Innovations in Incidence Geometry

Algebraic, Topological and Combinatorial

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msp

Vol. 17 No. 1 2019

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vol. 17, no. 1, 2019
dx.doi.org/10.2140/iig.2019.17.43

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Betten and Riesinger have shown that Clifford parallelism on real projective space is the only topological parallelism that is left invariant by a group of dimension at least 5. We improve the bound to 4. Examples of different parallelisms admitting a group of dimension ≤ 3 are known, so 3 is the "critical dimension".

Consider $\mathbb{R}^4$ as the quaternion skew field $\mathbb{H}$. Then the orthogonal group $\mathrm{SO}(4, \mathbb{R})$ may be described as the product of two commuting copies $\tilde{\Lambda}$, $\tilde{\Phi}$ of the unitary group $\mathrm{U}(2, \mathbb{C})$, consisting of the maps $q \mapsto aq$ and $q \mapsto qb$, respectively, where a, b are quaternions of norm one and multiplication is quaternion multiplication. The intersection of the two factors is of order two, containing the map $-\mathrm{id}$. Thus, passing to projective space, we get $\mathrm{PSO}(4, \mathbb{R}) = \Lambda \times \Phi$, a direct product of two copies of $\mathrm{SO}(3, \mathbb{R})$. The left and right Clifford parallelisms are defined as the equivalence relations on the line space of $\mathrm{PG}(3, \mathbb{R})$ formed by the orbits of Λ and Φ, respectively.

The two Clifford parallelisms are equivalent under quaternion conjugation $q \to \bar{q}$; this is immediate from their definition in view of the fact that conjugation does not change the norm and is an antiautomorphism, i.e., that $\overline{pq} = \bar{q}\,\bar{p}$. Note that both Λ and Φ are transitive on the point set of projective space. Since they centralize one another, each acts transitively on the parallelism defined by the other, and the group $\mathrm{PSO}(4, \mathbb{R})$ leaves both parallelisms invariant (we say that it consists of *automorphisms* of these parallelisms). For more information on Clifford parallels, see [Berger 1987; Klingenberg 1984; Betten and Riesinger 2012]. For generalizations to other dimensions, compare also [Tyrrell and Semple 1971].

The notion of a *topological parallelism* on real projective 3-space $\mathrm{PG}(3, \mathbb{R})$ generalizes this example. A *spread* is a set $\mathcal{C}$ of lines such that every point is incident with exactly one of them, and a topological parallelism may be defined

MSC2010: 51H10, 51A15, 51M30.
Keywords: Clifford parallelism, automorphism group, topological parallelism.

as a compact set Π of compact spreads such that every line belongs to exactly one of them; see, e.g., [Betten and Riesinger 2014b] for details. Many examples of different topological parallelisms have been constructed in a series of papers by Betten and Riesinger, see, e.g., [Betten and Riesinger 2009].

The group $\Sigma = \mathrm{Aut}\,\Pi$ of automorphisms of a topological parallelism is a closed subgroup of the Lie group $\mathrm{PGL}(4, \mathbb{R})$, hence it is a Lie group, as well. In particular, the identity component Σ^1 is an open subgroup of Σ and has the same (manifold) dimension as Σ. We know that Σ^1 is compact [Betten and Löwen 2017], and hence (conjugate to) a subgroup of $\mathrm{PSO}(4, \mathbb{R}) \cong \mathrm{SO}(3, \mathbb{R}) \times \mathrm{SO}(3, \mathbb{R})$. The group $\mathrm{SO}(3, \mathbb{R})$ does not have any 2-dimensional closed subgroups, because its Lie algebra is $\mathbb{R}^3$ with the vector product $\times$ and $x \times y$ is always orthogonal to both x and y. Moreover, the 1-dimensional closed subgroups of $\mathrm{SO}(3, \mathbb{R})$ form a single conjugacy class. It follows easily that there are no closed 5-dimensional subgroups of $\mathrm{SO}(3, \mathbb{R}) \times \mathrm{SO}(3, \mathbb{R})$ and all 4-dimensional ones are isomorphic to $\mathrm{SO}(3, \mathbb{R}) \times \mathrm{SO}(2, \mathbb{R})$.

