

## Rigidity of transversally biharmonic maps between foliated Riemannian manifolds

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(Received October 4, 2016; Revised October 24, 2016)

**Abstract.** On a smooth foliated map from a complete, possibly non-compact, foliated Riemannian manifold into another foliated Riemannian manifold of which transversal sectional curvature is non-positive, we will show that, if it is transversally biharmonic and has the finite energy and finite bienergy, then it is transversally harmonic.

*Key words:* foliation, divergence theorem, transversally harmonic, transversally biharmonic.

### 1. Introduction

Transversally biharmonic maps between two foliated Riemannian manifolds introduced by Chiang and Wolak (cf. [4]) are generalizations of transversally harmonic maps introduced by Konderak and Wolak (cf. [19], [20]).

Among smooth foliated maps  $\varphi$  between two Riemannian foliated manifolds, one can define the transversal energy and derive the Euler-Lagrange equation, and transversally harmonic map as its critical points, which are by definition the transversal tension field vanishes,  $\tau_b(\varphi) \equiv 0$ . The transverse bienergy can be also defined as  $E_2(\varphi) = (1/2) \int_M |\tau_b(\varphi)|^2 v_g$  whose Euler-Lagrange equation is that the transversal bitension field  $\tau_{2,b}(\varphi)$  vanishes and the transversally biharmonic maps which are, by definition, vanishing of the transverse bitension field.

Recently, S. D. Jung studied extensively the transversally harmonic maps and the transversally biharmonic maps on compact Riemannian foliated manifolds (cf. [14], [15], [17], [18]).

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*2010 Mathematics Subject Classification :* Primary 58E20; Secondary 53C43.

The second author was supported by the Grant-in-Aid for the Scientific Research, (C) No. 25400154 and (C) No. 26400073, Japan Society for the Promotion of Science.

The third author was supported by the Grant-in-Aid for the Scientific Reserch, (C) No. 25400154 and (C) No. 18K03352, Japan Society for the Promotion of Science.

In this paper, we study transversally biharmonic maps of a complete (possibly non-compact) Riemannian foliated manifold  $(M, g, \mathcal{F})$  into another Riemannian foliated manifold  $(M', g', \mathcal{F}')$  of which transversal sectional curvature is non-positive. Then, we will show that:

**Theorem 1.1** (cf. Theorem 2.11) *Let  $(M, g, \mathcal{F})$  and  $(M', g', \mathcal{F}')$  be two foliated Riemannian manifolds. Assume that the foliation  $\mathcal{F}$  is transversally volume preserving (cf. Definition 2.1) and the transversal sectional curvature of  $(M', g', \mathcal{F}')$  is non-positive. Let  $\varphi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}')$  be a  $C^\infty$  foliated map satisfying the conservation law. If  $\varphi$  is transversally biharmonic with the finite transversal energy  $E(\varphi) < \infty$  and finite transversal bienergy  $E_2(\varphi) < \infty$ , then it is transversally harmonic.*

This theorem can be regarded a natural analogue of B. Y. Chen's conjecture and the generalized Chen's conjecture (cf. [3], [12]).

B. Y. Chen's conjecture: *Every biharmonic submanifolds of the Euclidean space  $\mathbb{R}^n$  must be harmonic (minimal).*

The generalized B. Y. Chen's conjecture: *Every biharmonic submanifolds of a Riemannian manifold of non-positive curvature must be harmonic (minimal).*

Several authors has contributed to give partial answers to solve these problems (cf. [1], [5], [8], [11], [9], [10], [22], [23], [24]). For the first and second variational formula of the bienergy, see [13]. For the  $CR$  analogue of biharmonic maps, see also [2], [6], [31].

## 2. Preliminaries

We prepare the materials for the first and second variational formulas for the transversal energy of a smooth foliated map between two foliated Riemannian manifolds following [17], [18] and [32].

### 2.1. The Green's formula on a foliated Riemannian manifold

Let  $(M, g, \mathcal{F})$  be an  $n(= p + q)$ -dimensional foliated Riemannian manifold with foliation  $\mathcal{F}$  of codimension  $q$  and a bundle-like Riemannian metric  $g$  with respect to  $\mathcal{F}$  (cf. [29], [30]). Let  $TM$  be the tangent bundle of  $M$ ,  $L$ , the tangent bundle of  $\mathcal{F}$ , and  $Q = TM/L$ , the corresponding normal bundle of  $\mathcal{F}$ . We denote  $g_Q$  the induced Riemannian metric on the normal

bundle  $Q$ , and  $\nabla^Q$ , the transversal Levi-Civita connection on  $Q$ ,  $R^Q$ , the transversal curvature tensor, and  $K^Q$ , the transversal sectional curvature, respectively. Notice that the bundle projection  $\pi : TM \rightarrow Q$  is an element of the space  $\Omega^1(M, Q)$  of  $Q$ -valued 1-forms on  $M$ . Then, one can obtain the  $Q$ -valued bilinear form  $\alpha$  on  $M$ , called the second fundamental form of  $\mathcal{F}$ , defined by

