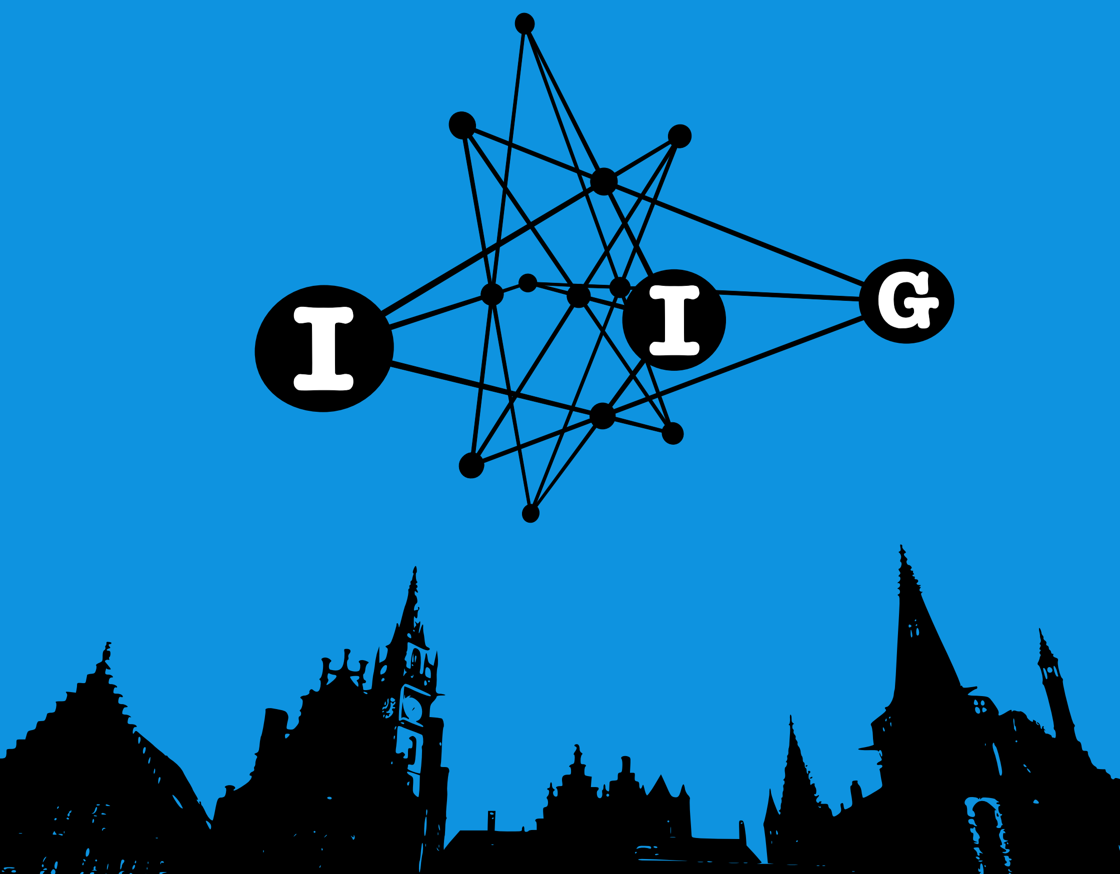
Innovations in Incidence Geometry

Algebraic, Topological and Combinatorial



A new family of 2-dimensional Laguerre planes that admit $\mathrm{PSL}_2(\mathbb{R}) \times \mathbb{R}$ as a group of automorphisms

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A new family of 2-dimensional Laguerre planes that admit $\mathrm{PSL}_2(\mathbb{R}) \times \mathbb{R}$ as a group of automorphisms

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We construct a new family of 2-dimensional Laguerre planes that differ from the classical real Laguerre plane only in the circles that meet a given circle in precisely two points. These planes share many properties with but are non-isomorphic to certain semiclassical Laguerre planes pasted along a circle in that they admit 4-dimensional groups of automorphisms that contain $\mathrm{PSL}_2(\mathbb{R})$ and are of Kleinewillinghöfer type I.G.1.

1. Introduction

A 2-dimensional Laguerre plane is an incidence structure on the cylinder $Z = \mathbb{S}^1 \times \mathbb{R}$ determined by a collection of graphs of continuous functions $\mathbb{S}^1 \to \mathbb{R}$; see the following section for a definition of and facts about Laguerre planes. The collection of all automorphisms of a 2-dimensional Laguerre plane is a Lie group of dimension at most 7. All 2-dimensional Laguerre planes whose automorphism groups have dimension at least 5 are known; see [Löwen and Pfüller 1987, Theorem 1]. The classification of 2-dimensional Laguerre planes whose automorphism groups are 4-dimensional is almost complete except when the automorphism group fixes no parallel class but is not transitive on the point set. Examples of 2-dimensional Laguerre planes which exhibit such groups of automorphisms can be found in [Steinke 1987; Löwen and Steinke 2007].

In this paper we contribute to the investigation of 2-dimensional Laguerre planes whose automorphism groups are 4-dimensional, and construct a new family of such planes that admit a group of automorphisms isomorphic to $\mathrm{PSL}_2(\mathbb{R}) \times \mathbb{R}$. It shares many circles with the classical real Laguerre plane (and the semiclassical Laguerre planes of group dimension 4 from [Steinke 1987]; see Section 5 for a brief description). Its full automorphism group fixes a distinguished circle and is 3-transitive on it. Derived projective planes at points on the distinguished circle are dual to

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the derived projective planes at corresponding points in the semiclassical Laguerre planes of group dimension 4 pasted along a circle. However, our Laguerre planes are not semiclassical. The new planes and the semiclassical Laguerre planes of group dimension 4 will play a prominent role in the classification of 2-dimensional Laguerre planes of group dimension 4 whose automorphism groups fix a circle.

Section 2 summarizes facts about 2-dimensional Laguerre planes. Section 3 describes the new family of 2-dimensional Laguerre planes. Section 4 proves that these are indeed 2-dimensional Laguerre planes. In the last section we determine isomorphism classes, full automorphism groups and Kleinewillinghöfer types of our planes. We further show that the Laguerre planes are not semiclassical and investigate the associated compact 3-dimensional generalized quadrangles.

2. Laguerre planes

A *Laguerre plane* $\mathscr{L} = (P, \mathscr{C}, \|)$ is an incidence structure consisting of a point set P, a circle set $\mathscr{C}$ and an equivalence relation $\|$ (parallelism) defined on the point set such that

- three mutually nonparallel points can be joined by a unique circle,
- given a point p on a circle C and a point q not parallel to p, there is a unique circle that contains both points and *touches C geometrically* at p, that is, intersects C only in p or coincides with C,
- each parallel class meets each circle in a unique point (parallel projection), and
- there are four points not on a circle and there is a circle that contains at least three points (richness);

compare [Groh 1968; 1969b].

In this paper we are only concerned with Laguerre planes whose common point set is the cylinder $Z = \mathbb{S}^1 \times \mathbb{R}$ (where the 1-sphere $\mathbb{S}^1$ usually is represented as $\mathbb{R} \cup \{\infty\}$), whose circles are graphs of functions $\mathbb{S}^1 \to \mathbb{R}$ and whose parallel classes of points are the generators of the cylinder. Notice that for an incidence structure on the cylinder with circles and parallel classes like this, the axioms of parallel projection and richness are automatically satisfied. In particular, we are interested in 2*-dimensional or flat Laguerre planes* on the cylinder. These Laguerre planes are characterized by the fact that all their circles are graphs of continuous functions from $\mathbb{S}^1$ to $\mathbb{R}$; cf. [Groh 1968; 1969b]. The axiom of joining and touching show that the collection of circle-describing functions of a 2-dimensional Laguerre plane solves the Hermite interpolation problem of rank 3.

The *classical real Laguerre plane* $\mathscr{L}_{\mathrm{cl}}$ is obtained as the geometry of nontrivial plane sections of a cylinder in $\mathbb{R}^3$ with an ellipse in $\mathbb{R}^2$ as base, or equivalently, as the geometry of nontrivial plane sections of an elliptic cone, in real projective three-space, with its vertex removed. The parallel classes are the generators of the cylinder or cone. By replacing the ellipse in this construction by arbitrary ovals in $\mathbb{R}^2$ (i.e., convex, differentiable simply closed curves), we also obtain 2-dimensional Laguerre planes. These are the so-called *2-dimensional ovoidal Laguerre planes*.

Circles of a 2-dimensional Laguerre plane, as described above, are homeomorphic to the unit circle $\mathbb{S}^1$. When the circle set is topologized by the Hausdorff metric with respect to a metric that induces the topology of the point set, then the plane is *topological* in the sense that the operations of joining three points by a circle, intersecting two circles, and touching are continuous with respect to the induced topologies on their respective domains of definition. For more information on topological Laguerre planes we refer to [Groh 1968; 1969b].

For each point p of $\mathscr{L}$ we form the incidence structure $\mathscr{A}_p = (A_p, \mathscr{L}_p)$ whose point set A_p consists of all points of $\mathscr{L}$ that are not parallel to p and whose line set $\mathscr{L}_p$ consists of all restrictions to A_p of circles of $\mathscr{L}$ passing through p and of all parallel classes not passing through p. It readily follows that $\mathscr{A}_p$ is an affine plane. We call $\mathscr{A}_p$ the *derived affine plane at* p. In fact, the axioms of a Laguerre plane are equivalent to each derived incidence structure being an affine plane. For example, each derived affine plane of an ovoidal Laguerre plane is Desarguesian.

