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E-polynomials associated to $\mathbf{Z}_4$-codes

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Abstract. Coding theory is connected with number theory via the invariant theory of some specified finite groups and theta functions. Under this correspondence we are interested in constructing, from a combinatorial point of view, an analogous theory of Eisenstein series. For this, we previously gave a formulation of E-polynomials based on the theory of binary codes. In the present paper we follow this direction and supply a new class of E-polynomials. To be precise, we introduce the E-polynomials associated to the $\mathbf{Z}_4$-codes and determine both the ring and the field structures generated by them. In addition, we discuss the zeros of the modular forms obtained from E-polynomials under the theta map.

Key words: E-polynomial, $\mathbf{Z}_4$-code.

1. Introduction

After the pioneering work of Gleason [7] and of Broué-Enguehard [4], the relations among coding theory, the invariant theory of some finite groups, and Siegel modular forms were clarified by Duke [6], Runge [14], [16]. Such studies give the correspondence between number theory and combinatorics. Our study follows this idea. In our previous papers [12], [13], we gave a formulation of E-polynomials based on the theory of binary codes and saw its fundamental properties. In the present paper we take up $\mathbf{Z}_4$-codes. This is the first interesting case we have to do after the binary case. More precisely, we define an E-polynomial in connection with the symmetrized weight enumerator of $\mathbf{Z}_4$-codes and determine the rings generated by them. It turns out that the ring generated by the E-polynomials almost coincides with the invariant ring for the finite group H which is defined below and is graded by w. The exceptions appear in lower w's. Moreover, we see the field of quotient of homogeneous invariants of the same weight can be generated by E-polynomials. These properties of E-polynomials can be seen in the theory of Eisenstein series (*cf.* [17]). In the last section, we consider the image of E-polynomials under the theta map and discuss the zeros of the

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resulting modular forms.

Notation We denote by $\mathbf{C}$ the field of complex numbers. Let A_w be a finite dimensional vector space over $A_0 = \mathbf{C}$ and

$$A = \bigoplus_{w=0}^{\infty} A_w$$

be the graded integral domain. The formal series

$$\sum_{w=0}^{\infty} (\dim A_w) t^w$$

is called the dimension formula of A. We shall denote by $F_0(A)$ the field of quotients of A which can be written as the quotient a/b of homogeneous elements of the same weight in A.

2. Preliminaries

In this section we recall some results in [9], [5], [2]. In the course of this, we introduce the notion of E-polynomials.

By η_8 we denote a primitive 8-th root of unity. Let H be a finite group generated by

$$\frac{\eta_8}{2} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

and $\mathrm{diag}[1, \eta_8, -1]$. This is of order 384. The group G generated by H and $\mathrm{diag}[\eta_8, \eta_8, \eta_8]$ is of order 768. Under a usual action of such matrices on the polynomial ring of three variables we denote by $\mathfrak{W}$, $\widetilde{\mathfrak{W}}$ the invariant rings of H, G, respectively:

$$\mathfrak{W} = \mathbf{C}[x_0, x_1, x_2]^H = \mathfrak{W}_0 \oplus \mathfrak{W}_1 \oplus \mathfrak{W}_2 \oplus \cdots,$$
$$\widetilde{\mathfrak{W}} = \mathbf{C}[x_0, x_1, x_2]^G = \widetilde{\mathfrak{W}}_0 \oplus \widetilde{\mathfrak{W}}_1 \oplus \widetilde{\mathfrak{W}}_2 \oplus \cdots.$$

The dimension formulae of these are given as

$$\sum_w (\dim \mathfrak{W}_w) t^w = \frac{1+t^{16}}{(1-t^8)^2(1-t^{12})},$$
$$\sum_w (\dim \widetilde{\mathfrak{W}}_w) t^w = \frac{1+t^{16}}{(1-t^8)^2(1-t^{24})}.$$

Coding theory helps us to give sets of generators of these invariant rings as we shall see next.