We see that in the case of the Clifford parallelism, Σ^1 is the 6-dimensional group $\mathrm{PSO}(4, \mathbb{R})$ that we used to define the parallelism. Betten and Riesinger [2014b] proved that no other topological parallelism has a group of dimension $\dim \Sigma \geq 5$. Examples of parallelisms with 1-, 2- or 3-dimensional automorphism groups are known; see [Betten and Riesinger 2014a; 2009; 2011] . Here we consider parallelisms with a 4-dimensional group.

Theorem 1. *Let Σ be the automorphism group of a topological parallelism Π on* $\mathrm{PG}(3, \mathbb{R})$. *If* $\dim \Sigma \geq 4$, *then Π is equivalent to the Clifford parallelism.*

Proof. Recall that a topological parallelism Π is homeomorphic to the real projective plane in the Hausdorff topology on the space of compact sets of lines, and that every equivalence class is a compact spread and homeomorphic to the 2-sphere; compare [Betten and Riesinger 2014b].

The remarks preceding the theorem show that a group Σ of dimension at least 4 contains a 4-dimensional connected closed subgroup Δ, and it will suffice for our proof to use this group. Further, up to equivalence, we may assume that $\Delta = \Lambda \cdot \Gamma$, where $\Gamma \leq \Phi$ is the subgroup defined by restricting the factor b to be a complex number (here we use the notation of the introduction). Since Λ does not have any one-dimensional coset spaces, we know that Λ acts on Π either transitively or trivially. If it acts trivially, then the classes of Π are the Λ-orbits of lines, and we have the Clifford parallelism. Observe here that every Λ-orbit is contained in a single class, and both the orbit and the class are 2-spheres.

In what follows, assume therefore that Λ acts transitively on Π. There is only one possibility for this action, namely, the standard transitive action of $\mathrm{SO}(3, \mathbb{R})$ on the real projective plane. Every 2-dimensional subgroup of Δ contains Γ. Hence, there is no effective action of Δ on the projective plane Π, and the kernel can only be Γ since the only other proper normal subgroup is Λ, which is transitive. If $\mathcal{C} \in \Pi$ is any equivalence class, then the stabilizer $\Lambda_{\mathcal{C}}$ is a product of a 1-torus and a group of order two. Hence $\Delta_{\mathcal{C}}$ contains a 2-torus T. There is only one conjugacy class of 2-tori in Δ, represented by the group

$$T_0 = \{\langle q \rangle \mapsto \langle aqb \rangle \mid a, b \in \mathbb{C}, |a| = |b| = 1\}.$$

Here, $\langle q \rangle$ denotes the 1-dimensional real vector space spanned by q. We may assume that $T = T_0$. Write quaternions as pairs of complex numbers with multiplication $(x, y)(u, v) = (xu - \bar{v}y, vx + y\bar{u})$; see 11.1 of [Salzmann et al. 1995]. Then complex numbers become pairs $(a, 0)$, and the elements of T are now given by

$$\langle (z, w) \rangle \mapsto \langle (azb, aw\bar{b}) \rangle.$$

The kernel of ineffectivity of T on the 2-sphere $\mathcal{C}$ must be a 1-torus Ξ, and the elements of the kernel other than the identity cannot have eigenvalue 1 — otherwise they would be axial collineations of the translation plane defined by the spread $\mathcal{C}$ and would act nontrivially on $\mathcal{C}$. There are only two subgroups of the 2-torus satisfying these conditions, given by $b = 1$ and by $a = 1$, respectively. In other words, the kernel Ξ is a subgroup either of Λ or of Φ. In both cases, $\mathcal{C}$ consists of the fixed lines of Ξ. If $\Xi \leq \Phi$, then Λ permutes these lines, contrary to the transitivity of Λ on Π. If $\Xi \leq \Lambda$, then Φ permutes the fixed lines, which means that $\mathcal{C}$ is a Φ-orbit. Now Λ is transitive both on Π and on the set of Φ-orbits, hence Π equals the Clifford parallelism formed by the Φ-orbits. □

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Received 17 Feb 2017.

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Innovations in Incidence Geometry: Algebraic, Topological and Combinatorial (ISSN pending) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

IIG peer review and production are managed by EditFlow® from MSP.

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Innovation in Incidence Geometry

Vol. 17 No. 1 2019