Each derived affine plane $\mathscr{A}_p$ of a 2-dimensional Laguerre plane is even a topological affine plane and extends to a 2-dimensional compact projective plane $\mathscr{P}_p$, which we call the *derived projective plane at* p; see [Salzmann 1967], [Salzmann et al. 1995] or [Polster and Steinke 2001, Chapter 2] for more information on topological 2-dimensional compact projective planes. Circles not passing through the distinguished point p induce closed ovals in $\mathscr{P}_p$ by removing the point parallel to p and adding in $\mathscr{P}_p$ the point ω at infinity of lines that come from parallel classes of $\mathscr{L}$. The line at infinity of $\mathscr{P}_p$ (relative to $\mathscr{A}_p$) is a tangent to this oval. According to [Polster and Steinke 1994, Proposition 2] there is a unique topology extending the natural topology of the affine plane such that one obtains a 2-dimensional Laguerre plane.

An *automorphism* of a Laguerre plane is a permutation of the point set such that parallel classes are mapped to parallel classes and circles are mapped to circles. Every automorphism of a 2-dimensional Laguerre plane is continuous and thus a homeomorphism of Z. The collection of all automorphisms of a 2-dimensional Laguerre plane $\mathscr{L}$ forms a group with respect to composition, the automorphism group Γ of $\mathscr{L}$. This group is a Lie group of dimension at most 7 with respect to the compact-open topology; see [Steinke 1986]. We call the dimension of Γ the *group dimension* of $\mathscr{L}$.

The maximum dimension is attained precisely in the classical real Laguerre plane. In fact, group dimension 6 does not occur. Furthermore, 2-dimensional

Laguerre planes of group dimension 5 must be special ovoidal Laguerre planes; see [Löwen and Pfüller 1987, Theorem 1].

We investigated 2-dimensional Laguerre planes admitting 4-dimensional point-transitive groups of automorphisms in [Steinke 1993]. It was shown that such planes must be classical. The 2-dimensional Laguerre planes admitting 4-dimensional groups of automorphisms that fix a parallel class were completely determined in [Steinke 2015]. These planes are covered by the families of Laguerre planes of generalized shear type, Laguerre planes of translation type and Laguerre planes of shift type; see [Steinke 2015, Corollary 3.5] for details and references to the various types of Laguerre planes.

The remaining open case is when a closed connected 4-dimensional group of automorphisms fixes a circle but no parallel class. Then the automorphism group contains a subgroup isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ or its universal (simply connected) covering group $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$; compare [Steinke 1990, Theorem B]. Examples of 2-dimensional Laguerre planes which admit such groups of automorphisms can be found in [Steinke 1987; Löwen and Steinke 2007].

The collection of all automorphisms of $\mathscr{L}$ that fix each parallel class is a closed normal subgroup of Γ, called the *kernel* of $\mathscr{L}$. The kernel of a 2-dimensional Laguerre plane has dimension at most 4. Furthermore, a kernel of dimension 4 characterizes the ovoidal Laguerre planes among 2-dimensional Laguerre planes, that is, a 2-dimensional Laguerre plane $\mathscr{L}$ is ovoidal if and only if its kernel is 4-dimensional; see [Groh 1969a].

3. The new models of 2-dimensional Laguerre planes

We construct a class of 2-dimensional Laguerre planes that admit a 4-dimensional group of automorphisms fixing a circle. This class depends on a real positive parameter k. To begin with, it is readily seen that a multiplicative homeomorphism of $\mathbb{R}$ is of the form

$$h_k(x) = x\,|x|^{k-1},$$

where $k > 0$. Furthermore, h_k is differentiable for all $x \neq 0$ and has derivative $h'_k(x) = k|x|^{k-1}$. We use h_k also when $k \leq 0$. Of course, in this case, h_k is not defined at 0, but still multiplicative on $\mathbb{R} \setminus \{0\}$.

Description of the models $\mathscr{L}_k$. We consider the following incidence structures $\mathscr{L}_k$, where $0 < k < 2$. For each such k we let $k' = 2 - k$, so that $0 < k' < 2$. The point set is the cylinder $Z = (\mathbb{R} \cup \{\infty\}) \times \mathbb{R}$. Two points $(x_1, y_1), (x_2, y_2) \in Z$ are parallel if and only if $x_1 = x_2$, and parallel classes in $\mathscr{L}$ are the sets $\{u\} \times \mathbb{R}$ for $u \in \mathbb{R} \cup \{\infty\}$. Circles are of one of the following forms:

- $C_{a,b,c} = \{(x, y) \in \mathbb{R}^2 \mid y = ax^2 + bx + c\} \cup \{(\infty, a)\}$, where $b^2 \leq 4ac$; these are circles of the classical real Laguerre plane and precisely those that do not meet $C_0 = C_{0,0,0}$ in exactly two points;

- $D_{0,b,c} = \{(x, y) \in \mathbb{R}^2 \mid y = bh_k(x - c)\} \cup \{(\infty, 0)\}$, where $b > 0$;
- $D_{0,b,c} = \{(x, y) \in \mathbb{R}^2 \mid y = bh_{k'}(x - c)\} \cup \{(\infty, 0)\}$, where $b < 0$; and
- $D_{a,b,c} = \{(x, y) \in \mathbb{R}^2 \mid y = ah_k(x-b)h_{k'}(x-c)\} \cup \{(\infty, a)\}$, where $a(b-c) > 0$.

We call a circle of the form $C_{a,b,c}$ a *C-circle* and a circle of the form $D_{a,b,c}$ a *D-circle*; see Figure 1 for the shape of D-circles. Note that unless $k = k' = 1$, the graph of $D_{a,b,c}$ for $a \neq 0$ has a vertical tangent line at one of its points on the x-axis.

The set of all circles (C- and D-circles as above) is denoted by $\mathscr{C}_k$. Then $\mathscr{L}_k = (Z, \mathscr{C}_k, \|)$ is the incidence structure with point set Z, set of circles $\mathscr{C}_k$ and equivalence relation $\|$ on Z as given above.

Sometimes it will be more convenient to use a slightly different parametrization of C-circles. We define

$$C'_{a,b,c} = \{(x, y) \in \mathbb{R}^2 \mid y = a(x - b)^2 + c\} \cup \{(\infty, a)\},$$

where $ac \geq 0$, $a \neq 0$. This uniquely covers all C-circles except the circles $C_{0,0,c}$ where $c \in \mathbb{R}$, the circles that touch C_0 at $(\infty, 0)$. (Extending the definition of $C'_{a,b,c}$ to include $a = 0$ would yield multiple descriptions of the latter touching circles.) Note that when the parameter c tends to b in a D-circle $D_{a,b,c}$ one just obtains $C'_{a,b,0}$. This is due to the fact that $h_k(x)h_{k'}(x) = x^2$ for all $x \in \mathbb{R}$.

We show in the next section that $\mathscr{L}_k$ is indeed a Laguerre plane. C-circles are the same as in the classical real Laguerre plane $\mathscr{L}_{\mathrm{cl}}$, which is obviously isomorphic to $\mathscr{L}_1$. So only the circles meeting C_0 in precisely two points have been replaced in $\mathscr{L}_{\mathrm{cl}}$ by the D-circles.

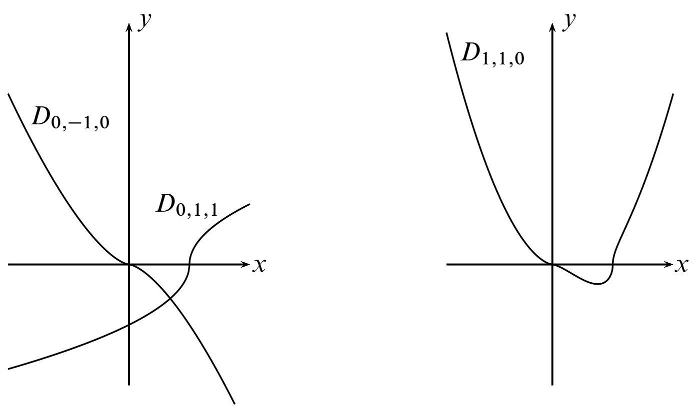

Figure 1. The circles $D_{0,1,1}$, $D_{0,-1,0}$ and $D_{1,1,0}$ in $\mathscr{L}_{1/2}$.

In [Polster and Steinke 1995, Proposition 6] it was proved that the set of circles that meet a given circle in exactly two points can be exchanged by a corresponding set of circles from a different 2-dimensional Laguerre plane so long as the two planes share the circles that touch the distinguished circle. However, the planes $\mathcal{L}_k$ are not examples for this construction as we do not have a 2-dimensional Laguerre plane (other than $\mathcal{L}_k$) that contains all D-circles of $\mathcal{L}_k$ and all circles touching C_0.

