A CHARACTERIZATION OF A PARTICULAR CLASS OF HELSON SETS

By

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Let G be a locally compact abelian group, Γ be its dual group, and $M(G)$ be the Banach algebra of all regular complex measures on G. A compact subset P of G is called a *Kronecker set* ($=K$-set) if every modular continuous function on P is uniformly approximable ($=u$-approximable) by continuous characters of G. P is called a *Helson set* if for some $\varepsilon>0$ and each $\mu\in M(P)$

$$(H,\varepsilon)\qquad \sup_{\gamma\in\Gamma}\left|\int_G \gamma(-x)d\mu(x)\right|\geqq\varepsilon\|\mu\|,\qquad \text{where}\quad \|\mu\|=|\mu|(P).$$

If P is a K-set, P has the property $(H, 1)$. W. Rudin asked in [1] whether the property $(H, 1)$ implies that P is a K-set, and I. Wik [2] and R. Kaufman [3] show counter examples.

The purpose of this paper is to prove the following theorem which is a characterization of the set with the property $(H, 1)$.

Theorem. *Let G be a locally compact abelian group, P be a metrizable compact subset of G. Then P satisfies $(H, 1)$ if and only if the following conditions are satisfied.*

1) *For every finite subset $\{x_0, x_1, x_2, \cdots, x_n\}$ of P, $\{x_0-x_1, x_0-x_2, x_0-x_3, \cdots, x_0-x_n\}$ is a K-set.*

2) *For each $\mu\in M(P)$, there exist open subsets C_i in P and $x_0\in P$ such that*

a) $C_1\supset C_2\supset C_3\supset C_4\supset\cdots\ni x_0$

b) $\lim_{i\to\infty}|\mu|(C_i)=|\mu|\ \{x_0\}$

c) *Every closed subset of $P-C_i$ $(i=1, 2, 3, \cdots)$ is a K-set.*

Later we show an example which shows that a closed subset of a K-set is not always a K-set.

Notations. Throughout this paper, G is used to denote a locally compact abelian group. we call a Borel set with the property $(H, 1)$ a $(H, 1)$-set, and by characters we mean continuous characters of G. T, $C(G)$ is used to denote the group of all complex numbers of absolute value 1, and the set of all continuous functions on G respectively. Let g be a function on G and P be a subset of G, then by $g|P$, we mean the restriction of g to P.

Definition. We call a compact subset P of G a *K'-set*, if for each modular continuous function g on P there exists $\beta\in T$ such that βg is *u*-approximable by characters.

Lemma 1. *Let B be a $(H, 1)$-set in G, and g be a modular continuous function on B. Then for each $\mu\in M(B)$ and $\{\delta_i:\delta_i>0,\ i=1,2,3,\cdots\}$, there exist $\beta\in T$, and open subsets C_i in B such that*

$$\begin{cases} 1)\ \ |\mu|(C_i)<\delta_i,\ C_1\supset C_2\supset C_3\supset C_4\supset\cdots, \\ 2)\ \ \beta g|B-C_i \text{ is } u\text{-approximable by characters.} \qquad (i=1,2,3,\cdots) \end{cases}$$

Proof. Since B is a $(H, 1)$-*set*, there exist $\gamma_1\in\Gamma$ and $\alpha_1\in T$ for which

$$(1.\ 1)\qquad \int_B \alpha_1\gamma_1(-x)g(x)d|\mu|(x)=\left|\int_B \gamma_1(-x)g(x)d|\mu|\right| \geqq |\mu|(B)-1/32$$

and we have

$$(1.\ 2)\qquad \int_B\left[1-Re\left\{\alpha_1\gamma_1(-x)g(x)\right\}\right]d|\mu|\leqq 1/32$$

