain hump-shaped and valley-shaped solitary-wave solutions and some periodic solutions. For the defocusing branch, the nonexistence of solitary traveling wave solutions is shown. Meanwhile, some periodic solutions are also obtained. The results presented in this paper supplement the previous results.

1. Introduction

To study the role of nonlinear dispersion in the formation of patterns in the liquid drops, in 1993, Rosenau and Hyman [1] suggested a family of KdV-like equations,

$$u_t + a(u^m)_x + (u^n)_{xxx} = 0, \quad m,n \geq 1, \tag{1.1}$$

which is the so-called $K(m,n)$ equation. In [2], the authors obtained topological soliton solution of the $K(m,n)$ equation by using the G'/G method.

For the focusing branch ($a = 1$), the $K(n,n)$ has compacton solutions [3–9] given by

$$\begin{aligned}
u(x,t) &= \begin{cases} \left\{\sqrt{\dfrac{2cn}{n+1}}\cos\left[\dfrac{n-1}{2n}(x-ct)\right]\right\}^{2/(n-1)}, & |x-ct| \leq \dfrac{n\pi}{n-1},\ n>1, \\ 0, & \text{otherwise}, \end{cases} \\
u(x,t) &= \begin{cases} \left\{\sqrt{\dfrac{2cn}{n+1}}\sin\left[\dfrac{n-1}{2n}(x-ct)\right]\right\}^{2/(n-1)}, & |x-ct| \leq \dfrac{n\pi}{n-1},\ n>1, \\ 0, & \text{otherwise}. \end{cases}
\end{aligned} \tag{1.2}$$

For the defocusing branch ($a = -1$), the $K(n,n)$ equation has solutions [7, 9] given by

$$
\begin{aligned}
u(x,t) &= \left\{\sqrt{\frac{2cn}{n+1}}\cosh\left[\frac{n-1}{2n}(x-ct)\right]\right\}^{2/(n-1)}, \\
u(x,t) &= \left\{\sqrt{\frac{2cn}{n+1}}\sinh\left[\frac{n-1}{2n}(x-ct)\right]\right\}^{2/(n-1)}.
\end{aligned}
\tag{1.3}
$$

Later, in 2000, Rosenau [5] introduced a variant of $K(m,n)$ equation defined by

$$
u_t + a(u^m)_x + (u(u^n)_{xx})_x = 0, \quad m,n \geq 1, \tag{1.4}
$$

which emerges in nonlinear lattices and was used to describe the dispersion of dilute suspensions for $m = 2$ and $n = 1$ [5]. In [10], (1.4) was investigated to support the feature of the dispersion-velocity method for $a = 1$, $m = 2$, and $n = 1$.

For the focusing branch ($a > 0$) and $m = n + 1$, Rosenau [5] obtained a compacton solution of (1.4) given by

$$
u(x,t) = \begin{cases} \left\{\dfrac{2c}{a}\cos^2\left[\dfrac{\sqrt{a}}{2}(x-ct)\right]\right\}^{1/n}, & |x-ct| \leq \pi, \\ 0, & \text{otherwise}. \end{cases} \tag{1.5}
$$

Later, Wazwaz [11] found another compacton solution given by

$$
u(x,t) = \begin{cases} \left\{\dfrac{2c}{a}\sin^2\left[\dfrac{\sqrt{a}}{2}(x-ct)\right]\right\}^{1/n}, & |x-ct| \leq \pi, \\ 0, & \text{otherwise}. \end{cases} \tag{1.6}
$$

For the defocusing branch ($a < 0$) and $m = n + 1$, Wazwaz [11] showed that (1.4) has solutions given by

$$
\begin{aligned}
u(x,t) &= \left\{-\frac{2c}{a}\cosh^2\left[\frac{\sqrt{-a}}{2}(x-ct)\right]\right\}^{1/n}, \\
u(x,t) &= \left\{-\frac{2c}{a}\sinh^2\left[\frac{\sqrt{-a}}{2}(x-ct)\right]\right\}^{1/n}.
\end{aligned}
\tag{1.7}
$$

Here we note that (1.3) and (1.7) are unbounded solutions and have no physical meaning. For more works regarding other variants of the $K(m,n)$ equation, we refer to [12, 13].

Rosenau and Wazwaz's work did not consider the periodic and general solitary-wave solutions for (1.4). So, in this paper, we attempt to employ the bifurcation method of planar dynamical systems and qualitative theory of polynomial differential systems to obtain such two types of solutions of (1.4) for $a = \pm 1$, $m = 3$, and $n = 2$, which are not obtained in [5, 11]. The equation reads

$$u_t \pm \left(u^3\right)_x + \left(u\left(u^2\right)_{xx}\right)_x = 0. \tag{1.8}$$

From Rosenau's point of view [5], we call (1.8) a variant of the $K(3,2)$ equation. $a = 1$ and $a = -1$ correspond to the focusing branch and defocusing branch of (1.8), respectively. It is known that periodic solution and solitary-wave solution are two important types of solutions for physical systems. They describe practical physical processes and can explain many physical phenomena. Here we must note that the bifurcation method was first used by Li and Liu [14] to obtain smooth and nonsmooth traveling-wave solutions to a nonlinearly dispersive equation and was later proved to be a powerful tool to solve a large number of integrable and nonintegrable partial differential equations [15–22].

The remainder of the paper is organized as follows. In Section 2, using the traveling wave transformation, we transform (1.8) into planar dynamical systems and then discuss bifurcations and phase portraits of the systems corresponding to the focusing and defocusing branch of (1.8), respectively. In Section 3, We obtain the solitary-wave solutions for the focusing branch of (1.8) and demonstrate the nonexistence of solitary-wave solutions for the defocusing branch of (1.8). In Section 4, we obtain some periodic solutions for both the focusing branch and the defocusing branch of (1.8). A short conclusion is given in Section 5.