It is readily verified that the permutations

$$\gamma_{a,b,c,d,r} : (x, y) \mapsto \begin{cases} \left(\dfrac{ax+b}{cx+d}, \dfrac{r(ad-bc)y}{(cx+d)^2}\right) & \text{if } x \in \mathbb{R},\ cx+d \neq 0, \\ \left(\infty, \dfrac{rc^2y}{ad-bc}\right) & \text{if } c \neq 0,\ x = -\dfrac{d}{c}, \\ \left(\dfrac{a}{c}, \dfrac{r(ad-bc)y}{c^2}\right) & \text{if } c \neq 0,\ x = \infty, \\ \left(\infty, \dfrac{rdy}{a}\right) & \text{if } c = 0,\ x = \infty \end{cases}$$

of the cylinder Z, where $a, b, c, d, r \in \mathbb{R}$, $ad - bc \neq 0$ and $r > 0$, are automorphisms of $\mathcal{L}_k$ (i.e., take circles to circles). Indeed, since each $\gamma_{a,b,c,d,r}$ is an automorphism of the classical real Laguerre plane, a C-circle is taken to a C-circle. For D-circles it suffices to consider the generating transformations $\gamma_{1,t,0,1,1}$ with $t \in \mathbb{R}$, $\gamma_{s,0,0,1,1}$ with $s \neq 0$, $\gamma_{1,0,0,1,r}$ with $r > 0$, and $\gamma_{0,-1,1,0,1}$. For example, in case $a \neq 0$ one finds that

$$\begin{aligned} \gamma_{1,t,0,1,1}(D_{a,b,c}) &= D_{a,b+t,c+t}, \\ \gamma_{s,0,0,1,1}(D_{a,b,c}) &= D_{a/s,bs,cs}, \\ \gamma_{1,0,0,1,r}(D_{a,b,c}) &= D_{ra,b,c}, \\ \gamma_{0,-1,1,0,1}(D_{a,b,c}) &= D_{ah_k(b)h_{k'}(c),-1/b,-1/c}, \end{aligned}$$

where also $bc \neq 0$ in the last case.

Let

$$\Gamma = \{\gamma_{a,b,c,d,r} \mid a, b, c, d, r \in \mathbb{R}, ad - bc \neq 0, r > 0\}.$$

Then Γ is a group of automorphisms of $\mathcal{L}_k$. Obviously,

$$\Sigma = \{\gamma_{a,b,c,d,1} \mid a, b, c, d \in \mathbb{R}, ad - bc \neq 0\}$$

is a subgroup of Γ. Furthermore, Σ is isomorphic to $\mathrm{PGL}_2(\mathbb{R})$ and Γ is isomorphic to $\mathrm{PGL}_2(\mathbb{R}) \times \mathbb{R}$. The action of Σ on C_0 is equivalent to the standard action of $\mathrm{PGL}_2(\mathbb{R})$ on $\mathbb{R} \cup \{\infty\}$. In particular, Σ is sharply 3-transitive on C_0. The subgroup $\{\gamma_{1,0,0,1,r} \mid r > 0\}$ of Γ comprises the kernel of Γ. Moreover, Σ and Γ have two orbits on Z, namely C_0 and $Z \setminus C_0$. On the circle space, Γ has four orbits: $\{C_0\}$, $\{C_{a,b,c} \mid b^2 = 4ac\}$, $\{C_{a,b,c} \mid b^2 < 4ac\}$, and the set of all D-circles.

We equip the cylinder Z with the natural Euclidean topology of $\mathbb{S}^1 \times \mathbb{R}$. On $\mathbb{R}^2 \subset Z$, the usual Euclidean topology is induced. In our representation, a neighbourhood of a point (∞, a) consists of all (x, y) such that either $x = \infty$ and y is sufficiently close to a, or $x \in \mathbb{R}$ is of sufficiently large modulus and y/x^2 is sufficiently close to a. It is readily checked that in this topology, circles of $\mathscr{L}_k$ are closed subsets of Z (in fact, are homeomorphic to $\mathbb{S}^1$) and that all transformations in Γ are continuous.

4. The geometric axioms

Since Γ has precisely two orbits on Z it suffices to verify that the derived incidence structures at $(\infty, 0)$ and $(\infty, 1)$ are affine planes in order to show that $\mathscr{L}_k$ is a Laguerre plane.

We first deal with the derived incidence structure $\mathscr{A}_0$ at $(\infty, 0)$. The point set of $\mathscr{A}_0$ is $\mathbb{R}^2$ and nonvertical lines come from $C_{0,0,c}$, $c \in \mathbb{R}$, and $D_{0,b,c}$, $b \neq 0$. Hence, nonvertical and nonhorizontal lines are given by

$$\begin{aligned} y &= bh_k(x-c), \quad b > 0, \quad \text{and} \\ y &= bh_{k'}(x-c), \quad b < 0. \end{aligned}$$

Lemma 4.1. *The derived incidence structure* $\mathscr{A}_0$ *of* $\mathscr{L}_k$ *at* $(\infty, 0)$ *is an affine plane. Furthermore*, $\mathscr{A}_0$ *is Desarguesian if and only if* $k = 1$.

Proof. We make the coordinate transformation

$$\mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (h_k^{-1}(y), x).$$

Then the nonvertical and nonhorizontal lines in the new (u, v)-coordinates become

$$\begin{aligned} v &= Bu + c, && \text{where } B > 0 \quad (B = 1/h_k^{-1}(b)), \quad \text{and} \\ v &= Bh_{k/k'}(u) + c, && \text{where } B < 0 \quad (B = 1/h_{k'}^{-1}(b)). \end{aligned}$$

One also has the vertical and horizontal lines $u = c$ and $v = c$, respectively. Since $h_{k/k'}$ is an orientation preserving homeomorphism of $\mathbb{R}$, one sees that A_0 is an affine plane; compare [Steinke 1985, Proposition 2.1]. In the notation of [Steinke 1985] the plane described above in the (u, v)-coordinates is the affine plane $\mathscr{A}_{h_{k/k'},\mathrm{id}}$. It is a plane over a Cartesian field — see [Salzmann et al. 1995, Section 37] — the affine part of the plane $\mathscr{P}_{1,k/k',1}$ in the notation of [Salzmann et al. 1995, 37.3]. Such a plane is Desarguesian if and only if $k/k' = 1$; compare [Steinke 1985, Corollary 3.2] or [Salzmann et al. 1995, 37.3 and Theorem 37.4]. However, $k = k'$ implies $k = 1$. □

Before we consider the derived incidence structure $\mathscr{A}_1$ at $(\infty, 1)$, we deal with the intersection of two general distinct circles in $\mathscr{L}_k$.

Lemma 4.2. *Two distinct circles in $\mathscr{L}_k$ have at most two points in common.*

Proof. The statement is obviously true for two distinct C-circles. Consider a C-circle and D-circle. By applying the group Γ we may assume that the D-circle is $D_{0,m,t}$, where $m \neq 0$ and the C-circle is $C_{1,0,c}$, where $c \geq 0$. The x-coordinates of points of intersection are found from the equation

$$x^2 + c = mh_k(x-t). \tag{1}$$

We apply $h_k^{-1} = h_{1/k}$ on both sides to obtain

$$h_{1/k}(x^2 + c) = Ax + B,$$

where $A = h_k^{-1}(m) \neq 0$ and $B = -h_k^{-1}(m)t$. However, the function $f_c : x \mapsto h_{1/k}(x^2 + c)$ on the left-hand side is strictly convex. This can be seen from the second derivative of f_c given by $f_c''(x) = \frac{2}{k}(x^2+c)^{\frac{1}{k}-2}\big(\frac{k'}{k}x^2 + c\big)$, which is positive except possibly when $x = 0$. Hence, (1) has at most two solutions and thus $C_{1,0,c}$ and $D_{0,m,t}$ have at most two points of intersection.

In the last case we consider two D-circles. By applying the group Γ and Lemma 4.1 we may assume that one circle is $D_{0,m,t}$, where $m \neq 0$ and the other circle is $D_{1,1,0}$. We first assume that $m > 0$. Then x-coordinates of points of intersection are found from the equation

$$h_k(x-1)h_{k'}(x) = mh_k(x-t). \tag{2}$$

We apply h_k^{-1} on both sides to obtain

$$(x-1)h_{k'/k}(x) = Ax + B,$$

where $A = h_k^{-1}(m) > 0$ and $B = -h_k^{-1}(m)t$. The function $f_+ : x \mapsto (x-1)h_l(x)$, where $l = k'/k$, on the left-hand side of the above equation has derivative

$$f_+'(x) = h_l(x) + l(x-1)|x|^{l-1} = ((l+1)x - l)|x|^{l-1}$$

and second derivative

$$\begin{aligned} f_+''(x) &= (l+1)|x|^{l-1} + (l-1)((l+1)x-l)h_{l-2}(x) \\ &= h_{l-2}(x)((l+1)x + (l-1)((l+1)x-l)) \\ &= lh_{l-2}(x)((l+1)x - l + 1). \end{aligned}$$

Hence f_+ is strictly decreasing on $(-\infty, x_{\min})$, where $x_{\min} = l/(l+1) > 0$, strictly increasing on $(x_{\min}, +\infty)$ and has an absolute minimum at $x_{\min}$. Furthermore, f_+ is strictly convex on the interval $(x_{\min}, +\infty)$; compare the diagram on the left in Figure 2. Since the restriction of f_+ to $(x_{\min}, +\infty)$ (the increasing branch of the graph of f_+) is convex, a Euclidean line of positive slope can meet the increasing branch in at most two points and the decreasing branch (the graph of