Put $A_1=\{x\in B:1-Re[\alpha_1\gamma_1(-x)g(x)]>1/16\}$ Then from the relation $|\mu|(A_1)\cdot(1/16)<1/32$, we have $|\mu|(A_1)<1/2$. On the other hand, the relation

$$(1.\ 3)\qquad 1-Re\left[\alpha_1\gamma_1(-x)g(x)\right]\leqq 1/16 \qquad (x\in B-A_1)$$

gives

$$(1.\ 4)\qquad \left|Im\left[\alpha_1\gamma_1(-x)g(x)\right]\right|\leqq 1/2\sqrt{2} \qquad (x\in B-A_1)$$

and hence

$$(1.\ 5)\qquad \left|\gamma_1(x)-\alpha_1 g(x)\right|\leqq\left|1-Re\left[\alpha_1\gamma_1(-x)g(x)\right]\right|+\left|Im\left[\alpha_1\gamma_1(-x)g(x)\right]\right| \leqq 1/16+1/2\sqrt{2}<1/2 \qquad (x\in B-A_1)$$

In the same way, we get open subsets A_i of B, $\gamma_i\in\Gamma$, $\alpha_i\in T$ $(i=2,3,4,\cdots)$ such that

$$(1.\ 6)\qquad \begin{cases} \left|\gamma_i(x)-\alpha_i g(x)\right|<1/i \qquad (x\in B-A_i) \\ |\mu|(A_i)<1/2^i \end{cases} \qquad (i=1,2,3,\cdots)$$

Let n_i be positive inetegers such that

$$(1.\ 7)\qquad \begin{cases} n_1<n_2<n_3<n_4<\cdots, \\ |\mu|\left(\bigcup_{k=n_i}^{\infty} A_k\right)<\delta_i \qquad (i=1,2,3,\cdots) \end{cases}$$

Choose a subsequence $\{\alpha_{j_i}\}$of $\{\alpha_i\}$ which converges to some $\beta\in T$. Then if we put $\bigcup_{k=n_i}^{\infty} A_k=C_i$ $(i=1,2,3,\cdots)$, $\beta g\,|\,B-C_i$ is u-approximable by characters, and the lemma is proved.

Proposition 2. *Let P be a compact metrizable $(H, 1)$-set in G. Then for each $\mu\in M(P)$, there exist open subsets C_i in P such that*

1) $C_1\supset C_2\supset C_3\supset C_4\supset\cdots,$
2) $\lim_{i\to\infty}|\mu|(C_i)=0,$
3) *Every closed subset of $P-C_i$ $(i=1,2,3,\cdots)$ is a K'-set.*

Proof. Since P is metrizable, $C(P)$ has a countable dense subset with respect to sup-norm, say $\{f_1, f_2, f_3, \cdots\}$. Put $B_i=\{x\in P: 1/2<|f_i(x)|<3/2\}$ $(i=1,2,3,\cdots)$, and since B_i is also $(H, 1)$-set, by Lemma 1 there exist open subsets C_{ij} in B_i and $\beta_i\in T$ such that

$$(2.1)\qquad \begin{cases} |\mu|(C_{ij})<1/2^{i+j}, & C_{i1}\supset C_{i2}\supset C_{i3}\supset C_{i4}\supset\cdots, \\ \beta_i f_i/|f_i|\,\big|\,B_i-C_{ij} \text{ is } u\text{-approximable by characters.} \end{cases}\qquad (i,j=1,2,\cdots)$$

Put $C_j=\bigcup_{i=1}^{\infty} C_{ij}$, and we have

$$(2.2)\qquad |\mu|(C_j)<1/2^j\ (j=1,2,3,\cdots),\ C_1\supset C_2\supset C_3\supset C_4\supset\cdots.$$