$$\alpha(X, Y) = -(D_X \pi)(Y) = \pi(\nabla_X^Q Y), \quad (X, Y \in \Gamma(L)),$$

where  $D$  is the torsion free connection on the bundle  $Q$  (cf. [32, p.240, Proposition 1]). See also Definition of  $\alpha$ , (6) in Page 241 of [32]. The trace  $\tau$  of  $\alpha$ , called the *tension field* of  $\mathcal{F}$  is defined by

$$\tau = \sum_{i,j=1}^p g^{ij} \alpha(X_i, X_j),$$

where  $\{X_i\}_{i=1}^p$  spans  $\Gamma(L|U)$  on a neighborhood  $U$  on  $M$ . The Green's theorem, due to Yorozu and Tanemura([32]), of a foliated Riemannian manifold  $(M, g, \mathcal{F})$  says that

$$\int_M \operatorname{div}_D(\nu) v_g = \int_M g_Q(\tau, \nu) v_g \quad (\nu \in \Gamma(Q)), \tag{2.1}$$

where  $\operatorname{div}_D(\nu)$  denotes the *transversal divergence* of  $\nu$  with respect to  $\nabla^Q$  given by  $\operatorname{div}_D(\nu) := \sum_{a,b=1}^q g^{ab} g_Q(D_{X_a} \nu, \pi(X_b))$ . Here  $\{X_a\}_{a=1}^q$  spans  $\Gamma(L^\perp|U)$  where  $L^\perp$  is the orthogonal complement bundle of  $L$  with a natural identification  $\sigma : Q \xrightarrow{\cong} L^\perp$ .

**Definition 2.1** A foliation  $\mathcal{F}$  is *transversally volume preserving* if  $\operatorname{div}(\tau) = 0$ .

Let us recall Gaffney's theorem ([7], [24]):

**Theorem 2.2** *Let  $(M, g)$  be a non-compact complete Riemannian manifold without boundary, If a  $C^1$  vector field  $X$  on  $M$  satisfies that*

$$\int_M |X| v_g < \infty \quad \text{and} \quad \int_M \operatorname{div}(X) v_g < \infty. \tag{2.2}$$

Then, it holds that

$$\int_M \operatorname{div}(X) v_g = 0. \quad (2.3)$$

Furthermore, if  $f \in C^1(M)$  and a  $C^1$  vector field  $X$  on  $M$  satisfy  $\operatorname{div}(X) = 0$ ,  $\int_M Xf v_g < \infty$ ,  $\int_M |f|^2 v_g < \infty$  and  $\int_M |X|^2 v_g < \infty$ , then it holds that

$$\int_M Xf v_g = 0. \quad (2.4)$$

For the sake of completeness, we give a proof of Theorem 2.2 in the appendix.

If  $\mathcal{F}$  is transversally volume preserving, it holds by definition that

$$\int_M \tau g_Q(\nu, \nu) v_g = 0 \quad (\nu \in \Gamma(Q) \text{ with compact support}). \quad (2.5)$$

## 2.2. The first and second variational formulas

Let  $(M, g, \mathcal{F})$ , and  $(M', g', \mathcal{F}')$  be two compact foliated Riemannian manifolds. The *transversal energy*  $E(\varphi)$  among the totality of smooth foliated maps from  $(M, g, \mathcal{F})$  into  $(M', g', \mathcal{F}')$  by

$$E(\varphi) = \frac{1}{2} \int_M |d_T \varphi|^2 v_g. \quad (2.6)$$

Here, a smooth map  $\varphi$  is a foliated map is, by definition, for every leaf  $\ell$  of  $\mathcal{F}$ , there exists a leaf  $\ell'$  of  $\mathcal{F}'$  satisfying  $\varphi(\ell) \subset \ell'$ . Then,  $d_T \varphi := \pi' \circ d\varphi \circ \sigma; Q \rightarrow Q'$  can be regarded as a section of  $Q^* \otimes \varphi^{-1}Q'$  where  $Q^*$  is a subspace of the cotangent bundle  $T^*M$ . Here,  $\pi, \pi'$  are the projections of  $TM \rightarrow Q = TM/L$  and  $TM' \rightarrow Q' = TM'/L'$ . Notice that our definition of the transversal energy is the same as the one of Jung's definition (cf. [18, p. 11, (3.4)]).

The first variational formula is given (cf. [17], the case  $f = 1$  in Theorem 4.1, (4.2)), for every smooth foliated variation  $\{\varphi_t\}$  with  $\varphi_0 = \varphi$  and  $d\varphi_t/dt|_{t=0} = V$  in which  $V$  being a section  $\varphi^{-1}Q'$ ,

$$\left. \frac{d}{dt} \right|_{t=0} E(\varphi_t) = - \int_M \langle V, \tau_b(\varphi) - d_T \varphi(\tau) \rangle v_g. \tag{2.7}$$

Here,  $\tau_b(\varphi)$  is the *transversal tension field* defined by

$$\tau_b(\varphi) = \sum_{a=1}^q (\tilde{\nabla}_{E_a} d_T \varphi)(E_a), \tag{2.8}$$

where  $\tilde{\nabla}$  is the induced connection in  $Q^* \otimes \varphi^{-1}Q'$  from the Levi-Civita connection of  $(M', g')$ , and  $\{E_a\}_{a=1}^q$  is a locally defined orthonormal frame field on  $Q$ .