Let $\mathbf{Z}_4 = \{0,1,2,3\}$ be the ring of integers modulo 4. In the following the elements of $\mathbf{Z}_4$ are sometimes regarded as those of $\mathbf{Z}$. By a $\mathbf{Z}_4$-code of length n, we shall mean an additive subgroup of $\mathbf{Z}_4^n$. We define an inner product on $\mathbf{Z}_4^n$ by $(a,b) = a_1b_1 + \cdots + a_nb_n \pmod 4$ where $a = (a_1, a_2, \ldots, a_n),\ b = (b_1, b_2, \ldots, b_n)$. We impose two conditions on $\mathbf{Z}_4$-codes treated in this paper. The first is self-duality which says that our code C coincides with its dual code $C^\perp$:

$$C = C^\perp := \{y \in \mathbf{Z}_4^n |\ (x,y) \equiv 0 \pmod 4,\ \forall x \in C\}.$$

The second is analogous to the doubly-evenness for binary case. In our $\mathbf{Z}_4$ case,

$$(x,x) \equiv 0 \pmod 8,\ \forall x \in C.$$

A $\mathbf{Z}_4$-code enjoying two conditions above is called Type II. The symmetrized weight enumerator of a $\mathbf{Z}_4$-code C is defined by

$$SW_C(x_0,x_1,x_2) = \sum_{a \in C} x_0^{n_0(a)} x_1^{n_1(a)+n_3(a)} x_2^{n_2(a)}$$

where $n_i(a) = \sharp\{j :\ a_j = i\}$. If C is a Type II $\mathbf{Z}_4$-code, $SW_C(x_0,x_1,x_2)$ is G-invariant. We denote by $p_8,\ q_8,\ p_{16},\ p_{24}$ the symmetrized weight enumerators of the octacode, the codes $\mathcal{K}_m$ for $m = 8,\ 16$, and the lifted Golay code, respectively, where $\mathcal{K}_m$ has the following generator matrix (*cf.* [9])

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ & 2 & 0 & \cdots & 0 & 2 \\ & & 2 & \cdots & 0 & 2 \\ & & & \ddots & \vdots & \vdots \\ & & & & 2 & 2 \end{pmatrix}.$$

The explicit forms are

$$p_8 = x^8 + 14x^4z^4 + 112x^3y^4z + 112xy^4z^3 + 16y^8 + z^8,$$

$$q_8 = x^8 + 28x^6z^2 + 70x^4z^4 + 28x^2z^6 + 128y^8 + z^8,$$

$$\begin{aligned} p_{16} = {} & x^{16} + 120x^{14}z^2 + 1820x^{12}z^4 + 8008x^{10}z^6 + 12870x^8z^8 + 8008x^6z^{10} \\ & + 1820x^4z^{12} + 120x^2z^{14} + 32768y^{16} + z^{16}, \end{aligned}$$

$$\begin{aligned} p_{24} = {} & x^{24} + 759x^{16}z^8 + 12144x^{14}y^8z^2 + 170016x^{12}y^8z^4 + 2576x^{12}z^{12} \\ & + 61824x^{11}y^{12}z + 765072x^{10}y^8z^6 + 1133440x^9y^{12}z^3 + 24288x^8y^{16} \\ & + 1214400x^8y^8z^8 + 759x^8z^{16} + 4080384x^7y^{12}z^5 + 680064x^6y^{16}z^2 \\ & + 765072x^6y^8z^{10} + 4080384x^5y^{12}z^7 + 1700160x^4y^{16}z^4 \\ & + 170016x^4y^8z^{12} + 1133440x^3y^{12}z^9 + 680064x^2y^{16}z^6 + 12144x^2y^8z^{14} \\ & + 61824xy^{12}z^{11} + 4096y^{24} + 24288y^{16}z^8 + z^{24}. \end{aligned}$$

The subscript denotes the weight of each polynomial. For the readers familiar with these topics it will become clear in the last section why we take up the symmetrized weight enumerators rather than the complete weight enumerators.