Let D be a closed subset of $P-C_j$, and g be a modular continuous function on D. We can extend g to a continuous function g_1 on P and there exists a sequence of positive integers $n_1, n_2, n_3, \cdots$, such that

$$(2.3)\qquad \|f_{n_i}-g_1\|_\infty<2^{-i}\qquad (i=1,2,3,\cdots)$$

and $\beta_{n_1}, \beta_{n_2}, \beta_{n_3}, \beta_{n_4}, \cdots$, converge to some $\beta\in T$. If $\varepsilon>0$, then there exist a positive integer k and $\gamma\in\Gamma$ such that

$$(2.4)\qquad \begin{cases} |\beta-\beta_{n_k}|<\varepsilon/4, \quad 2^{-k}<\varepsilon/4 \\ \Big|\gamma(x)-\beta_{n_k}f_{n_k}(x)/\big|f_{n_k}(x)\big|\Big|<\varepsilon/4 \qquad (x\in B_{n_k}-C_j) \end{cases}$$

From (2.3), $B_{n_k}\supset D$, and hence $B_{n_k}-C_j\supset D$.
From (2.3) and (2.4), we obtain

$$(2.5)\qquad \varepsilon/4>2^{-k}>\|f_{n_k}-g_1\|_\infty\geqq\big|f_{n_k}(x)\big|-1\qquad (x\in D)$$

and hence

$$(2.6)\qquad \Big|\beta_{n_k}f_{n_k}(x)-\beta_{n_k}f_{n_k}(x)/\big|f_{n_k}(x)\big|\Big|=\Big|\big|f_{n_k}(x)\big|-1\Big|<\varepsilon/4\qquad (x\in D)$$

From (2.3), (2.4) and (2.6) we have

$$(2.7)\qquad \left|\beta g(x)-\gamma(x)\right| \leqq \left|\beta g(x)-\beta_{n_k} g(x)\right| + \left|\beta_{n_k} g(x)-\beta_{n_k} f_{n_k}(x)\right|$$
$$+\left|\beta_{n_k} f_{n_k}(x)-\beta_{n_k} f_{n_k}(x)/\left|f_{n_k}(x)\right|\right| + \left|\beta_{n_k} f_{n_k}(x)/\left|f_{n_k}(x)\right|-\gamma(x)\right|$$
$$\leqq \varepsilon/4+\varepsilon/4+\varepsilon/4+\varepsilon/4=\varepsilon \qquad (x\in D)$$

Since $\varepsilon>0$ can be arbitrary small, this proves that D is a K'-set.

Lemma 3. *Let Q be a compact subset of G, and suppose every closed subset of Q is a K'-set. If every modular constant function on Q is u-approximable by characters, then every closed subset of Q is a K-set.*

Proof. Let Q_1 be a closed subset of Q, and Ω be the set of all continuous functions on Q_1 which are u-approximable by characters. Then Ω form a group with respect to multiplication. Let g be a modular continuous function on Q_1, then there exists $\alpha\in T$ such that $\alpha g\in\Omega$, and constant function $\alpha\in\Omega$ and so $\alpha^{-1}\alpha g=g\in\Omega$. This proves that Q_1 is a K-set.

Lemma 4. *Let Q be a compact subset of G, and suppose every closed subset of Q is a K'-set. Then for every pair (V, K), where V is open in Q, K is compact and $Q\supset V\supset K$, either of the following a) or b) is true.*

a) *Every closed subset of $Q-V$ is a K-set.*

b) *Every modular continuous function h on $Q-(V-K)$ of the form*

$$(4.1)\qquad h(x)=\begin{cases} 1 : x\in Q-V \\ c : x\in K, \end{cases} \qquad \textit{where}\quad c\in T,$$

is u-approximable by characters.

Proof. Let g be a modular continuous function on Q such that

$$(4.2)\qquad g(x)=\begin{cases} \alpha : x\in Q-V \\ 1 : x\in K \\ * : x\in V-K, \end{cases} \quad \text{where } \alpha\in T \text{ and } \alpha \text{ is of infinite order.}$$

Then there exists $\beta\in T$ such that βg is u-approximable by characters.

a) If $\alpha\beta$ is of infinite order, then every modular constant function on $Q-V$ is u-approximable by characters, and by lemma 3 every closed subset of $Q-V$ is a K-set.

b) If $\alpha\beta$ is of finite order, then there exists a positive integer m such that $(\alpha\beta)^m=1$, so the function

$$(4.2)\qquad (\beta g)^m\Big|_{Q-(V-K)}(x)=\begin{cases} 1 : x\in Q-V \\ \beta^m : x\in K \end{cases}$$

is u-approximable by characters. Since β^m is of infinite order, the function of the form (4.1) is u-approximable by characters, and the lemma is proved.