2. Plane Phase Analysis

Let $\xi = x - ct$, where c is the wave speed. By using the traveling wave transformation $u(x,t) = \varphi(x - ct) = \varphi(\xi)$, we can reduce (1.8) to the following ordinary differential equations:

$$-c\varphi' \pm \left(\varphi^3\right)' + \left(\varphi\left(\varphi^2\right)''\right)' = 0, \tag{2.1}$$

where $'$ denotes the derivative of the function with respect to ξ.

Integrating (2.1) once with respect to ξ, we have

$$-c\varphi \pm \varphi^3 + 2\varphi\varphi'^2 + 2\varphi^2\varphi'' = g, \tag{2.2}$$

where g is a constant of integration.

Let $\varphi' = y$, then (2.2) can be transformed into the following planar dynamical system:

$$\begin{aligned} \frac{d\varphi}{d\xi} &= y, \\ \frac{dy}{d\xi} &= \frac{g + c\varphi \mp \varphi^3 - 2\varphi y^2}{2\varphi^2}. \end{aligned} \tag{2.3}$$

We call it the traveling wave system of (1.8). It is a planar dynamical system with Hamiltonian function

$$H_{\pm}(\varphi, y) = \pm\frac{1}{4}\varphi^4 - \frac{c}{2}\varphi^2 - g\varphi + \varphi^2 y^2 = h, \tag{2.4}$$

where h is a constant.

Note that (2.3) has a singular line $\varphi = 0$. To avoid the line temporarily, we make transformation $d\xi = 2\varphi^2 d\zeta$. Under this transformation, system (2.3) becomes

$$\begin{aligned} \frac{d\varphi}{d\xi} &= 2\varphi^2 y, \\ \frac{dy}{d\xi} &= g + c\varphi \mp \varphi^3 - 2\varphi y^2. \end{aligned} \tag{2.5}$$

System (2.5) and system (2.3) have the same first integral as (2.4). Consequently, system (2.5) has the same topological phase portraits as system (2.3) except for the straight line $\varphi = 0$.

For a fixed h, (2.4) determines a set of invariant curves of system (2.5). As h is varied, (2.4) determines different families of orbits of system (2.5) having different dynamical behaviors. Let $M(\varphi_e, y_e)$ be the coefficient matrix of the linearized version of system (2.5) at the equilibrium point (φ_e, y_e), then

$$M(\varphi_e, y_e) = \begin{pmatrix} 4\varphi_e y_e & 2\varphi_e^2 \\ c \mp 3\varphi_e^2 - 2y_e^2 & -4\varphi_e y_e \end{pmatrix} \tag{2.6}$$

and, at this equilibrium point, we have

$$\begin{aligned} J(\varphi_e, y_e) &= \det M(\varphi_e, y_e) = -12\varphi_e^2 y_e^2 - 2\varphi_e^2\left(c \mp 3\varphi_e^2\right), \\ p(\varphi_e, y_e) &= \operatorname{trace}(M(\varphi_e, y_e)) = 0. \end{aligned} \tag{2.7}$$

By the theory of planar dynamical system (see [23]), for an equilibrium point of a planar dynamical system, if $J < 0$, then this equilibrium point is a saddle point; it is a center point if $J > 0$ and $p = 0$; if $J = 0$ and the Poincaré index of the equilibrium point is 0, then it is a cusp.

2.1. The Focusing Branch

The traveling wave system corresponding to the focusing branch of (1.8) is

$$\begin{aligned} \frac{d\varphi}{d\xi} &= 2\varphi^2 y, \\ \frac{dy}{d\xi} &= g + c\varphi - \varphi^3 - 2\varphi y^2. \end{aligned} \tag{2.8}$$

If the wave speed $c < 0$, then system (2.8) has only one equilibrium point, which is a center point ($g \neq 0$) or a degenerate center point ($g = 0$).

If the wave speed $c > 0$, then, by using the first integral value and properties of equilibrium points, we can obtain three bifurcation curves of system (2.8): they are $g_1(c) = (2c/9)\sqrt{3c}$, $g_2(c) = 0$, and $g_3(c) = (-2c/9)\sqrt{3c}$. Obviously, these three curves have no intersection point and $g_3(c) < g_2(c) < g_1(c)$ for arbitrary constant $c > 0$. They partition the (c, g) parameter plane into four regions. Using the bifurcation method of vector fields (e.g., [23–25]), we have the following result which describes the properties of the equilibrium points of system (2.8).

Proposition 2.1. *For given arbitrary constant $c > 0$, one has the following.*

(1) *If $g > g_1(c)$ or $g < g_3(c)$, then system* (2.8) *has one equilibrium point. It is a center point.*

(2) *If $g = g_1(c)$ or $g = g_3(c)$, then system* (2.8) *has two equilibrium points. One is a cusp, and the other is a center point.*

(3) *If $g_2(c) < g < g_1(c)$ or $g_3(c) < g < g_2(c)$, then system* (2.8) *has three equilibrium points. One is a saddle point, and the other two are all center points.*

(4) *If $g = g_2(c)$, then system* (2.8) *has three equilibrium points. One is a degenerate center point, and the other two are all center points.*

The phase portraits of system (2.8) in each region and on the bifurcation curves for the wave speed $c < 0$ and $c > 0$ are shown in Figures 1 and 2, respectively.

2.2. The Defocusing Branch

The traveling wave system corresponding to the defocusing branch of (1.8) is

$$\begin{aligned} \frac{d\varphi}{d\xi} &= 2\varphi^2 y, \\ \frac{dy}{d\xi} &= g + c\varphi + \varphi^3 - 2\varphi y^2. \end{aligned} \tag{2.9}$$

If the wave speed $c > 0$, then one can easily see that system (2.9) has only one equilibrium point, which is a saddle point ($g \neq 0$) or a degenerate center point ($g = 0$).

If the wave speed $c < 0$, then, by using the first integral value (2.4) and properties of equilibrium points, we can obtain three bifurcation curves of system (2.9) for $c < 0$: they are $g_4(c) = (-2c/9)\sqrt{-3c}$, $g_5(c) = 0$, and $g_6(c) = (2c/9)\sqrt{-3c}$. Obviously, the three curves have no intersection point and $g_6(c) < g_5(c) < g_4(c)$ for arbitrary $c < 0$. They partition the (c, g) parameter plane into four regions. Again using the bifurcation method of vector fields (e.g., [23–25]), we have the following result which describes the properties of the equilibrium points of system (2.9) for the wave speed $c < 0$.

