COUNTABLE DECOMPOSABILITY OF VECTOR LATTICES

By

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D. M. Topping conjectured in [1] that free (and therefore projective) vector lattices are countably decomposable, that is, they do not contain more than countable number of mutually orthogonal positive elements. The main purpose of this paper is to prove this fact. In the case of countable generators, this was essentially proved by E. C. Weinberg in [2].

First we reduce the problem to the case of finite generators making use of an argument which can be applied to any free algebraic system; it provides, for instance, a proof of the countable decomposability of free Boolean algebras.

Next we give a simple proof of the fact that every countably generated semi-simple vector lattice is countably decomposable. (The semi-simplicity of free vector lattices was shown in [1] and [2].)

Finally we show that the semi-simplicity is redundant in the above result in the case of at most two generators and otherwise necessary.

1. Reduction to the finite case.

Let $L(A)$ be the free vector lattice generated by a set A of generators and $\{x_\lambda\}_{\lambda\in\Lambda}$ an uncountable orthogonal system of positive elements in $L(A)$.

We assign, for every $\lambda\in\Lambda$, a finite subset A_λ of A such that x_λ is contained in $L(A_\lambda)$ considered as a sublattice of $L(A)$.

Replacing $\{A_\lambda\}$, if necessary, by a suitable uncountable subsystem, we can suppose that every A_λ is the disjoint union of a fixed subset B and C_λ where $\{C_\lambda\}$ is mutually disjoint system and each C_λ contains the same number of elements. To prove this, first take a subsystem for which A_λ contains the same number of elements. Then, take a maximal subset B of A with the property that A_λ includes B for more than countable unmber of λ.

Finally, choose a maximal mutually disjoint subsystem among $\{A_\lambda - B\}$. By virtue of the maximality of B, this subsystem must be uncountable.

Let C be a subset of A which is disjoint from B and contains the same number of elements with C_λ. Replacing the generators in C_λ by those in C, we obtain an isomorphism

$$\varphi_\lambda: \quad L(A_\lambda)\to L(B\cup C).$$

Since φ_λ and φ_μ can be extended at the same time to a homomorphism of $L(A_\lambda \smile A_\mu)$ to $L(B \smile C)$ $\{\varphi_\lambda(x_\lambda)\}_{\lambda \in \Lambda}$ constitues an orthogonal system in $L(B \smile C)$.

Thus our problem is reduced to the case of finite generators.

2. The case of countable generators.

Let L be a semi-simple vector lattice. Then L can be considered as a lattice of functions defined on a set X. Suppose L is generated by functions f_i $i=1,2,\cdots$.

L can be considered as a sublattice of all continuous functions on X if we define the topology of X as the weakest topology for which every f_i is continuous.

X is separable by this topology and hence can not contain uncountable mutually disjoint family of non-void open sets.

Thus L must be countably decomposable.

3. The case of two generators.

Every free vector lattice with two generators is isomorphic to the lattice L of functions on the real line which is generated by the constants and the identity. L consists of those continuous functions which are linear except at finite number of points.

Let J be an ideal of L and $\{f_\lambda\}_{\lambda \in \Lambda}$ an uncountable system of positive functions in L with the property that every intersection $f_\lambda \frown f_\mu$ with $\lambda \neq \mu$ belongs to J. We are going to prove that some f_λ itself belongs to J.

Let S be the set of those points at which every function in J vanishes. If S is empty, then every function in L with compact support belongs to J and hence, as can be seen easily, L/J is isomorphic to a quotient of the direct sum of a pair of two dimensional lexicographically ordered space; L/J allows at most only two mutually orthogonal elements.

Suppose $S \neq \phi$ then the complement of S is the union of countable or finite family of mutually disjoint open intervals I_i $i=1,2,\cdots$. Let φ be the homomorphism of L into the lattice $C(S)$ of continuous functions on S defined by restriction, then we have $\varphi(f_\lambda)=0$ except for a countable number of λ, since $C(S)$, as has been shown in 2, is countably decomposable. So we can suppose that every f_λ belongs to the Kernel L_0 of φ.

For every i and every mutually different triple λ, λ_2 and λ_3 in Λ, at least one of the restrictions of $f_{\lambda_1}, f_{\lambda_2}$ and f_{λ_3} to I_i is the restriction to I_i of a function in J, by a similar reason as has been described for the case where S is espty.

On the other hand, every function in L_0, having the polygonal graph with finite verteces, vanishes on I_i except for a finite number of i's. Therefore the number of those λ for which $f_\lambda \notin J$ is at most countable.

Thus we have proved that every quotient of L and therefore every lattice generated by two elements is countably decomposable.

4. A vector lattice which is generated by three elements and not countably decomposable.

Let L be the vector lattice of functions on the coordinated plane which is generated by the constants and the two coordinate functions g and h where $g(\alpha, \beta)=\alpha$ and $h(\alpha, \beta)=\beta$. (L is isomorphic to a free vector lattice with three generators.)

Let J be the ideal of L which consists of all functions in L whose supports are disjoint with the diaonal line.

For every real number α, we define the function f_α as

$$f_\alpha = \{(g-\alpha)_\frown(\alpha-h)\}^\smile 0 .$$

The intersection of the support of f_α with the diagonal line cosists of only one point (α, α) and every $f_\alpha{}_\frown f_\beta$ with $\alpha \neq \beta$ belongs to J.

So the images of f_α's in the quotient lattice L/J constitute an uncountable orthogonal system.

References

[1] D. M. TOPPING: Some homological pathology in vector lattices, Canad. J. Math. 17 (1965) 411–428.

[2] E. C. Weinberg: Free lattice ordered abelian groups, Math. Ann., 151 (1963) 187–199.

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