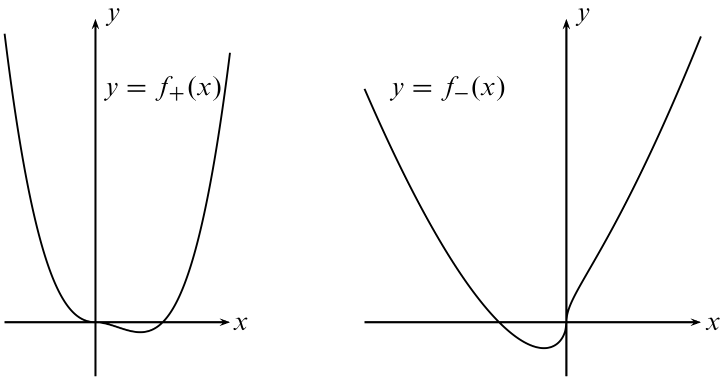

Figure 2. The graphs of $f_+(x) = (x-1)h_2(x)$ and $f_-(x) = (x+1)h_{1/2}(x)$.

the restriction of f_+ to $(-\infty, x_{\min})$) in at most one point. If such a line meets the increasing branch in two points, then because $\lim_{x\to+\infty} f_+(x)/x = +\infty$ the point $(x_{\min}, f_+(x_{\min}))$ lies above this line, so that the line cannot meet the graph of f_+ in any more points. In any case, we see that a Euclidean line of positive slope intersects the graph of f_+ at most twice. This shows that (2) has at most two solutions and thus that $D_{0,m,t}$, where $m > 0$, and $D_{1,1,0}$ have at most two points in common.

When $m < 0$ one similarly considers the equation

$$h_k(x-1)h_{k'}(x) = mh_{k'}(x-t), \tag{3}$$

from which one obtains

$$(x+1)h_{k/k'}(x) = Ax + B,$$

where $A = h_{k'}^{-1}(m) < 0$ and $B = h_{k'}^{-1}(m)(1-t)$. A similar straightforward analysis of the function $f_- : x \mapsto (x+1)h_l(x)$ on the left-hand side, where now $l = k/k'$, shows that the decreasing branch is strictly convex, so that a Euclidean line of negative slope intersects the graph of f_- at most twice; compare the diagram on the right in Figure 2. Therefore, (3) has at most two solutions and thus $D_{0,m,t}$, where $m < 0$, and $D_{1,1,0}$ have at most two points in common. This shows that in any case two distinct D-circles intersect in at most two points. □

We are now ready to deal with the derived incidence structure $\mathcal{A}_1$ at $(\infty, 1)$. The point set of $\mathcal{A}_1$ is $\mathbb{R}^2$ and nonvertical lines are induced by $C_{1,b,c}$, where $b^2 \leq 4c$, and $D_{1,b,c}$, where $b > c$. Explicitly, these lines are given by

$$y = x^2 + bx + c, \qquad b^2 \leq 4c, \quad \text{and}$$
$$y = h_k(x-b)h_{k'}(x-c), \qquad b > c.$$

We call them C-lines and D-lines, respectively, as they come from C- and D-circles of $\mathscr{L}_k$.

Lemma 4.3. *The derived incidence structure $\mathscr{A}_1$ of $\mathscr{L}_k$ at $(\infty, 1)$ is a linear space.*

Proof. By Lemma 4.2 we know that two different lines in $\mathscr{A}_1$ intersect in at most one point. This yields the uniqueness of a line joining two points if it exists.

Let $p_i = (x_i, y_i)$, $i = 1, 2$, be two distinct points of $\mathscr{A}_1$. If $x_1 = x_2$, then the vertical line $x = x_1$ (coming from a parallel class of the Laguerre plane) joins the two points. We therefore assume that $x_1 \neq x_2$. By the transitivity properties of the stabilizer $\Gamma_{(\infty,1)}$ we may assume that without loss of generality $x_1 = 0$ and $x_2 = 1$. Finally, because $\mathscr{A}_0$ (and thus each $\mathscr{A}_{(u,0)}$ where $u \in \mathbb{R}$) is an affine plane by Lemma 4.1, we may further assume that $y_1, y_2 \neq 0$.

In case $2(y_1 + y_2) \geq (y_2 - y_1)^2 + 1$ there is a unique C-line through p_1 and p_2. Indeed, the Euclidean parabola given by $y = x^2 + (y_2 - y_1 - 1)x + y_1$ passes through the two points, and this is a line of $\mathscr{A}_1$ if and only if

$$0 \geq (y_2 - y_1 - 1)^2 - 4y_1 = (y_2 - y_1)^2 + 1 - 2(y_1 + y_2).$$

In this case, the two points cannot be on a D-line by Lemma 4.2.

So now assume that $2(y_1 + y_2) < (y_2 - y_1)^2 + 1$. We must show that p_1 and p_2 are on a D-line $D_{1,b,c}$. The two parameters $b > c$ satisfy the equations

$$y_1 = h_k(b) h_{k'}(c), \qquad y_2 = h_k(b-1) h_{k'}(c-1).$$

After application of $h_{k'}^{-1}$ on both sides we obtain

$$v_1 := h_{k'}^{-1}(y_1) = h_l(b)c, \tag{4}$$

$$v_2 := h_{k'}^{-1}(y_2) = h_l(b-1)(c-1), \tag{5}$$

where $l = k/k'$. Hence

$$g(b) := h_l(b) h_l(b-1) - v_1 h_l(b-1) + v_2 h_l(b) = 0.$$

First note that $g(b) = (h_l(b) - v_1)(h_l(b-1) + v_2) + v_1 v_2$. From this equation one sees that $\lim_{b \to \pm\infty} g(b) = +\infty$.

When $y_2 < 0$, then $g(1) = v_2 < 0$. Thus, by the intermediate value theorem, there is a $b > 1$ such that $g(b) = 0$. From (5) it follows that $c < 1$. Similarly, when $y_1 < 0 < y_2$, then $g(0) = v_1 < 0$ and $g(1) = v_2 > 0$ so that there is some b, $0 < b < 1$, such that $g(b) = 0$. From (4) it then follows that $c < 0$. Hence, in these two cases, $b > c$ and we have a D-line through p_1 and p_2.

We finally assume that $y_1, y_2 > 0$. We compute

$$\begin{aligned} v_i &= h_{1/k'}(y_i) = (y_i)^{1/k'} = (\sqrt{y_i})^{2/k'} \\ &= (\sqrt{y_i})^{(k+k')/k'} = (\sqrt{y_i})^{l+1} = \sqrt{y_i}\, h_l(\sqrt{y_i}), \end{aligned}$$

where $i = 1, 2$. Hence,

$$g(\sqrt{y_1}) = h_l(\sqrt{y_1})[h_l(\sqrt{y_1}-1)(1-\sqrt{y_1})+v_2]$$
$$= h_l(\sqrt{y_1})((\sqrt{y_2})^{l+1} - |\sqrt{y_1}-1|^{l+1}).$$

One similarly obtains that

$$g(-\sqrt{y_1}) = h_l(-\sqrt{y_1})[h_l(-\sqrt{y_1}-1)(1+\sqrt{y_1})+v_2]$$
$$= h_l(-\sqrt{y_1})(\sqrt{y_2}^{l+1} - |\sqrt{y_1}+1|^{l+1}).$$

The inequality $2(y_1 + y_2) < (y_2 - y_1)^2 + 1$ can be rewritten as

$$(y_2 - y_1 - 1)^2 - 4y_1 > 0$$

from which we see that either $y_2 > (\sqrt{y_1}+1)^2$ or $y_2 < (\sqrt{y_1}-1)^2$. In the former case, $g(-\sqrt{y_1}) < 0$, and in the latter case, $g(\sqrt{y_1}) < 0$. Since $g(0) = v_1 > 0$ and $\lim_{b\to+\infty} g(b) = +\infty$, there must be a $b \in (-\sqrt{y_1}, 0)$ or $b \in (\sqrt{y_1}, +\infty)$, respectively, such that $g(b) = 0$. Finally, because

$$y_1 = (\sqrt{y_1})^2 = h_k(\sqrt{y_1})h_{k'}(\sqrt{y_1}),$$

one obtains from (4) that

$$h_{k'}(c/\sqrt{y_1}) = h_k(\sqrt{y_1}/b).$$

Hence $c < -\sqrt{y_1} < b$ when $b < 0$, and $0 < c < \sqrt{y_1} < b$ when $b > \sqrt{y_1}$. Hence, in any case, $b > c$ and we have a D-line through p_1 and p_2.

This proves that any two distinct points of $\mathscr{A}_1$ can be joined by a unique line, that is, $\mathscr{A}_1$ is a linear space as claimed. □

Lemma 4.4. *The derived incidence structure $\mathscr{A}_1$ of $\mathscr{L}_k$ at $(\infty, 1)$ is an affine plane.*

Proof. By Lemma 4.3 it only remains to show that through each point there is a unique line that is parallel to a given line. This is clearly the case for vertical lines.