**Definition 2.3** A smooth foliated map  $\varphi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}')$  is said to be *transversally harmonic* if  $\tau_b(\varphi) \equiv 0$ .

Then, for a transversally harmonic map  $\varphi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}')$ , the second variation formula of the transversal energy  $E(\varphi)$  is given as follows (cf. [18, p. 13], the case  $f = 1$  in Theorem 4.1, (4.2)): let  $\varphi_{s,t} : M \rightarrow M'$  ( $-\epsilon < s, t < \epsilon$ ) be any two parameter smooth foliated variation of  $\varphi$  with  $V = \partial\varphi_{s,t}/\partial s|_{(s,t)=(0,0)}$ ,  $W = \partial\varphi_{s,t}/\partial t|_{(s,t)=(0,0)}$  and  $\varphi_{0,0} = \varphi$ ,

$$\begin{aligned} \text{Hess}(E)_\varphi(V, W) &:= \left. \frac{\partial^2}{\partial s \partial t} \right|_{(s,t)=(0,0)} E(\varphi_{s,t}) \\ &= \int_M \langle J_{b,\varphi}(V), W \rangle v_g + \int_M V \langle W, d_T \varphi(\tau) \rangle v_g, \end{aligned} \tag{2.9}$$

where  $J_{b,\varphi}$  is a second order semi-elliptic differential operator acting on the space  $\Gamma(\varphi^{-1}Q')$  of sections of  $\varphi^{-1}Q'$  which is of the form:

$$\begin{aligned} J_{b,\varphi}(V) &:= \tilde{\nabla}^* \tilde{\nabla} V - \tilde{\nabla}_\tau V - \text{trace}_Q R^{Q'}(V, d_T \varphi) d_T \varphi \\ &= - \sum_{a=1}^q (\tilde{\nabla}_{E_a} \tilde{\nabla}_{E_a} - \tilde{\nabla}_{\nabla_{E_a} E_a}) V \\ &\quad - \sum_{a=1}^q R^{Q'}(V, d_T \varphi(E_a)) d_T \varphi(E_a) \end{aligned} \tag{2.10}$$

for  $V \in \Gamma(\varphi^{-1}Q')$ . Here,  $\nabla$  is the Levi-Civita connection of  $(M, g)$ , and recall also that:

$$\tilde{\nabla}^* \tilde{\nabla} V = - \sum_{a=1}^q (\tilde{\nabla}_{E_a} \tilde{\nabla}_{E_a} - \tilde{\nabla}_{\nabla_{E_a} E_a}) V + \tilde{\nabla}_\tau V, \quad (2.11)$$

$$\text{trace}_Q R^{Q'}(V, d_T \varphi) d_T \varphi := \sum_{a=1}^q R^{Q'}(V, d_T \varphi(E_a)) d_T \varphi(E_a). \quad (2.12)$$

Here,  $\tilde{\nabla}^*$  is the adjoint of the connection  $\tilde{\nabla}$  which satisfies (cf. [14, Proposition 3.1]) that

$$\int_M \langle \tilde{\nabla}^* V, W \rangle v_g = \int_M \langle V, \tilde{\nabla} W \rangle v_g \quad (V, W \in \Gamma(\varphi^{-1} Q')),$$

and for all  $V, W \in \Gamma(\varphi^{-1} Q')$ , it holds that

$$\int_M \langle \tilde{\nabla}^* \tilde{\nabla} V, W \rangle v_g = \int_M \langle \tilde{\nabla} V, \tilde{\nabla} W \rangle v_g = \int_M \langle V, \tilde{\nabla}^* \tilde{\nabla} W \rangle v_g.$$

**Definition 2.4** The *transversal bitension field*  $\tau_{2,b}(\varphi)$  of a smooth foliated map  $\varphi$  is defined by

$$\tau_{2,b}(\varphi) := J_{b,\varphi}(\tau_b(\varphi)). \quad (2.13)$$

**Definition 2.5** The *transversal bienergy*  $E_2$  of a smooth foliated map  $\varphi$  is defined by

$$E_2(\varphi) := \frac{1}{2} \int_M |\tau_b(\varphi)|^2 v_g. \quad (2.14)$$

Remark that this definition of the transversal bienergy is also the same as the one of Jung (cf. Jung [18, p. 16], the case  $f = 1$  in Definition 6.1, (6.2)) because  $\tau_b(\varphi) = \sum_{a=1}^q (\tilde{\nabla}_{E_a} d_T \varphi)(E_a) = -\tilde{\delta} d_T \varphi$  (cf. Jung [18, p. 11], the case  $f = 1$  in (3.3)). On the first variation formula of the transversal bienergy is given as follows. For a smooth foliated map  $\varphi$  and a smooth foliated variation  $\{\varphi_t\}$  of  $\varphi$ , it holds (cf. [18, p. 16], the case  $f = 1$  in (6.3)) that

$$\left. \frac{d}{dt} \right|_{t=0} E_2(\varphi_t) = - \int_M \{ \langle V, \tau_{2,b}(\varphi) \rangle + \langle \tilde{\nabla}_\tau V, \tau_b(\varphi) \rangle - \langle V, \tilde{\nabla}_\tau \tau_b(\varphi) \rangle \} v_g. \quad (2.15)$$