It is known that a Type II $\mathbf{Z}_4$-code of length n exists if and only if n is a multiple of 8. So the symmetrized weight enumerators of Type II $\mathbf{Z}_4$-codes are not enough to generate the invariant ring $\mathfrak{W}$. Now we give the definition of E-polynomials. An E-polynomial of weight k for H is, by definition,

$$\varphi_k^H = \varphi_k^H(x_0, x_1, x_2) = \frac{1}{|H|} \sum_{\sigma \in H} (\sigma x_0)^k = \frac{|K|}{|H|} \sum_{K \backslash H \ni \sigma} (\sigma x_0)^k$$

where

$$K = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in H \right\}$$

is a subgroup of H of order 8. If we apply the same definition of E-polynomials φ_k^G for G, the resulting polynomials are the same. Therefore we

simply denote by φ_k without specifying a group and call it an E-polynomial of weight k. The smallest nontrivial elements in both the symmetrized weight enumerators and E-polynomials are of weight 8. We have there the relation

$$\varphi_8 = \frac{5}{48}p_8 - \frac{1}{128}q_8. \qquad (\clubsuit)$$

All these being said, the module structures over the weighted polynomial rings of our invariant rings are described as follows.

$$\mathfrak{W} = \mathbf{C}[p_8, q_8, \varphi_{12}] \oplus \mathbf{C}[p_8, q_8, \varphi_{12}]p_{16},$$
$$\widetilde{\mathfrak{W}} = \mathbf{C}[p_8, q_8, p_{24}] \oplus \mathbf{C}[p_8, q_8, p_{24}]p_{16}.$$

The unique minimal relation of the ring $\mathfrak{W}$ is

$$\begin{aligned} &3^5 5^2 7^4 \cdot 11^2 p_{16}^2 + 2 \cdot 3^2 5^2 7^3 \cdot 11^2 (2^6 p_8^2 + 2^4 p_8 q_8 - 269 q_8^2) p_{16} \\ &\quad = 3^2 5^2 \cdot 11^2 (-2^{12} 3 p_8^4 - 2^{11} 11 p_8^3 q_8 + 2^7 3^2 \cdot 41 p_8^2 q_8^2 \\ &\qquad + 2^5 1039 p_8 q_8^3 - 110491 q_8^4) + 2^{30} 7^3 (p_8 - q_8)\varphi_{12}^2, \qquad (\spadesuit) \end{aligned}$$

the right-hand side of which contains no p_{16}. The structure of $\widetilde{\mathfrak{W}}$ is deduced from that of $\mathfrak{W}$ and from the identity

$$\begin{aligned} \frac{2^{26}7^3}{3^2 5 \cdot 11^2}\varphi_{12}^2 &= 2^6 3 p_8^3 - 2^3 13 \cdot 173 p_8^2 q_8 - 59113 p_8 q_8^2 + 37 \cdot 373 q_8^3 \\ &\quad + 2^6 3 \cdot 7^3 p_{24} + 7^2 (11 \cdot 59 p_8 - 5 \cdot 29 q_8) p_{16}. \end{aligned}$$

The unique relation of $\widetilde{\mathfrak{W}}$ is

$$\begin{aligned} &3^2 5 \cdot 7^4 p_{16}^2 + 2 \cdot 3 \cdot 7^2 (-2^3 41 p_8^2 + 2^8 3 p_8 q_8 - 5^2 47 q_8^2) p_{16} \\ &\quad = -2^{10} 19 p_8^4 - 2^7 1051 p_8^3 q_8 - 2^4 3 \cdot 29 \cdot 101 p_8^2 q_8^2 + 2^7 3 \cdot 13 \cdot 89 p_8 q_8^3 \\ &\qquad - 3 \cdot 151 \cdot 569 q_8^4 + 2^{10} 7^3 (p_8 - q_8) p_{24}. \end{aligned}$$

Here we notice that the invariant ring $\widetilde{\mathfrak{W}}$ is generated by the symmetrized weight enumerators of Type II $\mathbf{Z}_4$-codes. This remarkable fact, began with Gleason [7], is highly generalized in [10].