For a nonvertical line we define its slope s by

$$s(C_{1,b,c}) = -b \quad \text{and} \quad s(D_{1,b,c}) = kb + k'c.$$

We claim that two nonvertical lines of $\mathscr{A}_1$ are parallel if and only if they have the same slope. To see this and where the definition of s comes from, we apply the coordinate transformation induced by $\gamma_{0,1,-1,0,1}$, that is, $(x, y) \mapsto (-1/x, y/x^2)$ for x real and nonzero, suitably extended to Z. Then $C_{1,b,c}$ and $D_{1,b,c}$ are described by $v = cu^2 - bu + 1$ and $v = h_k(1 + bu)h_{k'}(1 + cu)$, respectively. Differentiation at $u = 0$ yields $-b$ and $kb + k'c$, that is, the slope of the corresponding line. Now, if the slopes of two nonvertical lines are different, then after the above coordinate transformation the resulting circles intersect transversally at $(0, 1)$. Hence these circles intersect in a second point in $Z \setminus \{0\} \times \mathbb{R}$. Therefore the original lines meet in a point of $\mathscr{A}_1$ and so are not parallel.

We now assume that two nonvertical lines of $\mathscr{A}_1$ have the same slope s. In case of two C-lines C_{1,b_1,c_1} and C_{1,b_2,c_2} this means that $b_1 = b_2 = -s$, and the two lines are clearly parallel.

A D-line of slope s is described by the function $f(c,x) = h_k(x-b)h_{k'}(x-c)$, where $b = (s-k'c)/k$. Differentiation with respect to c yields

$$\begin{aligned}\frac{\partial f(c,x)}{\partial c} &= k'|x-b|^{k-1}h_{k'}(x-c) - k'h_k(x-b)|x-c|^{k'-1}\\ &= k'|x-b|^{k-1}|x-c|^{k'-1}(b-c)\\ &= \frac{k'}{k'}|x-b|^{k-1}|x-c|^{k'-1}(s-2c).\end{aligned}$$

But $b > c$ if and only if $s-2c > 0$. Thus $\frac{\partial}{\partial c} f(c,x) > 0$, and $c \mapsto f(c,x)$ is strictly increasing on $\left(-\infty, \frac{s}{2}\right)$ for all $x \in \mathbb{R}$. It now follows that two D-lines D_{1,b_1,c_1} and D_{1,b_2,c_2} of the same slope $kb_1 + k'c_1 = kb_2 + k'c_2$ are parallel.

Note that $c < \frac{s}{2} < b$ for a D-line of slope s. Furthermore, as c tends to $\frac{s}{2}$, the D-line $D_{1,b,c}$, $kb + k'c = s$, converges to $D_{1,s/2,s/2} = C'_{1,s/2,0}$. In particular, $C'_{1,s/2,0}$ and $D_{1,b,c}$ are parallel, and $D_{1,b,c}$ lies below $C'_{1,s/2,0}$. Finally, a C-line $C'_{1,s/2,c}$, $c \geq 0$, of slope s lies above or coincides with $C'_{1,s/2,0}$. Hence, a C-line and a D-line of slope s are parallel.

Finally, given a point $p = (x_0, y_0)$ and a line of slope s there is a unique line of slope through p. Indeed, when $y_0 \geq \left(x_0 - \frac{s}{2}\right)^2$, the parallel through p must be a C-line $C'_{1,s/2,c}$, and c is uniquely determined by

$$c = y_0 - \left(x_0 - \frac{s}{2}\right)^2 \geq 0.$$

When $y_0 < \left(x_0 - \frac{s}{2}\right)^2$, the parallel through p must be a D-line $D_{1,b,c}$, $kb + k'c = s$. Since

$$\lim_{c\to-\infty}(y_0 - h_k(x-b)h_{k'}(x-c)) = +\infty \quad \text{and}$$
$$\lim_{c\to s/2}(y_0 - h_k(x-b)h_{k'}(x-c)) = y_0 - \left(x_0 - \tfrac{s}{2}\right)^2 < 0,$$

there is a c such that $D_{1,b,c}$ passes through p.

This shows that $\mathscr{A}_1$ satisfies the parallel axiom and that $\mathscr{A}_1$ is an affine plane. □

The following is a direct consequence of Lemmata 4.1 and 4.4, together with the transitivity properties of Γ and the fact that each derived plane of an ovoidal Laguerre plane is Desarguesian.

Corollary 4.5. *The incidence structure $\mathscr{L}_k$ where $0 < k < 2$ is a Laguerre plane. Furthermore, $\mathscr{L}_k$ is ovoidal if and only if $k = 1$. In this case the Laguerre plane is classical.*

Since in the topology on Z circles of $\mathscr{L}_k$ are closed Jordan curves on Z we have the following; compare [Groh 1969b, 3.10].

Theorem 4.6. *Each $\mathscr{L}_k$ where $0 < k < 2$ is a 2-dimensional Laguerre plane.*

5. Isomorphisms and other properties

Lemma 5.1. *Let ψ be an isomorphism from $\mathscr{L}_k$ to $\mathscr{L}_l$. If $k \neq 1$, then ψ takes C_0 in $\mathscr{L}_k$ to C_0 in $\mathscr{L}_l$.*

Proof. Suppose that $\psi(C_0) \neq C_0$. Then $\psi \Gamma_k \psi^{-1}$ is a group of automorphisms of $\mathscr{L}_l$ that has $Z \setminus \psi(C_0)$ and $\psi(C_0)$ as orbits. However, Γ_l has $Z \setminus C_0$ and C_0 as orbits, and it follows that the automorphism group of $\mathscr{L}_l$ must be transitive on Z. Hence, $\mathscr{L}_l$ is classical by [Steinke 1993]. But then $\mathscr{L}_k$ is also classical and $k = 1$ — a contradiction to our assumption. This shows that $\psi(C_0) = C_0$. □

Proposition 5.2. *Two Laguerre planes $\mathscr{L}_k$ and $\mathscr{L}_l$ are isomorphic if and only if $l \in \{k, k'\}$. In particular, each plane is isomorphic to exactly one plane $\mathscr{L}_k$, where $0 < k \leq 1$.*

Proof. Note that $\mu : Z \to Z$ given by $\mu(x, y) = (x, -y)$ is an automorphism of the classical real Laguerre plane; circles $C_{a,b,c}$ are taken to $C_{-a,-b,-c}$. In fact, μ induces an isomorphism from $\mathscr{L}_k$ onto $\mathscr{L}_{k'}$: one has

$$\mu\big(D^{(k)}_{a,b,c}\big) = D^{(k')}_{a,c,b}$$

when $a \neq 0$, and

$$\mu\big(D^{(k)}_{0,b,c}\big) = D^{(k')}_{0,-b,c}.$$

(Here the superscripts refer to the Laguerre planes the circles are from.) This verifies that $\mathscr{L}_k$ and $\mathscr{L}_{k'}$ are isomorphic.

Assume that $\mathscr{L}_k$ and $\mathscr{L}_l$ are isomorphic. If $k = 1$, then $\mathscr{L}_k$ is classical and so is $\mathscr{L}_l$. Thus $l = 1$, and $l = k = k'$.

Suppose that $k \neq 1$. Let ψ be an isomorphism from $\mathscr{L}_k$ to $\mathscr{L}_l$. By Lemma 5.1 we know that $\psi(C_0) = C_0$. Using the transitivity properties of Γ_l on $\mathscr{L}_l$ we may further assume that ψ takes $(\infty, 0)$, $(\infty, 1)$ and $(0, 0)$ in $\mathscr{L}_k$ to the corresponding points with the same coordinates in $\mathscr{L}_l$. Hence the derived affine planes $\mathscr{A}^{(k)}_0$ and $\mathscr{A}^{(l)}_0$ are isomorphic. As seen in the proof of Lemma 4.1 the projective extensions of $\mathscr{A}^{(k)}_0$ and $\mathscr{A}^{(l)}_0$ are isomorphic to cartesian planes $\mathscr{P}_{1,k/k',1}$ and $\mathscr{P}_{1,l/l',1}$, respectively. By [Salzmann et al. 1995, Theorem 37.3 and Proposition 37.6] we thus have that $l/l' = k/k'$ or $l/l' = k'/k$. In the former case $\frac{l}{2-l} = \frac{k}{2-k}$ so that $l = k$. In the latter case we similarly obtain $l = k'$. □

Proposition 5.3. *The group* Γ *from Section 3 is the full automorphism group of* $\mathscr{L}_k$ *when* $k \neq 1$.

Proof. Let $k \neq 1$ and let α be an automorphism of $\mathscr{L}_k$. By Lemma 5.1 the automorphism leaves C_0 invariant. The 3-transitivity of Σ on C_0 implies that there is a $\sigma \in \Sigma$ such that $\sigma\alpha$ fixes each of $(\infty, 0)$, $(0, 0)$ and $(1, 0)$. By using an automorphism $\gamma_{1,0,0,1,r}$, $r > 0$, we can furthermore achieve that $\gamma = \gamma_{1,0,0,1,r}\sigma\alpha$ fixes $(\infty, 0)$, $(0, 0)$, $(1, 0)$ and takes $(\infty, 1)$ to $(\infty, 1)$ or $(\infty, -1)$. In the former case γ fixes each of the four points $(\infty, 0)$, $(0, 0)$, $(1, 0)$, $(\infty, 1)$. Hence γ must be the identity by [Steinke 1990, Lemma 2.10] or [Salzmann 1967, Corollary 3.6]. Thus $\alpha = \gamma_{1,0,0,1,1/r}\sigma^{-1} \in \Gamma$.