**Definition 2.6** A smooth foliated map  $\varphi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}')$  is said to be *transversally biharmonic* if  $\tau_{2,b}(\varphi) \equiv 0$ .

Let us recall that

**Definition 2.7** A smooth foliated map  $\varphi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}')$  satisfies the conservation law if

$$\operatorname{div}_{\tilde{\nabla}} S(\varphi)(X) = 0 \quad (\forall X \in \Gamma(Q)). \tag{2.16}$$

Here,  $\operatorname{div}_{\tilde{\nabla}} S(\varphi)(X)$  is defined by

$$\operatorname{div}_{\tilde{\nabla}} S(\varphi)(X) := \sum_{a=1}^q (\tilde{\nabla}_{E_a} S(\varphi))(E_a, X), \quad (X \in \Gamma(Q)), \tag{2.17}$$

and recall (cf. [18, p.11]) the *transversal stress-energy tensor*  $S(\varphi) := (1/2)|d_T\varphi|^2 g_Q - \varphi^*g_{Q'}$ , and Jung showed (cf. Jung, [18, p11, Proposition 3.4]) that:

**Proposition 2.8** For every a  $C^\infty$  foliated map  $\varphi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}')$ , it holds that

$$\operatorname{div}_{\tilde{\nabla}} S(\varphi)(X) = -\langle \tau_b(\varphi), d_T\varphi(X) \rangle, \quad (X \in \Gamma(Q)). \tag{2.18}$$

Then, one can ask the following generalized B.Y. Chen’s conjecture:

**The generalized Chen’s conjecture:**

Let  $\varphi$  be a transversally biharmonic map from a foliated Riemannian manifold  $(M, g, \mathcal{F})$  into another foliated Riemannian manifold  $(M', g', \mathcal{F}')$  whose transversal sectional curvature  $K^{Q'}$  is non-positive. Then,  $\varphi$  must be transversally harmonic.

To this conjecture, Jung showed (cf. [18, p.19]) that

**Theorem 2.9 (Jung)** Assume that  $(M, g, \mathcal{F})$  is a compact foliated Riemannian manifold whose transversal Ricci curvature is non-negative and positive at some point, and  $(M', g', \mathcal{F}')$  has a positive constant transversal sectional curvature:  $K^{Q'} = C > 0$ . Then, every transversally stable, transversally biharmonic map  $\varphi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}')$  which satisfies the conservation law must be transversally harmonic.

Jung also showed (cf. [18, p. 5, Theorem 6.5]) that

**Theorem 2.10** (Jung) *Assume that  $(M, g, \mathcal{F})$  is a compact foliated Riemannian manifold whose transversal Ricci curvature is non-negative and positive at some point, and  $(M', g', \mathcal{F}')$  has non-positive transversal sectional curvature  $K^{Q'} \leq 0$ . Then, every transversally biharmonic map  $\varphi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}')$  must be transversally harmonic.*

Then, we can state our main theorem which gives an affirmative partial answer to the above generalized Chen’s conjecture under the additional assumption that  $\varphi$  has both the finite transversal energy and the finite transversal bienergy:

**Theorem 2.11** *Let  $\varphi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}')$  a smooth foliated map satisfying the conservation law. Assume that  $(M, g)$  is complete (possibly non-compact),  $\mathcal{F}$  is transversally volume preserving, i.e.,  $\operatorname{div}(\tau) = 0$ , and the transversal sectional curvature  $K^{Q'}$  of  $(M', g', \mathcal{F}')$  is non-positive:  $K^{Q'} \leq 0$ .*

*If  $\varphi$  is transversally biharmonic having both the finite transversal energy  $E(\varphi) < \infty$  and the finite transversal bienergy  $E_2(\varphi)$ , then it is transversally harmonic.*

Remark that in the case that  $M$  is compact, Theorem 2.11 is true due to Jung’s work (cf. [18, p. 17, Theorem 6.5]).

### 3. Proof of main theorem

In this section, we give a proof of Theorem 2.11.