3. Results

In this section we determine the generators of both the rings and the fields of E-polynomials. We denote by $\mathfrak{E}$ (resp. $\widetilde{\mathfrak{E}}$) the ring over $\mathbf{C}$ generated by the φ_ℓ's with $\ell \equiv 0 \pmod 4$ (resp. with $\ell \equiv 0 \pmod 8$).

Theorem 1 (1) $\mathfrak{E}$ *is minimally generated by the E-polynomials of weights*

$$8, 12, 16, 20, 24, 28, 32, 40, 48.$$

(2) $\widetilde{\mathfrak{E}}$ *is minimally generated by the E-polynomials of weights*

$$8, 16, 24, 32, 40, 48, 56, 64, 72, 80.$$

Proof. (1) We denote by $\mathfrak{E}^{sub}$ the ring generated by the φ_ℓ's of weights

$$\ell = 8, 12, 16, 20, 24, 28, 32, 40, 48.$$

We have $\mathfrak{E}^{sub} \subset \mathfrak{E} \subset \mathfrak{W}$. We compute the dimensions of each vector space $\mathfrak{E}^{sub}_w$ and get the following table.

w	8	12	16	20	24	28	32	36	40	44	48	52
$\dim \mathfrak{W}_w$	2	1	4	2	7	4	10	7	14	10	19	14
$\dim \mathfrak{E}^{sub}_w$	1	1	2	2	4	4	7	7	11	10	18	14

Also we can verify by direct calculations

$$\mathfrak{E}^{sub}_w = \mathfrak{W}_w \quad 52 \leq w \leq 96.$$

We shall show that the equality above holds for any $w \geq 100$. First we observe

$$\mathfrak{W} = \sum_{0 \leq m \leq 6, 0 \leq n \leq 1} \mathfrak{E}^{sub} p_8^m p_{16}^n.$$

Indeed this follows from the above calculations, (♣) and (♠). It is then enough to prove that any

$$\varphi = \varphi_8^a \varphi_{12}^b \varphi_{16}^c \varphi_{20}^d \varphi_{24}^e \varphi_{28}^f \varphi_{32}^g \varphi_{40}^h \varphi_{48}^i p_8^m p_{16}^n \quad (0 \leq m \leq 6,\ 0 \leq n \leq 1)$$

of weight greater than 96 lies in $\mathfrak{E}^{sub}$. Suppose that φ has the smallest weight greater than 96 such that $\varphi \notin \mathfrak{E}^{sub}$. We have $\varphi = \varphi_\ell F$ for some $\ell \in \{8, 12, 16, 20, 24, 28, 32, 40, 48\}$. We get

$$\deg \varphi > \deg F = \deg \varphi - \ell \geq 100 - 48 = 52$$

and by the choice of φ we conclude $F \in \mathfrak{E}^{sub}$. This completes the proof of (1).

(2) This can be proved by the same way as (1). We denote by $\widetilde{\mathfrak{E}}^{sub}$ the ringe generated by E-polynomials of weights

$$8, 16, 24, 32, 40, 48, 56, 64, 72, 80.$$

The dimensions we have to compare are the table below

w	8	16	24	32	40	48	56	64	72	80	88
$\dim \widetilde{\mathfrak{W}}$	2	4	7	10	14	19	24	30	37	44	52
$\dim \widetilde{\mathfrak{E}}^{sub}$	1	2	3	5	7	11	15	22	30	42	52

and it is enough to check $\widetilde{\mathfrak{E}}^{sub}_w = \widetilde{\mathfrak{W}}_w$ for $w = 88, 92, \ldots, 160$. We omit the details. □

Before proceeding to the next theorem, we give the *raison d'être* of it. We know the j-function* has many important aspects. Among other things, the field $\mathbf{C}(j(\tau))$ is the field of elliptic modular functions. The points we like to emphasize are that an elliptic modular function is written as a quotient of two modular forms of the same weight and that Eisenstein series are enough to write elliptic modular functions (*cf.* [17]). All these taken into account, the following are the very expected property for E-polynomials.