In the latter case there is an $s > 0$ such that $\gamma_{1,0,0,1,s}\gamma^2$ fixes each of the four points $(\infty, 0)$, $(0, 0)$, $(1, 0)$, $(\infty, 1)$. Therefore $\gamma_{1,0,0,1,s}\gamma^2 = \mathrm{id}$ so that γ^2 acts trivially on C_0. But γ fixes the three points $(\infty, 0)$, $(0, 0)$, $(1, 0)$ on C_0 and thus is an orientation preserving homeomorphism of C_0. This implies that γ is the identity on C_0.

Given a point p in the open upper half-cylinder Z^+ not parallel to $(\infty, 0)$, there are exactly two circles through $(\infty, 1)$ and p that touch C_0. Indeed, if $p = (x_0, y_0)$, where $y_0 > 0$, the two touching circles are $C'_{1,x_0+\sqrt{y_0},0}$ and $C'_{1,x_0-\sqrt{y_0},0}$. Since the point of touching on C_0 is fixed by γ and because $\gamma(\infty, 1) = (\infty, -1)$, these circles are taken to $C'_{-1,x_0+\sqrt{y_0},0}$ and $C'_{-1,x_0-\sqrt{y_0},0}$, respectively. Therefore, $\gamma(x_0, y_0) = (x_0, -y_0)$.

Now, the trace of a D-circle $D_{0,1,0}$ on Z^+ is taken by γ to the set

$$\{(x, -h_k(x)) \mid x > 0\},$$

which must be part of a D-circle through $(\infty, 0)$ and $(0, 0)$. Therefore, there must be an $m < 0$ such that $-h_k(x) = mh_{k'}(x)$ for all $x > 0$. When $x = 1$ we obtain $m = -1$. But then $x^k = x^{k'}$ for all $x > 0$, so that $k = k'$ — a contradiction to our assumption that $k \neq 1$. This shows that the latter case cannot occur, and we have $\alpha \in \Gamma$. □

Kleinewillinghöfer [1979; 1980] classified Laguerre planes with respect to *central automorphisms*, that is, automorphisms of the Laguerre plane such that at least one point is fixed and central collineations are induced in the derived projective plane at one of the fixed points. A subgroup of central automorphisms with the same "centre" and "axis" is said to be linearly transitive if the induced subgroup of central collineations of the derived projective plane is linearly transitive, that is, transitive on the points of each central line except the centre and its intersection with the axis. In [Polster and Steinke 2004], 2-dimensional Laguerre planes were considered and their so-called Kleinewillinghöfer types were investigated, that is, the Kleinewillinghöfer types with respect to the full automorphism group. The classification of those types that can occur in 2-dimensional Laguerre planes is almost complete except for two open cases; see [Steinke 2012] and the references to models of various types given there.

It turns out that the planes $\mathscr{L}_k$ constructed here are of type I.G.1 when $k \neq 1$, the same type as some semiclassical Laguerre planes pasted along a circle; see [Polster and Steinke 2004, Section 6] and below for a description of these semiclassical planes. This means that there is no circle for which the automorphism group of $\mathscr{L}_k$ is linearly transitive with respect to Laguerre homologies (type I, a Laguerre homology fixes a circle pointwise), that there is a circle C such that for each point p on C the group of Laguerre translations fixing p and the bundle of circles touching C at p is linearly transitive (type G, a Laguerre translation fixes a parallel class pointwise and induces a translation in a derived projective plane at one of the fixed points), and that there is no group of Laguerre homotheties that is linearly transitive (type 1, a Laguerre homothety fixes two nonparallel points and each circle through them). In type VII.K.13 all possible subgroups of central automorphisms with given centre and axis are linearly transitive. We refer to [Kleinewillinghöfer 1979] or [Polster and Steinke 2004] for a description of all types.

Proposition 5.4. *The Laguerre plane $\mathscr{L}_k$ is of Kleinewillinghöfer type* I.G.1 *when $k \neq 1$ and of type* VII.K.13 *when $k = 1$.*

Proof. When $k = 1$ we have the classical real Laguerre plane, which is of type VII.K.13; see [Polster and Steinke 2004, Corollaries 3.2 and 4.2,] and [Hartmann 1982, Satz 7]. Assume that $k \neq 1$. Then every automorphism of $\mathscr{L}_k$ fixes C_0, so that C_0 is the only possible axis of a Laguerre homology. Similarly, points on C_0 are the only possible centres of Laguerre homotheties, and Laguerre translations must be in direction of a tangent bundle to C_0. Hence, together with the 3-transitivity of Γ on C_0, only types I or II with respect to Laguerre homologies, types A or G with respect to Laguerre translations and types 1 or 6 with respect to Laguerre homotheties are possible as the types of $\mathscr{L}_k$. See [Kleinewillinghöfer 1979] or [Polster and Steinke 2004] for a full list of Kleinewillinghöfer types.

Now $\{\gamma_{1,t,0,1,1} \mid t \in \mathbb{R}\}$ is a linearly transitive group of Laguerre translations in direction of the tangent bundle to C_0 at $(\infty, 0)$. Conjugation by elements in Γ then shows that $\mathscr{L}_k$ has type G with respect to Laguerre translations. The automorphisms of $\mathscr{L}_k$ that fix each point of C_0 are $\gamma_{a,0,0,a,r}$, where $a \neq 0$, $r > 0$. However, the collection of these Laguerre homologies is not linearly transitive (because the open upper half-cylinder Z^+ is left invariant). Thus $\mathscr{L}_k$ has type I with respect to Laguerre homologies.

Similarly, the automorphisms of $\mathscr{L}_k$ that fix $(\infty, 0)$ and $(0, 0)$ are $\gamma_{a,0,0,d,r}$, where $ad \neq 0$, $r > 0$. Explicitly, these are the maps $(x, y) \mapsto (sx, rsy)$ extended to the parallel class at infinity, where $0 \neq s (= a/d)$, $r > 0$. A D-circle $D_{0,b,0}$

is taken to $D_{0,rb/|s|^k,0}$ when $b>0$ and $D_{0,rb/|s|^{k'},0}$ when $b<0$. However, a Laguerre homothety with centres $(\infty,0)$ and $(0,0)$ must fix each circle through the two centres, so that

$$r=|s|^k=|s|^{k'}$$

for all $s\neq 0$. This implies $k=k'$ — a contradiction to $k\neq 1$. This shows that $\mathscr{L}_k$ has type 1 with respect to Laguerre homotheties. □

In [Steinke 1987; 1988], semiclassical Laguerre planes were introduced. These are 2-dimensional Laguerre planes which are composed of two classical half-planes. By a half-plane we mean the closure of a connected component of the complement of two parallel classes or of a circle. Such a half-plane is called classical if, with its induced geometry, it is isomorphic to a half-plane of the same kind in the classical real Laguerre plane.

Some of the semiclassical planes also admit $\mathrm{PSL}_2(\mathbb{R})\times\mathbb{R}$ as a group of automorphisms and are of Kleinewillinghöfer type I.G.1. These are the planes $\mathscr{L}(h_m,\mathrm{id})$, where $m>0$, in the notation of [Steinke 1987]. They are obtained by pasting along a circle. According to [Steinke 1987, Theorem 4.8] in this case circles are of the form

$$K_{a,b,c}=\{(x,y)\in\mathbb{R}^2\mid y=ax^2+bx+c\}\cup\{(\infty,a)\},$$

where $a,b,c\in\mathbb{R}$, $b^2\le 4ac$ and

$$\begin{aligned}K_{a,b,c}=&\{(x,y)\in\mathbb{R}^2\mid y=ax^2+bx+c\ge 0\}\\&\cup\{(x,y)\in\mathbb{R}^2\mid y=(b^2-4ac)^{(m-1)/2}(ax^2+bx+c)\le 0\}\cup\{(\infty,\bar a)\},\end{aligned}$$

where $a,b,c\in\mathbb{R}$, $b^2>4ac$, $m>0$ and

$$\bar a=\begin{cases}a, & \text{if } a\ge 0,\\ (b^2-4ac)^{(m-1)/2}a, & \text{if } a<0.\end{cases}$$

(In case $m=1$ one just obtains the classical real Laguerre plane $\mathscr{L}_{\mathrm{cl}}$.)

These planes are semiclassical because the geometries and topologies on the closed upper half-cylinder $\overline{Z}_+=\mathbb{S}^1\times[0,+\infty)$ and the closed lower half-cylinder $\overline{Z}_-=\mathbb{S}^1\times(-\infty,0]$ are the same as on the corresponding subsets of the (topological) classical real Laguerre plane $\mathscr{L}_{\mathrm{cl}}$. The two classical geometries are pasted together along the circle $K_0:=K_{0,0,0}$.

Those permutations $\gamma_{a,b,c,d,r}$ of Z from Section 3 where $ad-bc=1$ and $r>0$ are in fact also automorphisms of $\mathscr{L}(h_m,\mathrm{id})$; see [Steinke 1987, 4.3 and Lemmata 4.4 and 4.5]. The collection of all these transformations is a group with respect to composition and is isomorphic to $\mathrm{PSL}_2(\mathbb{R})\times\mathbb{R}$.

Note that the circles that do not meet K_0 in precisely two points are the same as in $\mathscr{L}_{\mathrm{cl}}$ and thus as in our planes $\mathscr{L}_k$. However, our planes are not semiclassical except for the classical plane itself.