Proposition 5. *Let P be a compact metrizable $(H, 1)$-set in G. Then for each $\mu \in M(P)$, either of the following a) or b) is true.*

a) *There exist open subsets C_i in P and $x_0 \in P$* such that

1) $P \supset C_1 \supset C_2 \supset C_3 \supset C_4 \supset \cdots \ni x_0$,

2) $\lim_{i \to \infty} |\mu|(C_i) = |\mu|\{x_0\}$,

3) *Every closed subset of $P - C_i$ $(i = 1, 2, 3, \cdots)$ is a K-set.*

b) *There exist open subsets C_i in P such that*

1) $P \supset C_1 \supset C_2 \supset C_3 \supset C_4 \supset \cdots$,

2) $\lim_{i \to \infty} |\mu|(C_i) = 0$,

3) *Every closed subset of $P - C_i$ $(i = 1, 2, 3, \cdots)$ is a K-set.*

Proof. By Prop. 2 there exist open subsets C_i' of P such that

$$(5.1) \qquad \begin{cases} C_1' \supset C_2' \supset C_3' \supset C_4' \supset \cdots, \\ \lim_{i \to \infty} |\mu|(C_i') = 0 \\ \text{Every closed subset of } P - C_i' \ (i = 1, 2, 3, \cdots) \text{ is a } K'\text{-set.} \end{cases}$$

Put $Q_i' = P - C_i'$, then we have $P - \bigcap_{i=1}^{\infty} C_i' = \bigcup_{i=1}^{\infty} Q_i'$, $Q_1' \subset Q_2' \subset Q_3' \subset Q_4' \subset \cdots$, and every closed subset of Q_i' is a K'-set. For each $x \in \bigcup_{i=1}^{\infty} Q_i'$, put

$$(5.2) \qquad m_x = \sup\Big[\{0\} \cup \{i : x \in Q_i' \text{ and every closed subset of } Q_i' \text{ which don't contain } x \text{ is a } K\text{-set}\}\Big].$$

Case 1. Suppose there exists $x_0 \in \bigcup_{i=1}^{\infty} Q_i'$ for which $m_{x_0} = \infty$. Then there exist a sequence of open subsets U_i in P such that

$$(5.3) \qquad \begin{cases} U_1 \supset U_2 \supset U_3 \supset U_4 \supset \cdots \ni x_0 \\ \lim_{i \to \infty} |\mu|(U_i) = |\mu|\{x_0\} \end{cases}$$

If we set $C_i = (C_i' \cup U_i)$ $(i = 1, 2, 3, \cdots)$ we have a).

Case 2. Next consider the case $m_x < \infty$ for all $x \in \bigcup_{i=1}^{\infty} Q_i'$. In this case, we can correspond to each $x \in \bigcup_{j=1}^{\infty} Q_j'$ an open neiberhood U_x of x in P such that for every triple (Q_j', U, K), where $j \in \{m_x + 1, m_x + 2, m_x + 3, \cdots\}$, U is an open subset in $U_x \cap Q_j'$ and K is a compact subset of U, the case b) of the

Lemma 4. is true.