Proposition 2.2. *For arbitrary constant $c < 0$, one has the following.*

(1) *If $g > g_4(c)$ or $g < g_6(c)$, then system* (2.9) *has one equilibrium point. It is a saddle point.*

(2) *If $g = g_4(c)$ or $g = g_6(c)$, then system* (2.9) *has two equilibrium points. One is a cusp, and the other is a saddle point.*

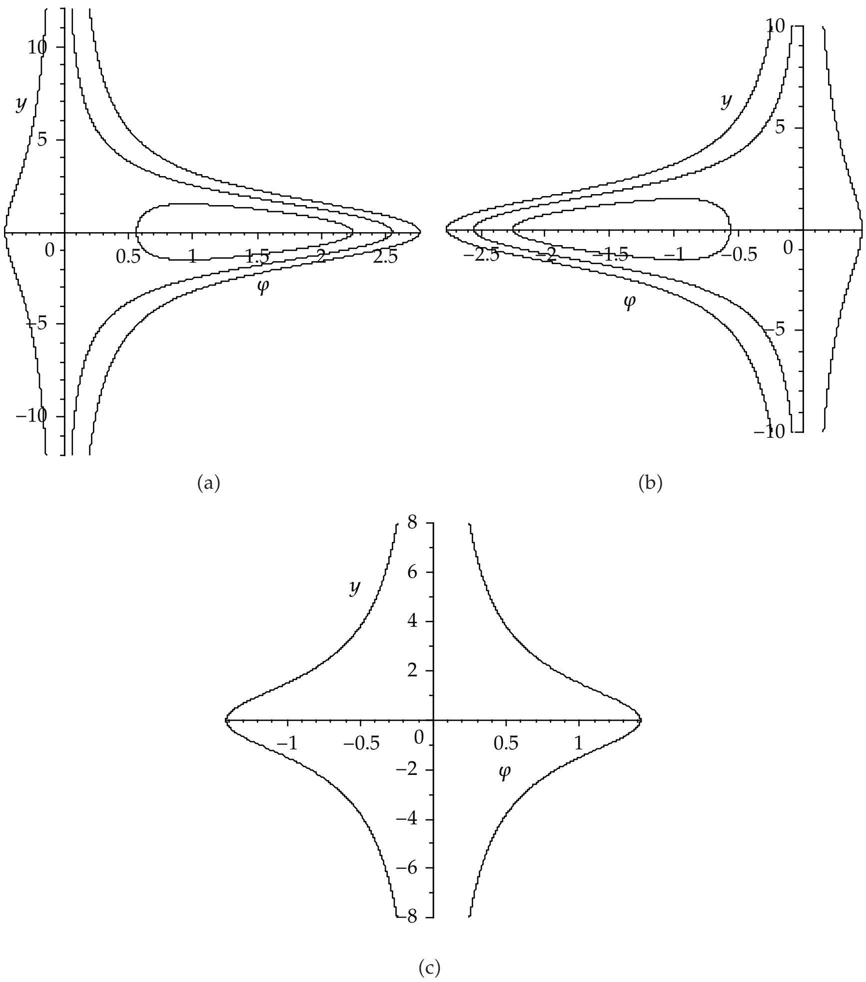


Figure 1: The phase portraits of system (2.8) for the wave speed $c < 0$. (a) $g > 0$; (b) $g < 0$; (c) $g = 0$.

(3) *If* $g_5(c) < g < g_4(c)$ *or* $g_6(c) < g < g_5(c)$, *then system* (2.9) *has three equilibrium points. One is a center point, and the other two are all saddle points.*

(4) *If* $g = g_5(c)$, *then system* (2.9) *has three equilibrium points. One is a degenerate center point, and the other two are all saddle points.*

The phase portraits of system (2.9) in each region and on the bifurcation curves for the wave speed $c > 0$ and $c < 0$ are given in Figures 3 and 4, respectively.

3. Solitary-Wave Solutions for (1.8)

Usually, a solitary-wave solution for (1.8) corresponds to a homoclinic orbit of system (2.8). From Figure 2, we can see that, when the parameter g satisfies $g_2(c) < g < g_1(c)$ and $g_3(c) < g < g_2(c)$ for $c > 0$, system (2.8) has homoclinic orbits. So the focusing branch of (1.8) has solitary-wave solutions accordingly.