Proposition 5.5. *No Laguerre plane $\mathscr{L}_k$, $k \neq 1$, is semiclassical.*

Proof. By [Steinke 1988, Proposition 5.1] an automorphism of a semiclassical Laguerre plane pasted along two parallel classes leaves invariant the union of the two parallel classes along which the pasting occurs, provided the Laguerre plane is nonclassical. Since the automorphism group of $\mathscr{L}_k$ is transitive on the set of parallel classes, $\mathscr{L}_k$ cannot be isomorphic to a semiclassical Laguerre plane of this kind unless $k = 1$.

Regarding semiclassical Laguerre planes pasted along a circle, only the plane $\mathscr{L}(h_m, \mathrm{id})$, where $m > 0$, pasted along the circle K_0, needs to be considered because other planes have lower group dimension; see [Steinke 1987, Theorem 4.8]. One first notes as in the proof of Lemma 5.1 that an isomorphism ψ from $\mathscr{L}(h_m, \mathrm{id})$ to $\mathscr{L}_k$, where $k \neq 1$, must take K_0 as in the description above to C_0.

As in the proof of Proposition 5.3 we may without loss of generality assume that ψ fixes $(\infty, 0)$, $(0, 0)$, $(1, 0)$ and takes $(\infty, 1)$ to $(\infty, 1)$ or $(\infty, -1)$. In the former case ψ fixes each of the four points $(\infty, 0)$, $(0, 0)$, $(1, 0)$, $(\infty, 1)$. Hence the circles $K_{1,0,0}$ and $K_{1,-2,1}$, which pass through $(\infty, 1)$ and touch K_0 at $(0, 0)$ and $(1, 0)$, respectively, are taken to the corresponding circles in $\mathscr{L}_k$, that is, to $C_{1,0,0}$ and $C_{1,-2,1}$. Therefore the point $\left(\frac{1}{2}, \frac{1}{4}\right)$ in the intersection of $K_{1,0,0}$ and $K_{1,-2,1}$ is taken to the point $\left(\frac{1}{2}, \frac{1}{4}\right)$ in the intersection of $C_{1,0,0}$ and $C_{1,-2,1}$. Moreover, the circle $K_{1,-1,0}$ through $(0, 0)$, $(1, 0)$, $(\infty, 1)$ is taken to the corresponding circle $D_{1,1,0}$ in $\mathscr{L}_k$. Finally, there is a unique circle through $(\infty, 0)$ that touches K_0 and $K_{1,-1,0}$. The latter point of touching is calculated to be $\left(\frac{1}{2}, -\frac{1}{4}\right)$. In $\mathscr{L}_k$ one calculates that the unique circle through $(\infty, 0)$ that touches C_0 and $D_{1,1,0}$ is $C_{0,0,c}$, where

$$c = -\tfrac{1}{2} h_k(k) h_{k'}(k'),$$

and that the common point between the latter two circles is $\left(\frac{k'}{2}, c\right)$. However, ψ preserves parallelity of points so that

$$\left(\tfrac{1}{2}, \tfrac{1}{4}\right) = \psi\left(\tfrac{1}{2}, \tfrac{1}{4}\right) \parallel \psi\left(\tfrac{1}{2}, -\tfrac{1}{4}\right) = \left(\tfrac{k'}{2}, c\right).$$

This shows that $\frac{k'}{2} = \frac{1}{2}$, that is, $k' = 1$ — a contradiction to our assumption $k \neq 1$.

In the case that ψ takes $(\infty, 1)$ to $(\infty, -1)$, we may apply the isomorphism $\mu : \mathscr{L}_k \to \mathscr{L}_{k'}$ from the proof of Proposition 5.2. Then the map $\mu\psi$ fixes each of the four points $(\infty, 0)$, $(0, 0)$, $(1, 0)$, $(\infty, 1)$. Hence we conclude as before that $k = (k')' = 1$ — again a contradiction.

This proves that $\mathscr{L}_k$, $k \neq 1$, is not semiclassical. □

Remark 5.6. In the proof of Lemma 4.1 we already mentioned that the derived projective plane of $\mathscr{L}(k)$ at $(\infty, 0)$ is isomorphic to a cartesian plane $\mathscr{P}_{1,k/k',1}$. It is readily seen that the derived projective plane of a semiclassical plane $\mathscr{L}(h_m, \mathrm{id})$ at $(\infty, 0)$ is isomorphic to a cartesian plane $\mathscr{P}_{m,1,1}$. As mentioned in [Salzmann et al.

1995, Proof of 37.6] the plane $\mathcal{P}_{\alpha,\beta,c}$ is dual to $\mathcal{P}_{\beta,\alpha,c}$. Hence, when $m = k/k'$, the derived projective plane at $(\infty, 0)$ of a Laguerre plane $\mathcal{L}(k)$ and of a semiclassical plane $\mathcal{L}(h_m, \mathrm{id})$ are dual to each other. However, there does not seem to be an extension of this duality to the level of the Laguerre planes (for example, via associated generalized quadrangles, see below).

It is well known that 2-dimensional Laguerre planes correspond to certain compact 3-dimensional generalized quadrangles, compare [Schroth 1993a], [Schroth 1993b] or [Schroth 1995b]. In a compact 3-dimensional generalized quadrangle the point and line spaces are compact and 3-dimensional. These generalized quadrangles are also characterized by having topological parameter 1 (so that all lines and line pencils are homeomorphic to the 1-dimensional sphere $\mathbb{S}^1$). More precisely, the Lie geometry associated with a 2-dimensional Laguerre plane is an antiregular compact generalized quadrangle with topological parameter 1. Up to duality, every compact 3-dimensional generalized quadrangle is the Lie geometry of a 2-dimensional Laguerre plane; see [Schroth 1995b, Corollary 2.16 and Chapter 3]. Recall that the *Lie geometry* of a Laguerre plane $\mathcal{L}$ has as points the points of $\mathcal{L}$ plus the circles of $\mathcal{L}$ plus one additional point at infinity, denoted by $\overline{\infty}$. (The bar helps distinguish this from other uses of the symbol ∞.) The lines of the Lie geometry are the augmented parallel classes, that is, the parallel classes to which the point $\overline{\infty}$ is adjoined, and the augmented tangent pencils, that is, the collections of all circles that touch a given circle at a given point p together with the point p, called the support of the tangent pencil. Incidence is the natural one. So “collinear” in the Lie geometry corresponds to “on the same parallel class or incident or touching” in the Laguerre plane. The generalized quadrangle obtained from the classical real Laguerre plane $\mathcal{L}_{\mathrm{cl}}$ is the real orthogonal quadrangle $Q(4, \mathbb{R})$ over $\mathbb{R}$. Points are the 1-dimensional isotropic subspaces of $\mathbb{R}^5$, with respect to a symmetric form of Witt index 2; lines are the 2-dimensional totally isotropic subspaces of $\mathbb{R}^5$.

Conversely, for every point p of an antiregular generalized quadrangle $\mathfrak{Q}$, one obtains a Laguerre plane $\mathfrak{Q}'_p$, called the *derivation of* $\mathfrak{Q}$ *at* p, whose points are the points of $\mathfrak{Q}$ that are collinear with p except p itself and whose circles are of the form $p^\perp \cap q^\perp$ for points q not collinear with p, where $x^\perp$ denotes the set of all points collinear with the point x. See also [Joswig 1999, Theorem 3.1], where it is shown that it suffices to have a strongly antiregular point of the generalized quadrangle in order to obtain a Laguerre plane as derivation at that point. Each derived Laguerre plane of the real orthogonal quadrangle $Q(4, \mathbb{R})$ over $\mathbb{R}$ is isomorphic to the classical real Laguerre plane.

Starting with a 2-dimensional Laguerre plane $\mathcal{L}$ one obtains an antiregular compact 3-dimensional generalized quadrangle $\mathfrak{Q}(\mathcal{L})$. One can then derive at any point p of $\mathfrak{Q}(\mathcal{L})$ to obtain another 2-dimensional Laguerre plane $\mathcal{L}'_p = (\mathfrak{Q}(\mathcal{L}))'_p$. In [Schroth 1995a] and [Schroth 1995b, Chapter 6] this Laguerre plane $\mathcal{L}'_p$ is called a *sister* of $\mathcal{L}$. The process of going from $\mathcal{L}$ to its sister $\mathcal{L}'_p$ can be completely described within $\mathcal{L}$ without explicitly using the associated generalized quadrangle; see [Schroth 1995a, Section 3]. In case one derives $\mathfrak{Q}(\mathcal{L})$ at a point that comes from a circle K of $\mathcal{L}$, the points of $\mathcal{L}'_K$ are the circles of $\mathcal{L}$ that touch K and the points of $\mathcal{L}$ on K. The parallel classes of $\mathcal{L}'_K$ are obtained from the tangent pencils with support on K.

Circles of $\mathcal{L}'_K$ correspond to the points of $\mathcal{L}$ not on K (more precisely, such a point q represents the collection of all circles of $\mathcal{L}$ through q that touch K) and to the circles of $\mathcal{L}$ not touching K (more precisely, such a circle C represents the collection of all circles of $\mathcal{L}$ that touch C and K), and the extra point $\overline{\infty}$. Incidence is the natural one; compare [Schroth 1995a, Section 3].

Note that an automorphism α of $\mathcal{L}$ extends to an automorphism $\bar{\alpha}$ of $\mathfrak{Q}(\mathcal{L})$. Furthermore, $\bar{\alpha}$ fixes $\overline{\infty}$. If α fixes a point or circle of $\mathcal{L}$, then $\bar{\alpha}$ induces an automorphism in the derived Laguerre plane of $\mathfrak{Q}(\mathcal{L})$ at that point or circle.