Case 2. a. Suppose the set $\left\{m_x : x \in \bigcup_{j=1}^{\infty} Q'_j\right\}$ is not bounded. Then we can choose $x_i \in \bigcup_{j=1}^{\infty} Q'_j$ and open nelberhoods V_i of x_i in P such that

$$(5, 4) \qquad \begin{cases} 0 < m_{x_1} < m_{x_2} < m_{x_3} < m_{x_4} < m_{x_5} < \cdots, \\ |\mu|(V_i) < 2^{-i} \qquad (i=1, 2, 3, \cdots). \end{cases}$$

If we put $C_j = C'_j \cup \left(\bigcup_{i=j}^{\infty} V_i\right)$ $(j=1, 2, 3, \cdots)$, then we have the case b).

Case 2. b. Finally consider the case $\left\{m_x : x \in \bigcup_{i=1}^{\infty} Q'_i\right\}$ is bounded. In this case, we can suppose without loss of generality that $m_x = 0$ for all $x \in \bigcup_{i=1}^{\infty} Q'_i$. Since Q'_i is compact, we can choose a sequence $x_1, x_2, x_3, \cdots \in \bigcup_{i=1}^{\infty} Q'_i$ and an increasing sequence of positive integers $t_1, t_2, t_3, \cdots$, such that

$$(5. 5) \qquad \begin{cases} \{x_1, x_2, x_3, \cdots, x_{t_k}\} \subset Q'_k \\ \bigcup_{i=1}^{t_k} U_{x_i} \supset Q'_k \qquad (k=1, 2, 3 \cdots) \end{cases}$$

Since μ is regular, we can choose compact subsets $K_i^{(n)}$ such that

$$(5. 6) \qquad \begin{cases} K_i^{(n)} \subset U_{x_i} \cap Q'_n \quad (i=1, 2, \cdots, t_n ;\ n=1, 2, \cdots), \\ Q_1 \subset Q_2 \subset Q_3 \subset Q_4 \subset Q_5 \subset \cdots, \\ \lim_{i\to\infty} |\mu|(Q_i) = |\mu|(P). \end{cases}$$

where $Q_n = \bigcup_{j=1}^{t_n} \left(K_j^{(n)} - \bigcup_{i=0}^{j-1} U_{x_i}\right)$, $U_{x_0} = \phi$ $(n=1, 2, 3, \cdots)$.

By the property of U_x, every modular constant function on Q_n is u-approximable by characters, and by Lemma 3 every closed subset of Q_n is a K-set. If we set $C_i = P - Q_i$, we have the case b) and this completes the proof.

Corollary. *Let P be a metrizable $(H, 1)$-set in G. Then for each $\mu \in M(P)$ there exist open subsets C_i in P and $x_0 \in P$ such that*

1) $P \supset C_1 \supset C_2 \supset C_3 \supset C_4 \supset \cdots \ni x_0$,
2) $\lim_{i\to\infty} |\mu|(C_i) = |\mu|\{x_0\}$,
3) *Every closed subset of $P - C_i$ $(i=1, 2, 3, \cdots)$ is a K-set.*

Proof. By prop. 5, a) or b) of the prop. is true. In the case a) we have nothing to prove. In the case b) we can select $x_0 \in P$ and open neiberhoods W_i of x_0 in P such that

$$\begin{cases} W_1 \supset W_2 \supset W_3 \supset W_4 \supset \cdots \ni x_0 \\ |\mu|(W_i) < 1/2^i \end{cases} \qquad (i=1, 2, 3, \cdots).$$

If we rewrite $C_i \cup W_i$ by C_i, $\{C_i : i=1, 2, 3, \cdots\}$ and x_0 satisfies 1), 2), 3).

Lemma 6. *Let Q be a compact subset of G and suppose every closed subset of Q is a K-set. If G-$Q \ni x_0$, $Q \supset V \supset K$, where V is open in Q and K is compact, either of the following a) or b) is true.*

a) *$(Q-V) \cup \{x_0\}$ is a K-set.*

b) *Every continuous function of the form*

$$k(x) = \begin{cases} c : x \in K \\ 1 : x \in (Q-V) \cup \{x_0\}, \text{ where } c \in T, \end{cases} \tag{6.1}$$

is u-approximable by characters.