(The first step) First, let us take a cut off function  $\eta$  from a fixed point  $x_0 \in M$  on  $(M, g)$ , i.e.,

$$\begin{cases} 0 \leq \eta(x) \leq 1 & (x \in M), \\ \eta(x) = 1 & (x \in B_r(x_0)), \\ \eta(x) = 0 & (x \notin B_{2r}(x_0)), \\ |\nabla^g \eta| \leq \frac{2}{r} & (x \in M), \end{cases}$$

where  $B_r(x_0) := \{x \in M | r(x) < r\}$ ,  $r(x)$  is a distance function from  $x_0$  on  $(M, g)$ ,  $\nabla^g$  is the Levi-Civita connection of  $(M, g)$ , respectively.

Assume that  $\varphi$  is a transversally biharmonic map of  $(M, g, \mathcal{F})$  into

$(M', g', \mathcal{F}')$ , i.e.,

$$\begin{aligned} \tau_{2,b}(\varphi) &= J_{b,\varphi}(\tau_b(\varphi)) \\ &= \tilde{\nabla}^* \tilde{\nabla} \tau_b(\varphi) - \tilde{\nabla}_\tau \tau_b(\varphi) - \text{trace}_Q R^{Q'}(\tau_b(\varphi), d_T \varphi) d_T \varphi \\ &= 0, \end{aligned} \tag{3.1}$$

where recall  $\tilde{\nabla}$  is the induced connection on  $\varphi^{-1}Q' \otimes T^*M$ .

(The second step) Since  $\tau_b(\varphi) \in \Gamma(Q)$  satisfies that  $\int_M |\tau_b(\varphi)|^2 v_g < \infty$ , it holds that as  $r \rightarrow \infty$ ,

$$\begin{aligned} \int_M \langle \tilde{\nabla}_\tau \tau_b(\varphi), \eta^2 \tau_b(\varphi) \rangle v_g &= \frac{1}{2} \int_M \tau \langle \tau_b(\varphi), \tau_b(\varphi) \rangle \eta^2 v_g \\ &\longrightarrow \frac{1}{2} \int_M \tau \langle \tau_b(\varphi), \tau_b(\varphi) \rangle v_g = 0 \end{aligned} \tag{3.2}$$

due to the completeness of  $(M, g)$ ,  $\text{div}(\tau) = 0$ ,  $\int_M |\tilde{\nabla}_\tau \tau_b(\varphi)|^2 v_g < \infty$  and Gaffney's theorem (cf. Theorem 2.2).

Furthermore, by (3.1), we obtain that

$$\begin{aligned} &\int_M \langle \tilde{\nabla}^* \tilde{\nabla} \tau_b(\varphi), \eta^2 \tau_b(\varphi) \rangle v_g \\ &= \int_M \eta^2 \langle \text{trace}_Q R^{Q'}(\tau_b(\varphi), d_T \varphi) d_T \varphi, \tau_b(\varphi) \rangle v_g \\ &= \int_M \eta^2 \sum_{a=1}^q \langle R^{Q'}(\tau_b(\varphi), d_T \varphi(E_a)) d_T \varphi(E_a), \tau_b(\varphi) \rangle v_g \\ &= \int_M \eta^2 \sum_{a=1}^q K^{Q'}(\Pi_{\varphi,a}) v_g \\ &\leq 0, \end{aligned} \tag{3.3}$$

where the sectional curvature  $K^{Q'}(\Pi_{\varphi,a})$  of  $(M', g', \mathcal{F}')$  corresponding to the plane spanned by  $\tau_b(\varphi)$  and  $d_T \varphi(E_a)$  is non-positive.

(The third step) On the other hand, by the properties of the adjoint  $\tilde{\nabla}^*$  of  $\tilde{\nabla}$ , the left hand side of (3.3) is equal to

$$\begin{aligned}
 & \int_M \langle \tilde{\nabla} \tau_b(\varphi), \tilde{\nabla}(\eta^2 \tau_b(\varphi)) \rangle v_g \\
 &= \int_M \sum_{a=1}^q \langle \tilde{\nabla}_{E_a} \tau_b(\varphi), \tilde{\nabla}_{E_a}(\eta^2 \tau_b(\varphi)) \rangle v_g \\
 &= \int_M \eta^2 \sum_{a=1}^q |\tilde{\nabla}_{E_a} \tau_b(\varphi)|^2 v_g + 2 \int_M \sum_{a=1}^q \langle \eta \tilde{\nabla}_{E_a} \tau_b(\varphi), (E_a \eta) \tau_b(\varphi) \rangle v_g
 \end{aligned} \tag{3.4}$$

since

$$\tilde{\nabla}_{E_a}(\eta^2 \tau_b(\varphi)) = \eta^2 \tilde{\nabla}_{E_a} \tau_b(\varphi) + 2 \eta (E_a \eta) \tau_b(\varphi).$$

Together (3.3) and (3.4), we obtain

$$\begin{aligned}
 & \int_M \eta^2 \sum_{a=1}^q |\tilde{\nabla} \tau_b(\varphi)|^2 v_g \leq -2 \int_M \sum_{a=1}^q \langle \eta \tilde{\nabla}_{E_a} \tau_b(\varphi), (E_a \eta) \tau_b(\varphi) \rangle v_g \\
 & \leq \frac{1}{2} \int_M \eta^2 \sum_{a=1}^q |\tilde{\nabla}_{E_a} \tau_b(\varphi)|^2 v_g + 2 \int_M \sum_{a=1}^q |E_a \eta|^2 |\tau_b(\varphi)|^2 v_g.
 \end{aligned} \tag{3.5}$$