We carry out the above procedure for the Laguerre planes $\mathcal{L}_k$ and the distinguished circle C_0. Since C_0 is fixed by Γ, this group is again a group of automorphisms of $(\mathcal{L}_k)'_{C_0}$. Note that $\mathcal{L}_k$ shares many circles with the classical real Laguerre plane $\mathcal{L}_{\mathrm{cl}}$ and, in particular, all the circles that touch C_0. So we expect that $(\mathcal{L}_k)'_{C_0}$ has many circles in common with $\mathcal{L}_{\mathrm{cl}}$, and looks like one of the Laguerre planes constructed in this paper or a semiclassical Laguerre plane obtained by pasting along a circle. In fact, we have the following.

Proposition 5.7. *The Laguerre plane $(\mathcal{L}_k)'_{C_0}$ obtained by deriving the generalized quadrangle $\mathfrak{Q}(\mathcal{L}_k)$ at C_0 is isomorphic to $\mathcal{L}_k$.*

Proof. A circle of $\mathcal{L}_k$ touching C_0 is $C'_{a,b,0}$, where $a, b \in \mathbb{R}$, $a \neq 0$, or $C_{0,0,c}$, where $c \in \mathbb{R}$, $c \neq 0$. We identify such a circle with $\left(b, \frac{1}{a}\right) \in Z$ and $\left(\infty, \frac{1}{c}\right)$, respectively. A point $(x, 0)$ on C_0 is identified with $(x, 0) \in Z$. This coordinatization maps all points of $(\mathcal{L}_k)'_{C_0}$ onto the cylinder Z. Parallel classes are still the generators of Z.

The point $\overline{\infty}$ gives rise to the set C_0, which thus is again a circle of $(\mathcal{L}_k)'_{C_0}$. If (x_0, y_0), $y_0 \neq 0$, is a point not on C_0, then for each $b \in \mathbb{R}$, $b \neq x_0$, there is a unique circle through (x_0, y_0) that touches C_0 at $(b, 0)$; this circle is $C'_{y_0/(x_0-b)^2,b,0}$, which yields the point $(b, (x_0-b)^2/y_0)$ according to the above rule. One further obtains $(x_0, 0)$ (from the parallel class through (x_0, y_0)) and $(\infty, 1/y_0)$ (from the circle $C_{0,0,y_0}$ touching C_0 at $(\infty, 0)$). Put together we thus obtain all the points on $C'_{1/y_0,x_0,0}$, so that this is again a circle of $(\mathcal{L}_k)'_{C_0}$.

Next consider a circle not meeting C_0. Such a circle is of the form $C'_{a,b,c}$, where $ac > 0$. The circle of $\mathcal{L}_k$ touching C_0 at $(u, 0)$ and also touching $C'_{a,b,c}$ is $C'_{\tilde{a},u,0}$, where $u \in \mathbb{R}$ and $\tilde{a} = ac/(a(u-b)^2 + c$. Hence we obtain the point

$$\left(u, \frac{1}{c}(u-b)^2 + \frac{1}{a}\right)$$

in $(\mathscr{L}_k)'_{C_0}$. When $u=\infty$ we find the circle $C_{0,0,c}$, which yields the point $\left(\infty, \frac{1}{c}\right)$. Thus we have recovered the C-circle $C'_{1/c,b,1/a}$ as a circle of $(\mathscr{L}_k)'_{C_0}$.

Finally consider a circle meeting C_0 in two points. Such a circle is a D-circle. In this case the calculations are a bit more involved. To find the circle $C'_{v,u,0}$ that touches $D_{a,b,c}$, $a \neq 0$, and also touches C_0 at $(u,0)$, where $u \neq b, c, \infty$, it is necessary that the equations

$$v(x-u)^2 = ah_k(x-b)h_{k'}(x-c), \tag{6}$$

$$\begin{aligned} 2v(x-u) &= a(h_k(x-b)h_{k'}(x-c))' \\ &= a(2x-k'b-kc)|x-b|^{k-1}|x-c|^{k'-1} \end{aligned} \tag{7}$$

are satisfied. Dividing (6) by (7) one finds that

$$x = \frac{u(k'b+kc)-2bc}{2u-kb-k'c}.$$

Substitution into (6) then yields

$$\frac{1}{v} = -\frac{4}{a(b-c)^2h_k(k)h_{k'}(k')}h_k(u-c)h_{k'}(u-b).$$

In the coordinates of $(\mathscr{L}_k)'_{C_0}$ as introduced above the two points $(b,0)$ and $(c,0)$ of intersection of C_0 and $D_{a,b,c}$ yield the points $(b,0)$ and $(c,0)$ on the circle induced by $D^{\perp}_{a,b,c}$. When $u=\infty$ one similarly obtains from $(h_k(x-b)h_{k'}(x-c))'=0$ that $x=\frac{1}{2}(k'b+kc)$ and thus $v=-\frac{1}{4}a(b-c)^2h_k(k)h_{k'}(k')$. In total we have recovered all the points of $D_{\tilde{a},c,b}$, where

$$\tilde{a} = -\frac{4}{a(b-c)^2h_k(k)h_{k'}(k')}.$$

The cases when $a=0$ are dealt with in a similar way. $\square$

In case one derives the generalized quadrangle $\mathfrak{Q}(\mathscr{L})$ at a point that comes from a point p of $\mathscr{L}$ then the points of $\mathscr{L}'_p$ are the circles of $\mathscr{L}$ that pass through p, the points of $\mathscr{L}$ on the parallel class $|p|$ of p but not p itself, and the extra point $\overline{\infty}$. The parallel classes of $\mathscr{L}'_p$ are obtained from the parallel class $|p|$ and the tangent pencils with support p. The circles of $\mathscr{L}'_p$ correspond to the points of $\mathscr{L}$ not on $|p|$ (more precisely, such a point q represents the collection of all circles of $\mathscr{L}$ through p and q) and to the circles of $\mathscr{L}$ not passing through p (more precisely, such a circle C represents the collection of all circles of $\mathscr{L}$ through p that touch C). Thus the affine part of $\mathscr{L}'_p$ with respect to the parallel class containing $\overline{\infty}$ is made up of the nonvertical lines of the derived affine plane $\mathscr{A}_p$ of $\mathscr{L}$ at p, and points of $\mathscr{A}_p$ represent circles of $\mathscr{L}'_p$ through $\overline{\infty}$. Hence the derived projective plane $\mathscr{P}'_{\overline{\infty}}$ of $\mathscr{L}'_p$ at $\overline{\infty}$ is the dual of $\mathscr{P}_p$, the derived projective plane of $\mathscr{L}$ at p. A circle of $\mathscr{L}$ not passing through p induces an oval $\mathbb{O}$ in $\mathscr{P}_p$. Since this circle also represents a circle of $\mathscr{L}'_p$, we just obtain the dual oval $\mathbb{O}^*$ of $\mathbb{O}$ in $\mathscr{P}'_{\overline{\infty}}$. Hence, the whole process involves forming the dual of the derived projective plane $\mathscr{P}_p$ plus all duals of the ovals in $\mathscr{P}_p$ that are induced by circles of $\mathscr{L}$; we then remove one line to obtain the affine part of the sister $\mathscr{L}'_p$ and add one parallel class at infinity in order to complete the Laguerre plane. Although applying this process to a point p on K_0 of a nonclassical semiclassical Laguerre plane $\mathscr{L}(h_m, \mathrm{id})$ yields the dual of the derived plane at p, other circles of $\mathscr{L}(h_m, \mathrm{id})'_p$ do not match circles of $\mathscr{L}_k$. Since the point p has a 1-dimensional orbit we also expect the automorphism group of $\mathscr{L}(h_m, \mathrm{id})'_p$ to be at most 3-dimensional, and so $\mathscr{L}(h_m, \mathrm{id})'_p$ cannot be isomorphic to a plane $\mathscr{L}_k$.

Schroth [2000] used a provisional classification of 2-dimensional Laguerre planes of group dimension 4 to show that a compact 3-dimensional generalized quadrangle is the real orthogonal quadrangle $Q(4, \mathbb{R})$, or its dual if the group of automorphisms of the quadrangle has dimension at least 6. Since the new Laguerre planes $\mathscr{L}_k$ do not appear on the list used in [Schroth 2000], this can potentially affect Schroth's result. However, as noted in [Schroth 2000, Section, 3.7], in case of a 4-dimensional group of automorphisms of a 2-dimensional Laguerre plane such that a circle is fixed, the information on the groups involved is enough to see that the dimension of the automorphism group of the associated quadrangle does not become larger; see also [Schroth 2000, Section 4.6]. The automorphism group of $\mathscr{L}_k$ has at most as many orbits on the circle set and point set as the automorphism group of semiclassical Laguerre planes pasted along a circle. This implies that the same dimensions of orbits occur as stated in [Schroth 2000, Section 4.6]. Hence we have the following result; compare [Schroth 2000, Theorem 4.8].

Corollary 5.8. *The automorphism group of the* 3*-dimensional compact generalized quadrangle* $\mathfrak{Q}(\mathscr{L}_k)$ *is* 4*-dimensional when* $k \neq 1$.

As a consequence, the planes constructed here are not counterexamples to the main theorem of [Schroth 2000].

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