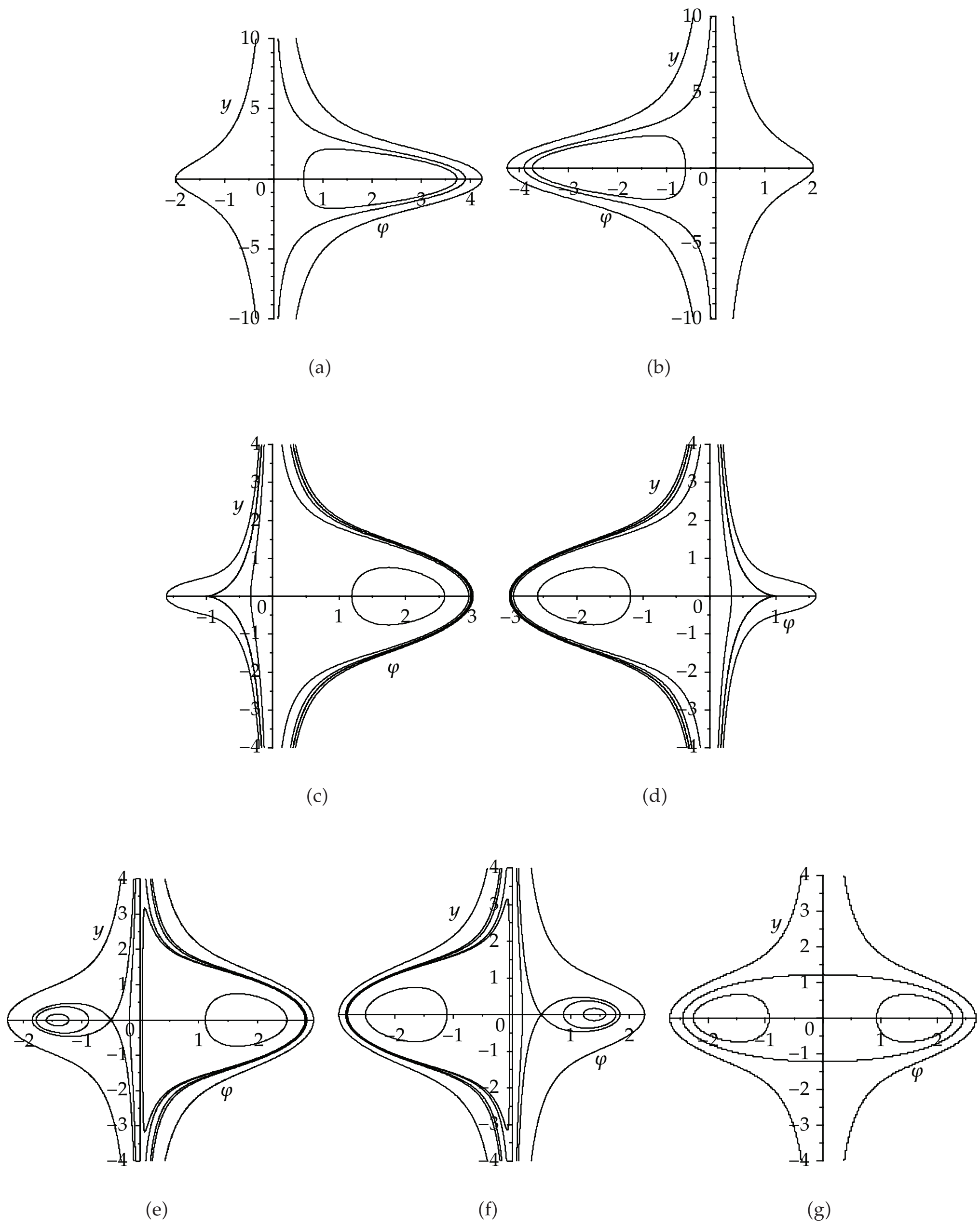


Figure 2: The phase portraits of system (2.8) for the wave speed $c > 0$. (a) $g > g_1(c)$; (b) $g < g_3(c)$; (c) $g = g_1(c)$; (d) $g = g_3(c)$; (e) $g_2(c) < g < g_1(c)$; (f) $g_3(c) < g < g_2(c)$; (g) $g = g_2(c)$.

Now, take $g = (11c/72)\sqrt{3c}$ for $c > 0$, which satisfies $g_2(c) < g < g_1(c)$. Then we can figure out three equilibrium points of system (2.8). Denote them as $(\varphi_{1-}, 0)$, $(\varphi_2, 0)$, and $(\varphi_{1+}, 0)$, where $\varphi_{1\pm} = (\sqrt{3c} \pm 3\sqrt{15c})/12$ and $\varphi_2 = -(\sqrt{3c}/6)$. $(\varphi_{1-}, 0)$ and $(\varphi_{1+}, 0)$ are two center points, while $(\varphi_2, 0)$ is a saddle point. There is a homoclinic orbit connecting with the

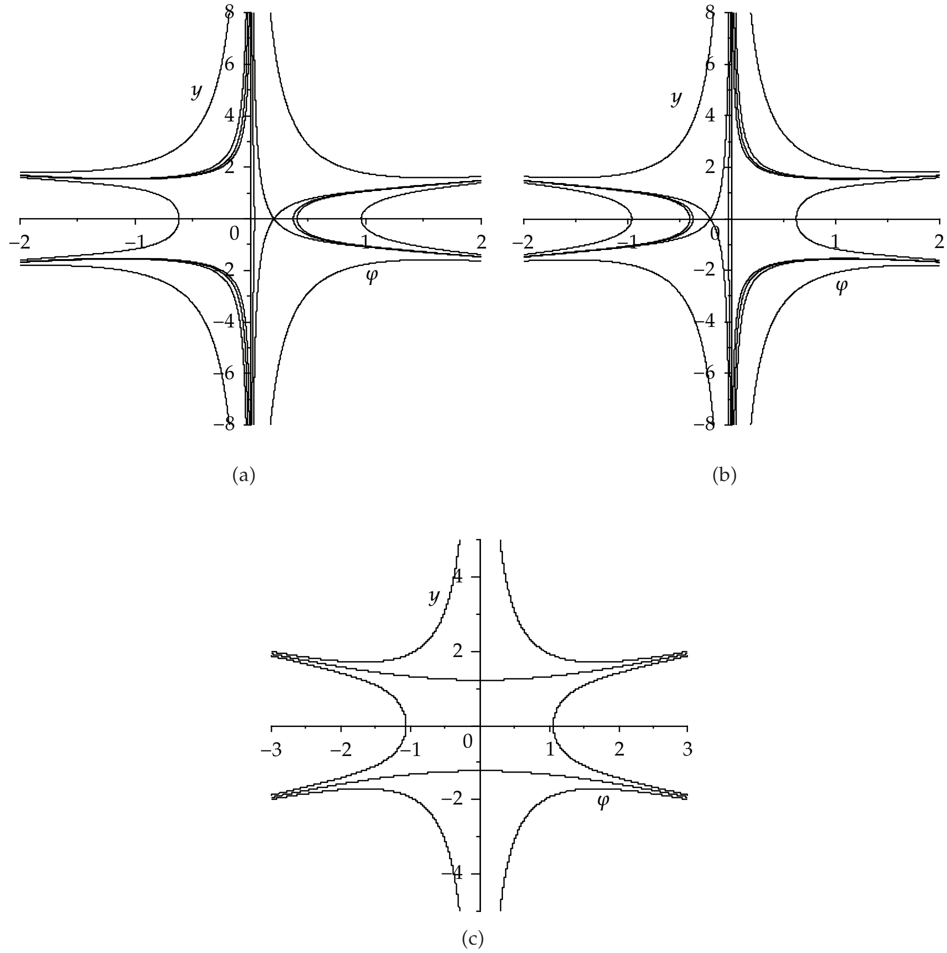


Figure 3: The phase portraits of system (2.9) for the wave speed $c > 0$. (a) $g > 0$; (b) $g < 0$; (c) $g = 0$.