Because, putting  $V_a := \eta \tilde{\nabla}_{E_a} \tau_b(\varphi)$ ,  $W_a := (E_a \eta) \tau_b(\varphi)$  ( $a = 1, \dots, q$ ), we have

$$0 \leq \left| \sqrt{\epsilon} V_a \pm \frac{1}{\sqrt{\epsilon}} W_a \right|^2 = \epsilon |V_a|^2 \pm 2 \langle V_a, W_a \rangle + \frac{1}{\epsilon} |W_a|^2$$

which is

$$\mp 2 \langle V_a, W_a \rangle \leq \epsilon |V_a|^2 + \frac{1}{\epsilon} |W_a|^2. \tag{3.6}$$

If we put  $\epsilon = 1/2$  in (3.6), then we obtain

$$\mp 2 \langle V_a, W_a \rangle \leq \frac{1}{2} |V_a|^2 + 2 |W_a|^2 \quad (a = 1, \dots, q). \tag{3.7}$$

By (3.7), we have the second inequality of (3.5).

(The fourth step) Noticing that  $\eta = 1$  on  $B_r(x_0)$  and  $|E_a\eta|^2 \leq 2/r$  in the inequality (3.5), we obtain

$$\begin{aligned} \int_{B_r(x_0)} \sum_{a=1}^q |\tilde{\nabla}_{E_a} \tau_b(\varphi)|^2 v_g &= \int_{B_r(x_0)} \eta^2 \sum_{a=1}^q |\tilde{\nabla}_{E_a} \tau_b(\varphi)|^2 v_g \\ &\leq \int_M \eta^2 \sum_{a=1}^q |\tilde{\nabla}_{E_a} \tau_b(\varphi)|^2 v_g \\ &\leq 4 \int_M \sum_{a=1}^q |E_a\eta|^2 |\tau_b(\varphi)|^2 v_g \\ &\leq \frac{16}{r^2} \int_M |\tau_b(\varphi)|^2 v_g. \end{aligned} \tag{3.8}$$

Letting  $r \rightarrow \infty$ , the right hand side of (3.8) converges to zero since  $E_2(\varphi) = (1/2) \int_M |\tau_b(\varphi)|^2 v_g < \infty$ . But due to (3.8), the left hand side of (3.8) must converge to  $\int_M \sum_{a=1}^q |\tilde{\nabla}_{E_a} \tau_b(\varphi)|^2 v_g$  since  $B_r(X_0)$  tends to  $M$  because  $(M, g)$  is complete. Therefore, we obtain that

$$0 \leq \int_M \sum_{a=1}^q |\tilde{\nabla}_{E_a} \tau_b(\varphi)|^2 v_g \leq 0,$$

which implies that

$$\tilde{\nabla}_{E_a} \tau_b(\varphi) = 0 \quad (a = 1, \dots, q), \text{ i.e., } \tilde{\nabla}_X \tau_b(\varphi) = 0 \quad (\forall X \in \Gamma(Q)). \tag{3.9}$$

(The fifth step) Let us define a 1-form  $\alpha$  on  $M$  by

$$\alpha(X) := \langle d\varphi(\pi(X)), \tau_b(\varphi) \rangle, \quad (X \in \mathfrak{X}(M)), \tag{3.10}$$

and a canonical dual vector field  $\alpha^\# \in \mathfrak{X}(M)$  on  $M$  by  $\langle \alpha^\#, Y \rangle := \alpha(Y)$ ,  $(Y \in \mathfrak{X}(M))$ . Then, its divergence  $\text{div}(\alpha^\#)$  written as  $\text{div}(\alpha^\#) = \sum_{i=1}^p g(\nabla^{g_{E_i}} \alpha^\#, E_i) + \sum_{a=1}^q g(\nabla^{g_{E_a}} \alpha^\#, E_a)$ , can be given as follows. Here,  $\{E_i\}_{i=1}^p$  and  $\{E_a\}_{a=1}^q$  are locally defined orthonormal frame fields on leaves  $L$  of  $\mathcal{F}$  and  $Q$ , respectively,  $(\dim L_x = p, \dim Q_x = q, x \in M)$ .