Proof. Let h be a modular continuous function on $Q-(V-K)$ of the form

$$h(x) = \begin{cases} \alpha : x \in K \\ 1 : x \in Q-V, \quad \text{where } \alpha \in T \text{ is of infinite order.} \end{cases} \tag{6.2}$$

Then there exist $\gamma_i \in \Gamma$ $(i=1, 2, 3, \cdots)$ such that

$$\left|\gamma_i(x) - h(x)\right| < 1/i \qquad \big(x \in Q-(V-K)\big) \tag{6.3}$$

and we can choose subsequense $\{\gamma_{n_1}(x_0), \gamma_{n_2}(x_0), \cdots\}$ of $\{\gamma_1(x_0), \gamma_2(x_0), \cdots\}$ which converge to some $\beta \in T$.

a) If β is of infinite order, then $(Q-V) \cup \{x_0\}$ is a K-set.

b) If β is of finite order, there exists a positive integer m such that $\beta^m = 1$ and $\{(\gamma_{n_i})^m | (Q \cup \{x_0\} - (V-K) : i=1, 2, 3, \cdots\}$ converge uniformly to h_1, where

$$h_1(x) = \begin{cases} \alpha^m : x \in K \\ 1 \quad : x \in (Q-V) \cup \{x_0\} \end{cases} \tag{6.4}$$

and so the function of the form (6.1) is u-approximable by characters.

Proposition 7. *Let P be a compact subset of G. Suppose*

1) *Every finite subset of P is a K'-set.*

2) *For each $\mu \in M(P)$, there exist open subsets C_i of P and $x_0 \in P$ such that*

a) $C_1 \supset C_2 \supset C_3 \supset C_4 \supset \cdots \ni x_0$

b) $\lim |\mu|(C_i) = |\mu|\{x_0\}$

c) *Every closed subset of $P - C_i$ $(i=1, 2, 3, \cdots)$ is a K-set.*

Then P is a $(H, 1)$-set.

Proof. If $\mu\in M(P)$, by condition 2) there exist C_i $(i=1, 2, 3, \cdots)$ and $x_0\in P$ for which a), b), c) hold.

(I) *The case when* $\mu\{x_0\}=0$. In this case, the fact a K-set is a $(H, 1)$-set and a simple calculation lead us to

$$|\mu|(P)=\sup_{\gamma\in\Gamma}\left|\int_G \gamma(-x)d\mu(x)\right| \tag{7.1}$$

(II) *The case when* $\mu\{x_0\}\neq 0$. Fix $\varepsilon>0$ and positive integer i_0, such that

$$|\mu|(C_{i_0}-x_0)<\varepsilon/2^2. \tag{7.2}$$

We can choose $\{x_1, x_2, x_3, \cdots, x_n\}\subset P-C_{i_0}$, and open neiberhoods U_i of x_i $(i=1, 2, 3, \cdots, n)$ in $P-C_{i_0}$, such that

$$\begin{cases} |\mu|\{x_i\}\geqq\varepsilon/2 & (i=1, 2, 3, \cdots, n) \\ |\mu|\{x\}<\varepsilon/2 & \left(x\in P-C_{i_0}\cup\{x_1, x_2, x_3, \cdots, x_n\}\right) \\ |\mu|\left(\bigcup_{i=1}^{n}U_i-\left\{x_1, x_2, x_3, \cdots, x_n\right\}\right)<\varepsilon/2^3 & \end{cases} \tag{7.3}$$

We set $\bigcup_{i=1}^{n}U_i=U$ and $P-C_{i_0}=Q$.