saddle point $(\varphi_2, 0)$, which lies on the left side of the singular line $\varphi = 0$ (see Figure 2(e)). This homoclinic orbit is determined by $H_+(\varphi, y) = H_+(\varphi_2, 0)$ and can be expressed as

$$y = \pm \frac{(\varphi - \varphi_2)\sqrt{(\varphi - \varphi_2^-)(\varphi_2^+ - \varphi)}}{2\varphi} \quad \text{for } \varphi_2^- \le \varphi \le \varphi_2 < 0, \tag{3.1}$$

where

$$\varphi_2^{\pm} = \frac{\sqrt{3c} \pm \sqrt{66c}}{6}. \tag{3.2}$$

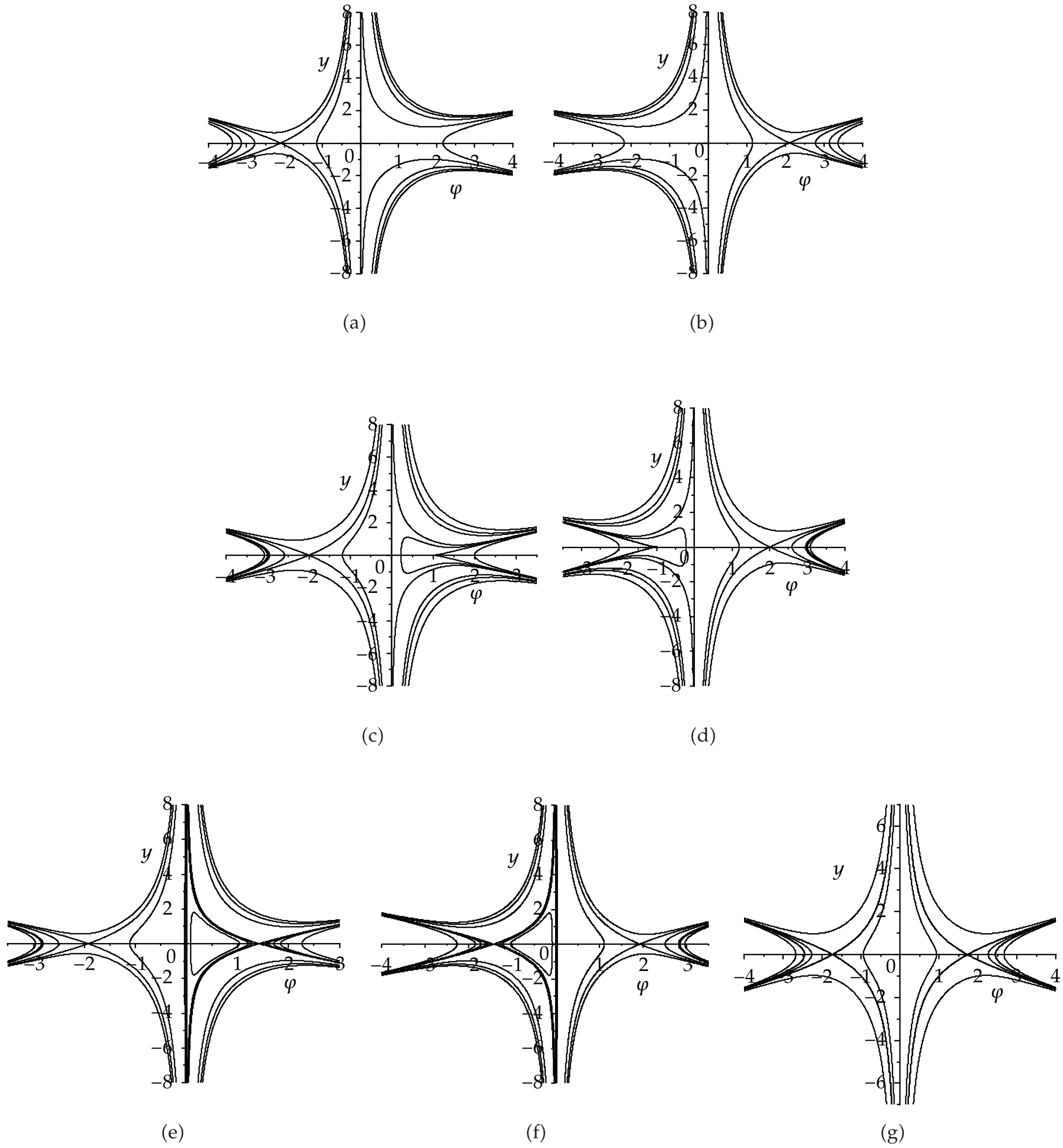
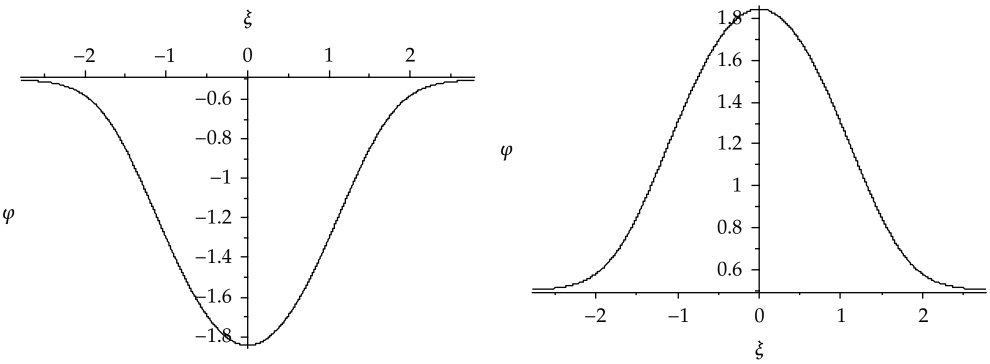


Figure 4: The phase portraits of system (2.9) for the wave speed $c < 0$. (a) $g > g_4(c)$; (b) $g < g_6(c)$; (c) $g = g_4(c)$; (d) $g = g_6(c)$; (e) $g_5(c) < g < g_4(c)$; (f) $g_6(c) < g < g_5(c)$; (g) $g = g_4(c)$.