Then, we can calculate  $\text{div}(\alpha^\#)$  as follows:

$$\begin{aligned}
\operatorname{div}(\alpha^\#) &= \sum_{i=1}^p \{E_i(\alpha(E_i)) - \alpha(\nabla^g_{E_i} E_i)\} \\
&\quad + \sum_{a=1}^q \{E_a(\alpha(E_a)) - \alpha(\nabla^g_{E_a} E_a)\} \\
&= \left\langle d\varphi \left( \pi \left( - \sum_{i=1}^p \nabla^g_{E_i} E_i \right) \right), \tau_b(\varphi) \right\rangle \\
&\quad + \sum_{a=1}^q \{E_a \langle d\varphi(E_a), \tau_b(\varphi) \rangle - \langle d\varphi(\pi(\nabla^g_{E_a} E_a)), \tau_b(\varphi) \rangle\} \\
&= \left\langle d\varphi \left( \pi \left( - \sum_{i=1}^p \nabla^g_{E_i} E_i \right) \right), \tau_b(\varphi) \right\rangle \\
&\quad + \sum_{a=1}^q \{ \langle \tilde{\nabla}_{E_a}(d\varphi(E_a)), \tau_b(\varphi) \rangle + \langle d\varphi(E_a), \tilde{\nabla}_{E_a} \tau_b(\varphi) \rangle \\
&\quad \quad - \langle d\varphi(\pi(\nabla^g_{E_a} E_a)), \tau_b(\varphi) \rangle \} \\
&= \left\langle d\varphi \left( \pi \left( - \sum_{i=1}^p \nabla^g_{E_i} E_i \right) \right) \right. \\
&\quad \left. + \sum_{a=1}^q \{ \tilde{\nabla}_{E_a}(d\varphi(E_a)) - d\varphi(\pi(\nabla^g_{E_a} E_a)) \}, \tau_b(\varphi) \right\rangle. \quad (3.11)
\end{aligned}$$

since  $\tilde{\nabla}_{E_a} \tau_b(\varphi) = 0$  in the last equality of (3.11). Integrating the both hands of (3.11) over  $M$ , we have

$$\begin{aligned}
&\int_M \left\langle d\varphi \left( \pi \left( \sum_{i=1}^p \nabla^g_{E_i} E_i \right) \right), \tau_b(\varphi) \right\rangle v_g \\
&= \int_M \left\langle \sum_{a=1}^q \{ \tilde{\nabla}_{E_a}(d\varphi(E_a)) - d\varphi(\pi(\nabla^g_{E_a} E_a)) \}, \tau_b(\varphi) \right\rangle v_g. \quad (3.12)
\end{aligned}$$

because of  $\int_M \operatorname{div}(\alpha^\#) v_g = 0$ . Notice that the both hands in (3.12) are well defined because of  $E(\varphi) < \infty$  and  $E_2(\varphi) < \infty$ .

Since  $\kappa^\# := \pi(\sum_{i=1}^p \nabla^g_{E_i} E_i)$  is the second fundamental form of each leaf  $L$  in  $(M, g)$  and

$$\begin{aligned}
 \tau_b(\varphi) &= \sum_{a=1}^q \{ \tilde{\nabla}_{E_a}(d\varphi(E_a)) - d\varphi(\nabla^g_{E_a} E_a) \} \\
 &= \sum_{a=1}^q \{ \tilde{\nabla}_{E_a}(d\varphi(E_a)) - d\varphi(\pi(\nabla^g_{E_a} E_a)) \} - d\varphi \left( \left( \sum_{a=1}^q \nabla^g_{E_a} E_a \right)^\perp \right),
 \end{aligned}
 \tag{3.13}$$

the right hand side of (3.12) coincides with

$$\int_M \left\langle \tau_b(\varphi) + d\varphi \left( \left( \sum_{a=1}^q \nabla^g_{E_a} E_a \right)^\perp \right), \tau_b(\varphi) \right\rangle v_g,
 \tag{3.14}$$

(3.12) is equivalent to that

$$\begin{aligned}
 &\int_M \langle d\varphi(\kappa^\#), \tau_b(\varphi) \rangle v_g \\
 &= \int_M \langle \tau_b(\varphi), \tau_b(\varphi) \rangle v_g + \int_M \left\langle d\varphi \left( \left( \sum_{a=1}^q \nabla^g_{E_a} E_a \right)^\perp \right), \tau_b(\varphi) \right\rangle v_g.
 \end{aligned}
 \tag{3.15}$$

Finally,  $\varphi : (M, g) \rightarrow (M', g')$  satisfies the conservation law, then it holds due to Proposition 2.6 that  $\langle d\varphi_x(Q_x), \tau_b(\varphi) \rangle = 0$ . Furthermore, recall that  $X^\perp$  ( $X \in \mathfrak{X}(M)$ ) is the  $Q$ -component of  $X \in \mathfrak{X}(M)$  relative to the decomposition  $TM = L \oplus Q$  of the bundles. Therefore, these imply that both the left hand side and the second term of the right hand side of (3.15) must vanish. That is, we obtain that  $\int_M \langle \tau_b(\varphi), \tau_b(\varphi) \rangle v_g = 0$ . Therefore  $\tau_b(\varphi) \equiv 0$ . We have Theorem 2.11.  $\square$

#### 4. Appendix

Here, we give a proof of Theorem 2.2. For the first part of the proof, see Appendix, Page 271 in [24]. We give a proof of the latter half.