Theorem 2 (1) $F_0(\mathfrak{W})$ *can be generated over* $\mathbf{C}$ *by*

$$\frac{q_8}{p_8},\ \frac{p_{16}}{p_8^2}$$

and coincides with $F_0(\mathfrak{E})$.

(2) $F_0(\widetilde{\mathfrak{W}})$ *can be generated over* $\mathbf{C}$ *by*

*See Section 4 for the discussion on modular forms.

$$\frac{q_8}{p_8},\ \frac{p_{24}}{p_8^3}$$

and coincides with $F_0(\widetilde{\mathfrak{E}})$.

Proof. Since the proof of (2) is similar to that of (1), we give the proof of (1) only. We shall show the first part of (1). Consider an element

$$\frac{(A_{abc}p_8^a q_8^b \varphi_{12}^c + \cdots) + (B_{\alpha\beta\gamma}p_8^\alpha q_8^\beta \varphi_{12}^\gamma + \cdots)p_{16}}{(C_{abc}p_8^{a'} q_8^{b'} \varphi_{12}^{c'} + \cdots) + (D_{\alpha'\beta'\gamma'}p_8^{\alpha'} q_8^{\beta'} \varphi_{12}^{\gamma'} + \cdots)p_{16}} \qquad (\heartsuit)$$

of $F_0(\mathfrak{W})$. If we look at the weights of the numerator, we have

$$8a + 8b + 12c = 8\alpha + 8\beta + 12\gamma + 16$$

or

$$2(a+b) + 3c = 2(\alpha+\beta) + 3\gamma + 4.$$

This gives the parities of c and γ are the same. If we look at the weights of the numerator and the denominator, the equation

$$8a + 8b + 12c = 8a' + 8b' + 12c'$$

gives the parities of c and c' are the same. Consequently we know the parities of c, γ, c', γ' coincide. So we find that in the expression ($\heartsuit$), we only need to consider the even power of φ_{12}. As a consequence we have only to show that φ_{12}^2/p_8^3 is in $F_0(\mathfrak{W})$. This is obtained from ($\spadesuit$).

We shall show the latter part of (1). We know $\dim \mathfrak{W}_{20} = \dim \mathfrak{E}_{20} = 2$. So $p_8\varphi_{12}$ and $q_8\varphi_{12}$ are respectively linear combinations of $\varphi_8\varphi_{12}, \varphi_{20}$. Thus $q_8/p_8 = q_8\varphi_{12}/p_8\varphi_{12}$ is an element of $F_0(\mathfrak{E})$. Similarly, if we consider $p_{16}\varphi_{12},\ p_8^2\varphi_{12}$, we get $p_{16}/p_8^2 = p_{16}\varphi_{12}/p_8^2\varphi_{12}$ is in $F_0(\mathfrak{E})$. This completes the proof of (1). □

4. Concluding Remarks

We conclude this paper giving the observations on the zeros of the mapped E-polynomials. Let τ be an element of the upper-half plane, that is, $\tau \in \mathbf{C}$ with the positive imaginary part. We recall the following functions

$$\theta_{ab}(\tau) = \sum_{n\in\mathbf{Z}} \exp 2\pi\sqrt{-1}\left\{\tau\left(n+\frac{a}{4}\right)^2 + \left(n+\frac{a}{4}\right)\frac{b}{4}\right\}$$

where $a, b \in \{0,1,2,3\}$. Put $f_a(\tau) = \theta_{a0}(2\tau)$. Here we have $f_1(\tau) = f_3(\tau)$ and this is the reason why we are interested in the symmetrized weight enumerators rather than the complete weight enumerators (*cf.* [15], [1]). At any rate it is known that, for an element $F \in \mathfrak{W}$ of weight n,

$$Th(F(x_0,x_1,x_2)) = F(f_0(\tau), f_1(\tau), f_2(\tau))$$

is a modular form of weight $n/2$ for $SL(2,\mathbf{Z})$. A typical example of a modular form of weight k is an Eisenstein series defined by

$$E_k(\tau) = \frac{1}{2}\sum_{\substack{c,d\in\mathbf{Z}\\(c,d)=1}} \frac{1}{(c\tau+d)^k}$$

for even $k \geq 4$, where $(c,d) = 1$ means that c, d are coprime. We put $q = e^{2\pi\sqrt{-1}\tau}$. Then $E_k(\tau)$ is normalized, that is, the constant term of the q-expansion of $E_k(\tau)$ is equal to 1. In particular, it is known that

$$E_4(\tau) = 1 + 240q + 2160q^2 + \cdots,$$
$$E_6(\tau) = 1 - 504q - 16632q^2 + \cdots.$$