Substituting (3.1) into the first equation of system (2.3) and integrating along the corresponding homoclinic orbit, we have

$$\int_{\varphi_2^-}^{\varphi} \frac{s}{(s-\varphi_2)\sqrt{(s-\varphi_2^-)(\varphi_2^+-s)}}ds = \frac{1}{2}|\xi|, \quad \varphi_2^- \le \varphi \le \varphi_2. \tag{3.3}$$



(a) (b)

Figure 5: Profiles of solitary-wave solutions for the focusing branch of (1.8). (a) $c = 3,\ g = -1.375,\ h = -0.328125$; (b) $c = 3,\ g = 1.375,\ h = -0.328125$.

It follows from (3.3) that

$$\begin{aligned} &\arcsin(\alpha_1(\varphi)) - \arcsin(\alpha_1(\varphi_2^-)) - \frac{2(\varphi_2^- + \varphi_2^+)}{\sqrt{(3\varphi_2^- + \varphi_2^+)(\varphi_2^- + 3\varphi_2^+)}} \\ &\quad \times(\arctan(\beta_1(\varphi)) - \arctan(\beta_1(\varphi_2^-))) = \frac{1}{2}|\xi|, \quad \varphi_2^- \le \varphi \le \varphi_2, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \alpha_1(\varphi) &= \frac{2\varphi - \varphi_2^- - \varphi_2^+}{\varphi_2^+ - \varphi_2^-}, \\ \beta_1(\varphi) &= \frac{(\varphi_2^-)^2 - (\varphi_2^+)^2 + 2(\varphi_2^- - \varphi_2^+)\varphi + 4(\varphi_2^- + \varphi_2^+)\sqrt{(\varphi - \varphi_2^-)(\varphi_2^+ - \varphi)}}{(-2\varphi + \varphi_2^- + \varphi_2^+)\sqrt{(3\varphi_2^- + \varphi_2^+)(\varphi_2^- + 3\varphi_2^+)}}. \end{aligned} \tag{3.5}$$

Equation (3.4) is the implicit expression of a valley-shaped solitary-wave solution for the focusing branch of (1.8). In physics, it can be called dark soliton (see [26]). A typical such solution is shown in Figure 5(a).

Similarly, if taking $g = (-11c/72)\sqrt{3c}$ for $c > 0$, which satisfies $g_3(c) < g < g_2(c)$, then we can obtain a hump-shaped solitary-wave solution for the focusing branch of (1.8) given by

$$\begin{aligned} &\arcsin(\alpha_2(-\varphi_2^-)) - \arcsin(\alpha_2(\varphi)) + \frac{2(\varphi_2^- + \varphi_2^+)}{\sqrt{(3\varphi_2^- + \varphi_2^+)(\varphi_2^- + 3\varphi_2^+)}} \\ &\quad \times(\arctan(\beta_2(-\varphi_2^-)) - \arctan(\beta_2(\varphi))) = \frac{1}{2}|\xi|, \quad -\varphi_2 \le \varphi \le -\varphi_2^-, \end{aligned} \tag{3.6}$$

where

$$\alpha_2(\varphi) = \frac{2\varphi + \varphi_2^- + \varphi_2^+}{\varphi_2^+ - \varphi_2^-},$$
$$\beta_2(\varphi) = \frac{(\varphi_2^-)^2 - (\varphi_2^+)^2 - 2(\varphi_2^- - \varphi_2^+)\varphi + 4(\varphi_2^- + \varphi_2^+)\sqrt{(\varphi + \varphi_2^+)(-\varphi_2^- - \varphi)}}{(2\varphi + \varphi_2^- + \varphi_2^+)\sqrt{(3\varphi_2^- + \varphi_2^+)(\varphi_2^- + 3\varphi_2^+)}}, \tag{3.7}$$

and $\varphi_2^\pm$ are as in (3.2).

In physics, (3.6) can be called bright soliton (see [26]). A typical such solution is shown in Figure 5(b).

For the defocusing branch of (1.8), one can see from Figures 3 and 4 that system (2.9) has no homoclinic orbits. So the defocusing branch of (1.8) has no solitary-wave solutions of traveling-wave type accordingly.

4. Periodic Solutions for (1.8)

Usually, a periodic solution for (1.8) corresponds to a periodic orbit of system (2.8). From Figures 1 and 2, we can see that system (2.8) has periodic orbits except for the case $c < 0$, $g = 0$. So the focusing branch of (1.8) has periodic solutions accordingly. In the following we take some parametric conditions to obtain the exact expressions of some periodic solutions.