(II) a. At first consider the case there exists an open subset V in Q such that $|\mu|V<\varepsilon/2$ and $(Q-V)\cup\{x_0\}$ is a K-set. In this case, as in the case (I) there exists $\gamma\in\Gamma$ for which

$$|\mu|(P)-\varepsilon<\left|\int_G \gamma(-x)d\mu(x)\right| \tag{7.4}$$

(II) b. Next consider the case, that for every open set V in Q with $|\mu|(V)<\varepsilon/2$, $(Q-V)\cup\{x_0\}$ is not a K-set. Then by Lemma 6, for every pair (V, K), where V is open in Q with $|\mu|(V)<\varepsilon/2$ and K is a compact subset of V, a continuous function of the form (6.1) is u-approximable by characters. Since $\{x_0, x_1, x_2, \cdots, x_n\}$ is a K'-set, there exist $d_1\in T$, $\gamma_1\in\Gamma$, such that

$$\sum_{i=0}^{n}d_1\gamma_1(-x_i)\mu\{x_i\}>\sum_{i=0}^{n}|\mu|\{x_i\}-\varepsilon/2^5 \tag{7.5}$$

Also since $Q-U$ is a K-set, there exists $d_2\in T$, $\gamma_2\in\Gamma$, such that

$$\int_{Q-U}d_1d_2(\gamma_1+\gamma_2)(-x)d\mu(x)>|\mu|(Q-U)-\varepsilon/2^6. \tag{7.6}$$

From (7.3), to each $x\in Q-U$ we can correspond an open neiberhood V_x of x in Q with

(7. 7) $\quad |\mu|(V_x)<\varepsilon/2,\ \left|\gamma_2(x)-\gamma_2(y)\right|<\varepsilon/2^9\|\mu\| \qquad (y\in V_x)$

and since $Q-U$ is compact and μ is reguler, we can choose $\{y_1, y_2, \cdots, y_m\}$ $\subset Q-U$ and compact subset K_i of $V_{y_i}\cap(Q-U)$ such that

(7. 8) $$\bigcup_{i=1}^{m} V_{y_i}\supset Q-U,\quad |\mu|\left(\bigcup_{i=1}^{m}(K_i-\bigcup_{k=0}^{i-1}V_{y_k})\right)>|\mu|(Q-U)-\varepsilon/2^4,$$

where $V_{y_0}=\phi$. We set $\bigcup_{i=1}^{m}\left(K_i-\bigcup_{k=0}^{i-1}V_{y_k}\right)=K$. By the properties of V_{y_i} and K_i, we can choose $\gamma_3\in\Gamma$, such that

(7. 9) $$\begin{cases} \left|\gamma_3(-x)-d_2\gamma_2(-x)\right|<\varepsilon/2^8\|\mu\| & (x\in K) \\ \left|\gamma_3(-x)-1\right|<\varepsilon/2^8\|\mu\| & \left(x\in\{x_0, x_1, x_2, \cdots, x_n\}\right) \end{cases}$$

Hence from (7.2), (7.3), (7.5), (7.6), (7.8), (7.9), we obtain

(7. 10) $$\begin{aligned} \left|\int_G(\gamma_1+\gamma_3)(-x)d\mu\right| &= \left|\int_G d_1(\gamma_1+\gamma_3)(-x)d\mu\right| \\ &> \left|\int_{K\cup\{x_0,x_1,\cdots,x_n\}} d_1(\gamma_1+\gamma_3)(-x)d\mu\right| -(\varepsilon/2^2+\varepsilon/2^3+\varepsilon/2^4) \\ &\geqq \left|\int_{\{x_0,x_1,\cdots,x_n\}} d_1\gamma_1(-x)d\mu\right| + \left|\int_K d_1d_2(\gamma_1+\gamma_2)(-x)d\mu\right| \\ &\quad -\left|\int_{\{x_0,x_1,\cdots,x_n\}} \left\{d_1(\gamma_1+\gamma_3)(-x)-d_1\gamma_1(-x)\right\}d\mu\right| \\ &\quad -\left|\int_K \left\{d_1(\gamma_1+\gamma_3)(-x)-d_1d_2(\gamma_1+\gamma_2)(-x)\right\}d\mu\right| \\ &\quad -(\varepsilon/2^2+\varepsilon/2^3+\varepsilon/2^4) \\ &> |\mu|\{x_0,x_1,x_2,\cdots,x_n\}+|\mu|(K)-\sum_{i=2}^{6}\left(\varepsilon/2^i+\varepsilon/2^8+\varepsilon/2^8\right) \\ &> |\mu|(P)-\left(\sum_{i=2}^{7}\varepsilon/2^i+\varepsilon/2^2+\varepsilon/2^3+\varepsilon/2^4\right)>|\mu|(P)-\varepsilon. \end{aligned}$$