As usual, we put

$$\Delta(\tau) = \frac{1}{1728}(E_4^3(\tau) - E_6^2(\tau)),$$
$$j(\tau) = \frac{E_4^3(\tau)}{\Delta(\tau)}.$$

Then we have

$$\Delta(\tau) = q - 24q^2 + 252q^3 + \cdots,$$
$$j(\tau) = \frac{1}{q} + 744 + 196884q + \cdots.$$

In addition our modular forms are also normalized as

$$e_\ell(\tau) = (\text{const})Th(\varphi_\ell) = 1 + \cdots .$$

For any even integer $k > 2$ we can write k uniquely in the form

$$k = 12m + 4\delta + 6\varepsilon \quad \text{with} \quad m \in \mathbf{Z}_{\geq 0},\ \delta \in \{0, 1, 2\},\ \varepsilon \in \{0, 1\}$$

and then any modular form $f(\tau)$ of weight k can be written uniquely as

$$f(\tau) = \Delta(\tau)^m E_4(\tau)^\delta E_6(\tau)^\varepsilon \widetilde{f}(j(\tau))$$

for some polynomial $\widetilde{f}$ of degree $\leq m$ in $j(\tau)$. Since zeros or $\Delta(\tau)$, $E_4(\tau)$, $E_6(\tau)$ are well understood, additional zeros of $f(\tau)$ can be read off from the polynomial $\widetilde{f}(j)$. For example a zero τ (in the fundamental domain) of $f(\tau) = 0$ is in

$$|\tau| = 1,\ -\frac{1}{2} < \operatorname{Re}\tau < 0,\ \operatorname{Im}\tau > 0$$

if and only if the root j of $\widetilde{f}(j) = 0$ is in the interval $(0, 1728)$. We examined our modular forms $e_\ell(\tau)$ of lower weights.

1. The zeros of the associated polynomials of $e_\ell(\tau)$ are in $(0, 1728)$.
2. We shall denote the zeros of the associated polynomial of $e_\ell(\tau)$ by $a_1, a_2, \ldots, a_m$ and those of $e_{\ell+24}(\tau)$ by $b_1, b_2, \ldots, b_{m+1}$. Then we get $b_j < a_j < b_{j+1}$ for $j = 1, 2, \ldots, m$ (*cf.* [11]).
3. If $k = \ell/2 = p - 1$ where $p \geq 5$ is prime, then the coefficients of the associated polynomial of $e_\ell(\tau)$ are p-integral (*cf.* [8]).

These are not proved generally but to be investigated. We give a graph in the last page which shows the zeros of $e_{4k+28}(\tau)$ for $k = 1, 2, \ldots, 58$.

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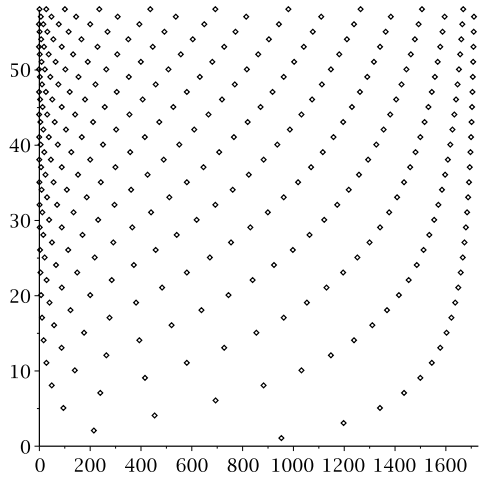

Figure 1. Zeros $e_{4k+28}(\tau)$.