Firstly, take $g = (11c/72)\sqrt{3c}$ for $c > 0$, which satisfies $g_2(c) < g < g_1(c)$. Then $H_+(\varphi, y) = H_+(\varphi_{1-}, 0)$ determines a periodic orbit of system (2.8) on the right-hand side of the singular line $\varphi = 0$ (see Figure 2(e)). This periodic orbit can be expressed as

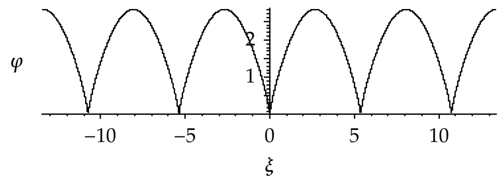
$$y = \pm\frac{(\varphi - \varphi_{1-})\sqrt{(\varphi - \varphi_3^-)(\varphi_3^+ - \varphi)}}{2\varphi} \quad \text{for } 0 < \varphi_3^- \le \varphi \le \varphi_3^+, \tag{4.1}$$

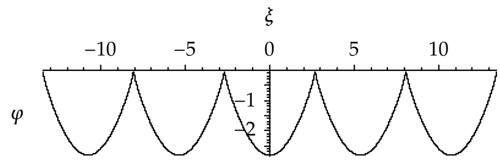
where

$$\varphi_3^\pm = \frac{1}{12}\left(-\sqrt{3c} + 3\sqrt{15c} \pm 2\sqrt{3c + 9\sqrt{5c}}\right). \tag{4.2}$$

Substituting (4.1) into the first equation of system (2.3) and integrating along the corresponding periodic orbit, we have

$$\int_{\varphi_3^-}^{\varphi} \frac{2s}{(s - \varphi_{1-})\sqrt{(s - \varphi_3^-)(\varphi_3^+ - s)}}\,ds = |\xi|, \quad \varphi_3^- \le \varphi \le \varphi_3^+. \tag{4.3}$$





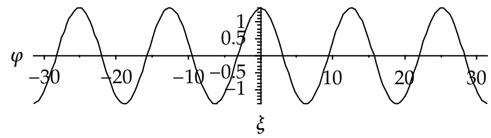


Figure 6: Profiles of periodic solutions for the focusing branch of (1.8). (a) $c = 3,\ g = 1.375,\ h = 0.0557$; (b) $c = 3,\ g = -1.375,\ h = 0.0557$; (c) $c = 3,\ g = 0,\ h = 0$.

It follows from (4.3) that

$$
\begin{aligned}
&2\left(\arcsin(\alpha_3(\varphi)) - \arcsin(\alpha_3(\varphi_3^-))\right) - \frac{4(\varphi_3^- + \varphi_3^+)}{\sqrt{(3\varphi_3^- + \varphi_3^+)(\varphi_3^- + 3\varphi_3^+)}} \\
&\quad \times \left(\arctan(\beta_3(\varphi)) - \arctan(\beta_3(\varphi_3^-))\right) = \pm\xi + nT_1, \quad n = 0, \pm1, \pm2, \ldots,
\end{aligned}
\tag{4.4}
$$

where

$$
\begin{aligned}
\alpha_3(\varphi) &= \frac{2\varphi - \varphi_3^- - \varphi_3^+}{\varphi_3^+ - \varphi_3^-}, \\
\beta_3(\varphi) &= \frac{(\varphi_3^-)^2 - (\varphi_3^+)^2 + 2(\varphi_3^- - \varphi_3^+)\varphi + 4(\varphi_3^- + \varphi_3^+)\sqrt{(\varphi - \varphi_3^-)(\varphi_3^+ - \varphi)}}{(-2\varphi + \varphi_3^- + \varphi_3^+)\sqrt{(3\varphi_3^- + \varphi_3^+)(\varphi_3^- + 3\varphi_3^+)}},
\end{aligned}
\tag{4.5}
$$

$$
T_1 = 4\pi\left(1 - \frac{\varphi_3^- + \varphi_3^+}{\sqrt{(3\varphi_3^- + \varphi_3^+)(\varphi_3^- + 3\varphi_3^+)}}\right), \tag{4.6}
$$

and $\varphi_3^\pm$ is as (4.2).

Equation (4.4) is the implicit expression of a periodic solution with the period T_1 for the focusing branch of (1.8). A typical such solution is shown in Figure 6(a).

Next take $g = (-11c/72)\sqrt{3c}$ for $c > 0$, which satisfies $g_3(c) < g < g_2(c)$. Similar to the above discussion, we can obtain another periodic solution with the same period T_1 for the focusing branch of (1.8) as follows:

$$\begin{aligned}
&2\left(\arcsin(\alpha_4(\varphi)) - \arcsin\left(\alpha_4\left(-\varphi_3^-\right)\right)\right) + \frac{2\left(\varphi_3^- + \varphi_3^+\right)}{\sqrt{\left(3\varphi_3^- + \varphi_3^+\right)\left(\varphi_3^- + 3\varphi_3^+\right)}} \\
&\quad\times \left(\arctan(\beta_4(\varphi)) - \arctan\left(\beta_4\left(-\varphi_3^+\right)\right)\right) = \pm\xi + nT_1, \quad n = 0, \pm 1, \pm 2, \ldots,
\end{aligned} \tag{4.7}$$

where

$$\alpha_4(\varphi) = \frac{2\varphi + \varphi_3^+ + \varphi_3^-}{\varphi_3^+ - \varphi_3^-}, \tag{4.8}$$

$$\beta_4(\varphi) = \frac{\left(\varphi_3^-\right)^2 - \left(\varphi_3^+\right)^2 - 2\left(\varphi_3^- - \varphi_3^+\right)\varphi + 4\left(\varphi_3^- + \varphi_3^+\right)\sqrt{\left(\varphi + \varphi_3^+\right)\left(-\varphi_3^- - \varphi\right)}}{\left(2\varphi - \varphi_3^- - \varphi_3^+\right)\sqrt{\left(3\varphi_3^- + \varphi_3^+\right)\left(\varphi_3^- + 3\varphi_3^+\right)}}, \tag{4.9}$$

$\varphi_3^{\pm}$ and T_1 are as (4.2) and (4.6), respectively. A typical such solution is shown in Figure 6(b).

Last, take $g = g_2(c)$. Then $H_+(\varphi, y) = 0$ determines a periodic orbit of system (2.9) (see Figure 2(g)). It can be expressed as

$$y = \pm\frac{1}{2}\sqrt{2c - \varphi^2} \quad \text{for} \ -\sqrt{2c} \le \varphi \le \sqrt{2c}. \tag{4.10}$$

Substituting (4.10) into the first equation of system (2.3) and integrating along the corresponding periodic orbit, we have

$$\int_{\varphi}^{\sqrt{2c}} \frac{1}{\sqrt{2c - s^2}} ds = \frac{1}{2}|\xi|, \quad -\sqrt{2c} \le \varphi \le \sqrt{2c}. \tag{4.11}$$

It follows from (4.11) that

$$\varphi = \sqrt{2c}\cos\left(\frac{1}{2}\xi\right). \tag{4.12}$$

Equation (4.12) is obviously a periodic solution with the period $T = 4\pi$ for the focusing branch of (1.8). A typical such solution is shown in Figure 6(c).