Since in (7.4), (7.10), $\varepsilon>0$ can be arbitrary small, (7.1) is also holds in the case (II), and this completes the proof.

Theorem. *Let P be a metrizable compact subset of a locally compact abelina group G, and Γ be its dual group. Then following a) and b) are equivalent.*

a) For each $\mu\in M(P)$, $|\mu|(P)=\sup_{\gamma\in\Gamma}\left|\int_G\gamma(-x)d\mu(x)\right|$ holds.

b) 1) For every finite subset $\{x_0, x_1, \cdots, x_n\}$ of P, $\{x_0-x_1, x_0-x_2, \cdots, x_0-x_n\}$ is a K-set.

2) For each $\mu\in M(P)$, there exist open subsets C_i in P and $y_0\in P$

such that

(i) $P \supset C_1 \supset C_2 \supset C_3 \supset \cdots \ni y_0$,

(ii) $\lim_{i \to \infty} |\mu| (C_i) = |\mu| \{y_0\}$

(iii) *Every closed subset of* $P - C_i$ $(i=1, 2, 3, \cdots)$ *is a* K*-set.*

Proof. a) implies b): a) implies that P is a $(H, 1)$-set, and by Cor. of prop. 2, 2) of b) holds. Let $x_i \in P$ $(i=0, 1, 2, \cdots, n)$, $c_i \in T$ $(i=1, 2, 3, \cdots, n)$ and $\mu \in M(P)$, with $\mu\{x_0\} = 1$, $\mu\{x_i\} = c_i$ $(i=1, 2, 3, \cdots. n)$, $\mu(E)=0$ if $E \cap \{x_0, x_1, \cdots, x_n\} = \phi$.

For $\varepsilon > 0$, there exists $\gamma \in \Gamma$ for which

$$\left| \sum_{i=0}^{n} \gamma(-x_i)\, \mu\{x_i\} \right| \geqq \sum_{i=0}^{n} |\mu| \{x_i\} - \varepsilon$$

and we have

$$\left| 1 + \sum_{i=1}^{n} \gamma(x_0 - x_i) c_i \right| = \left| \sum_{i=1}^{n} \gamma(-x_i) c_i + \gamma(-x_0) \right|$$

$$= \left| \sum_{i=0}^{n} \gamma(-x_i)\, \mu\{x_i\} \right| \geqq \sum_{i=0}^{n} |\mu| \{x_i\} - \varepsilon$$

Since $c_i \in T$ and $\varepsilon > 0$ is arbitrary, this implies that $\{x_0 - x_1, x_0 - x_2, \cdots, x_0 - x_n\}$ is a K-set.

b) implies a): This is true from prop. 7 and the fact a finite subset $\{x_0, x_1, \cdots, x_n\}$ of G is K'-set if and only if $\{x_0 - x_1, x_0 - x_2, \cdots, x_0 - x_n\}$ is a K-set.

Example. Let S be the closed unit disk with 0 in the center in the complex plane, and Γ be the set of all continuous functions from S into T. Γ form a discreat group with respect to multiplication. Let G be the dual group of Γ, and then S can be naturally imbedded in G (cf. Kaufman [2]). S is a Kronecker set, but $T(\subset S)$ is not so, since the identity function from T to T is not u-approximable by characters, and this shows a closed subset of a Kroneker set is not always a Kronecker set.

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