**Theorem 4.1** (cf. Theorem 2.2) *Let  $(M, g)$  be a non-compact complete Riemannian manifold without boundary, If a  $C^1$  vector field  $X$  on  $M$  satisfies that*

$$\int_M |X| v_g < \infty \quad \text{and} \quad \int_M \operatorname{div}(X) v_g < \infty.
 \tag{4.1}$$

Then, it holds that

$$\int_M \operatorname{div}(X) v_g = 0. \tag{4.2}$$

Furthermore, if  $f \in C^1(M)$  and a  $C^1$  vector field  $X$  on  $M$  satisfy  $\operatorname{div}(X) = 0$ ,  $\int_M Xf v_g < \infty$ ,  $\int_M |f|^2 v_g < \infty$  and  $\int_M |X|^2 v_g < \infty$ , then it holds that

$$\int_M Xf v_g = 0. \tag{4.3}$$

*Proof.* (The first step) For  $f \in C_c^2(M)$  ( $f \in C^2(M)$  with compact support) and a  $C^1$  vector field  $X$  on  $M$  satisfying  $\operatorname{div}(X) = 0$ , let us define  $m$ -form  $\omega = f v_g$ , ( $m = \dim M$ ). Then, the Lie derivative  $L_X \omega$  of  $\omega$  by  $X$  is calculated as follows:

$$\begin{cases} L_X \omega = Xf v_g + f L_X v_g = Xf v_g + f \operatorname{div}(X) v_g = Xf v_g, \\ L_X \omega = i_X d\omega + d i_X \omega = d i_X \omega \end{cases} \tag{4.4}$$

due to  $\operatorname{div}(X) = 0$ , the H. Cartan's identity and  $d\omega = 0$ , where  $i_X K$  is the interior product of a tensor field  $K$  by  $X$ . By (4.1), we have

$$\int_M Xf v_g = \int_M L_X \omega = \int_M d i_X \omega = \int_{\partial M} i_X \omega = 0 \tag{4.5}$$

because each integral is finite due to  $f \in C_c^2(M)$ , and  $\partial M = \emptyset$ .

(The second step) Let us take  $f \in C^1(M)$  and a  $C^1$  vector field  $X$  on  $M$  satisfying  $\operatorname{div}(X) = 0$  and  $\int_M Xf v_g < \infty$ . Then there exists a sequence  $f_n \in C^2(M)$  ( $n = 1, 2, \dots$ ) such that  $f_n \rightarrow f$  in the  $C^1$  topology in a Riemannian manifold  $(M, g)$ . Then, it holds that

$$\int_M Xf_n v_g \rightarrow \int_M Xf v_g \tag{4.6}$$

in the  $C^0$  topology in  $(M, g)$ .

(The third step) Let us take a cutoff function  $\mu$  from a fixed point  $x_0 \in M$  on  $(M, g)$  as in the first step of the proof of Theorem 2.11 in Section Three.

Applying the first step to the functions  $f_n \mu \in C_c^2(M)$ , it holds that

$$\int_M X(f_n \mu) v_g = 0. \tag{4.7}$$

But, we have

$$\int_M X(f_n \mu) v_g = \int_M (X f_n) \mu v_g + \int_M f_n (X \mu) v_g. \tag{4.8}$$

By (4.4) and (4.5), we have,

$$\begin{aligned} \left| \int_M (X f_n) \mu v_g \right| &= \left| - \int_M f_n (X \mu) v_g \right| \leq \int_M |f_n| |X \mu| v_g \\ &\leq \int_M |f_n| |X| |\nabla \mu| v_g \\ &\leq \frac{2}{r} \int_M |f_n| |X| v_g \\ &\leq \frac{2C}{r} \int_M |f| |X| v_g \leq \frac{2C}{r} \|f\| \|X\| \end{aligned} \tag{4.9}$$

with  $\|f\|^2 = \int_M |f|^2 v_g < \infty$  and  $\|X\|^2 = \int_M |X|^2 v_g < \infty$  for a certain positive constant  $C > 0$ . Tending  $r \rightarrow \infty$  in (4.6), since the right hand side of (4.6) goes to zero,

$$\int_M (X f_n) \mu v_g \longrightarrow 0 \quad (\text{as } r \rightarrow \infty). \tag{4.10}$$

On the other hand, as  $r \rightarrow \infty$ ,

$$\int_M (X f_n) \mu v_g \longrightarrow \int_M X f_n v_g \tag{4.11}$$

which implies that

$$\int_M X f_n v_g = 0. \tag{4.12}$$

Due to (4.3), as  $n \rightarrow \infty$ , we have

$$\int_M Xf v_g = 0 \quad (4.13)$$

which is the desired.  $\square$

**Acknowledgement** The last author would like to express his gratitude to Professor Seoung Dal Jung who invited him at Jeju National University at January, 2016, and noticed to the authors the errors in the manuscripts of [17] and [18]. This work has been started during the period of this period. Finally not the least, the authors express their thanks to the referee who pointed several errors in the first draft.

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