For the defocusing branch of (1.8), we can see from Figures 4(e) and 4(f) that, in the parametric regions $c < 0$, $g_5(c) < g < g_4(c)$ and $c < 0$, $g_6(c) < g < g_5(c)$, system (2.9) has periodic orbits. So the defocusing branch of (1.8) has periodic solutions accordingly.

Now, take $g = (-11c/72)\sqrt{(-3c)}$ for $c < 0$, which satisfies $g_5(c) < g < g_4(c)$. Then system (2.9) has three equilibrium points $(\varphi_1^*, 0)$, $(\varphi_2^*, 0)$, and $(\varphi_3^*, 0)(\varphi_1^* < 0 < \varphi_2^* < \varphi_3^*)$. For fixed h $(0 < h < H_-(\varphi_2^*, 0))$, the curve $H_+(\varphi, y) = h$ intersects the φ-axis with four points a, b, d, and f, which satisfy that $d < 0 < f < a < b$. This curve determines a periodic orbit of system (2.9) (see Figure 4(e)). It can be expressed as

$$y = \pm\frac{\sqrt{(\varphi-d)(\varphi-f)(a-\varphi)(b-\varphi)}}{2\varphi} \quad \text{for } f \le \varphi \le a. \tag{4.13}$$

Substituting (4.13) into the first equation of system (2.3) and integrating along the corresponding periodic orbit, we have

$$\int_f^{\varphi} \frac{2s}{\sqrt{(s-d)(s-f)(a-s)(b-s)}}\,ds = |\xi|, \quad f \le \varphi \le a. \tag{4.14}$$

It follows from (4.14) that

$$\begin{aligned} &\frac{-4}{\sqrt{(a-d)(b-f)}}\Big(dF(\arcsin(\gamma(\varphi)) \mid m) + (f-d)\Pi\Big(l; sn^{-1}(\gamma(\varphi) \mid m) \mid m\Big)\Big) \\ &\quad = \pm\xi + nT, \quad n = 0, \pm1, \pm2, \ldots, \end{aligned} \tag{4.15}$$

where

$$\gamma(\varphi) = \sqrt{\frac{(a-d)(\varphi-f)}{(a-f)(\varphi-d)}}, \tag{4.16}$$

$$\begin{aligned} m &= \frac{(b-d)(a-f)}{(b-f)(a-d)}, \\ l &= \frac{a-f}{a-d}, \\ T_2 &= \frac{8}{\sqrt{(a-d)(b-f)}}\big(dK(m) + (f-d)\Pi(n \mid m)\big), \end{aligned} \tag{4.17}$$

and $F(\cdot|\cdot)$ is the elliptic integral of the first kind, and the notation is as used in Section 17.2.6 of [27]. $\Pi(\cdot;\ \cdot|\cdot)$ is the elliptic integral of the third kind, and the notation is as used in Section 17.2.15 of [27]. $sn^{-1}(\cdot|\cdot)$ is the inverse of the Jacobian elliptic function $sn(\cdot|\cdot)$, where the notation is as used in Chapter 16 of [27]. $K(\cdot)$ is the complete elliptic integral of the first kind, and $\Pi(\cdot|\cdot)$ is the complete elliptic integral of the third kind.

Equation (4.15) is a periodic solution with the period T_2 for the defocusing branch of (1.8). A typical such solution is shown in Figure 7(a).

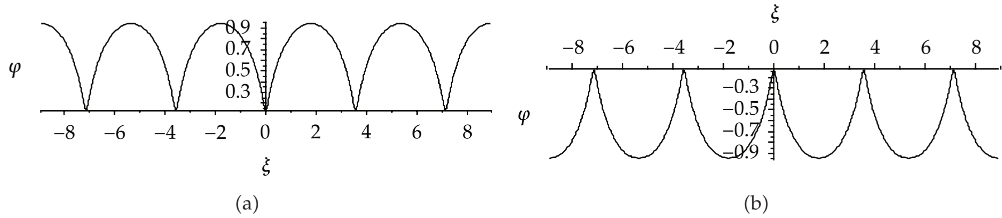


Figure 7: Profiles of periodic solutions for the defocusing branch of (1.8). (a) $c = -3$, $g = 1.375$, $h = 0.16$; (b) $c = 3$, $g = -1.375$, $h = 0.16$.

Next take $g = (11c/72)\sqrt{3c}$ for $c > 0$, which satisfies $g_6(c) < g < g_5(c)$. For the same fixed h as above, we can obtain another periodic solution with the same period T_2 as follows:

$$\begin{aligned}&\frac{4}{\sqrt{(a-d)(b-f)}}\Big(dF(\arcsin(\gamma(\varphi)) \mid m) + (f-d)\Pi\Big(n; sn^{-1}(\gamma(\varphi) \mid m) \mid m\Big)\Big)\\&\quad = \pm\xi + nT_2, \quad n = 0, \pm 1, \pm 2, \ldots,\end{aligned} \tag{4.18}$$

where $\gamma(\varphi)$, m, l, and T_2 are as (4.16)-(4.9), respectively.

A typical such solution is shown in Figure 7(b).

5. Conclusion

In summary, by using the bifurcation method, we investigate two new types of bounded traveling-wave solutions for a variant of the $K(3, 2)$ equation. For the focusing branch, we obtain its solitary-wave solutions and some periodic solutions. For the defocusing branch, we demonstrate the nonexistence of solitary-wave solutions of traveling type and obtain some periodic solutions in terms of elliptic integrals. Our work supplements the results in [5, 11] and may help people to know deeply the described physical process and possible applications of this nonlinear dispersive equation.

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