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A New Characterization of Some Simple Groups by Order and Degree Pattern of Solvable Graph

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Abstract. The solvable graph of a finite group G, denoted by $\Gamma_{\mathrm{s}}(G)$, is a simple graph whose vertices are the prime divisors of $|G|$ and two distinct primes p and q are joined by an edge if and only if there exists a solvable subgroup of G such that its order is divisible by pq. Let $p_1 < p_2 < \cdots < p_k$ be all prime divisors of $|G|$ and let $\mathrm{D}_{\mathrm{s}}(G) = (d_{\mathrm{s}}(p_1), d_{\mathrm{s}}(p_2), \ldots, d_{\mathrm{s}}(p_k))$, where $d_{\mathrm{s}}(p)$ signifies the degree of the vertex p in $\Gamma_{\mathrm{s}}(G)$. We will simply call $\mathrm{D}_{\mathrm{s}}(G)$ the degree pattern of solvable graph of G. In this paper, we determine the structure of any finite group G (up to isomorphism) for which $\Gamma_{\mathrm{s}}(G)$ is star or bipartite. It is also shown that the sporadic simple groups and some of projective special linear groups $L_2(q)$ are characterized via order and degree pattern of solvable graph.

Key words: solvable graph, degree pattern, simple group, OD_{s}-characterization of a finite group.

1. Introduction

All groups considered in this paper will be finite. Let G be a finite group, $\pi(G)$ the set of all prime divisors of its order and $\mathrm{Spec}(G)$ be the spectrum of G, that is the set of its element orders. The prime graph $\mathrm{GK}(G)$ of G (or Gruenberg-Kegel graph) is a simple graph whose vertex set is $\pi(G)$ and two distinct vertices p and q are joined by an edge if and only if $pq \in \mathrm{Spec}(G)$. The prime graph of a group can be generalized in the following way (see [1], [2]).

Let $\mathcal{P}$ be a group-theoretic property. Given a finite group G, we define $S_{\mathcal{P}}(G)$ to be the set of all $\mathcal{P}$-subgroups of G. Let σ be a mapping of $S_{\mathcal{P}}(G)$ to the the set of natural numbers. Following the notation of [1], [2], we define its $(\mathcal{P}, \sigma)$-graph as follows: its vertices are the primes dividing an element of $\sigma(S_{\mathcal{P}}(G))$ and two vertices p and q are joined by an edge if there is a natural number in $\sigma(S_{\mathcal{P}}(G))$ which can be divided by pq. We illustrate this with the following examples.

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(1) $\mathcal{P} \equiv$ *cyclic* and $\sigma(H) \equiv$ *order of* H for each $H \in S_{\mathcal{P}}(G)$. In this case, $S_{\mathcal{P}}(G)$ is the set of all cyclic subgroups of G and the $(\mathcal{P}, \sigma)$-graph is called the "*cyclic graph*" of G (see [2]). In fact, in the cyclic graph of G, the vertices are the prime numbers dividing the order of G and two different vertices p and q are joined by an edge (and we write $p \sim q$) when G has a cyclic subgroup whose order is divisible by pq. We will denote by $\Gamma_{\rm c}(G)$ the cyclic graph of a group G. It is worth noting that $\sigma(S_{\mathcal{P}}(G)) = \mathrm{Spec}(G)$ and the cyclic graph and the prime graph of a group are exactly one thing. Also, if we take $\mathcal{P} \equiv$ *abelian* or *nilpotent*, then $(\mathcal{P}, \sigma)$-graph of G and the cyclic graph of G coincide.

(2) $\mathcal{P} \equiv$ *solvable* and $\sigma(H) \equiv$ *order of* H for each $H \in S_{\mathcal{P}}(G)$. Here $S_{\mathcal{P}}(G)$ is the set of all solvable subgroups of G and the $(\mathcal{P}, \sigma)$-graph of G is called the "*solvable graph*" of G (see [2]). We will denote by $\Gamma_{\rm s}(G)$ the solvable graph of a group G. Note that the solvable graph of G is a generalization of the cyclic graph of G. In fact, the vertices are, like in the cyclic graph, the prime numbers dividing the order of G, but two different vertices p and q are adjacent (we write $p \approx q$) when G has a solvable subgroup of order divisible by pq.

(3) $\mathcal{P} \equiv$ *commutativity of an element* and $\sigma(H) \equiv$ *index of* H *in* G for each $H \in S_{\mathcal{P}}(G)$. In this case, $S_{\mathcal{P}}(G)$ is the set of centralizers of all elements of G and the $(\mathcal{P}, \sigma)$-graph of G is called the "*conjugacy class graph*" of G (see [6]).

In this paper we will focus our attention on the solvable graph associated with a finite group. Especially, we will determine the structure of any finite group G (up to isomorphism) for which $\Gamma_{\rm s}(G)$ is star or bipartite.

In the case of a generic group G, it is sometimes convenient to represent the graph $\Gamma_{\rm c}(G)$ (resp. $\Gamma_{\rm s}(G)$) in a compact form. By the compact form we mean a graph whose vertices are labeled with disjoint subsets of $\pi(G)$. Actually, a vertex labeled U represents the complete subgraph of $\Gamma_{\rm c}(G)$ (resp. $\Gamma_{\rm s}(G)$) on U. Moreover, an edge connecting U and W represents the set of edges of $\Gamma_{\rm c}(G)$ (resp. $\Gamma_{\rm s}(G)$) that connect each vertex in U with each vertex in W. For instance, we draw in the following the compact form of the cyclic and solvable graph of some simple groups.

- $R(q) = {}^2G_2(q)$: the simple Ree group defined over the field with $q = 3^{2m+1} \geqslant 27$ elements. Figures 1 and 2 depict the compact forms of the cyclic and solvable graphs of the Ree group $R(q)$. In constructing these graphs, we used the following facts:

The spectrum of $R(q)$ is as follows (see [5, Lemma 4]):

$$\mathrm{Spec}(R(q)) = \big\{3, 6, 9, \text{all factors of } q-1, (q+1)/2, \\ q-\sqrt{3q}+1 \text{ and } q+\sqrt{3q}+1\big\}.$$

The list of maximal subgroups of $R(q)$ in [12] can be summarized as follows. Here, $[q^3]$ denotes an unspecified group of order q^3 and $A : B$ denotes a split extension.

Structure	Order	Structure	Order
$[q^3] : \mathbb{Z}_{q-1}$	$q^3(q-1)$	$\mathbb{Z}_{q+\sqrt{3q}+1} : \mathbb{Z}_6$	$6(q+\sqrt{3q}+1)$
$\mathbb{Z}_2 \times L_2(q)$	$q(q^2-1)$	$\mathbb{Z}_{q-\sqrt{3q}+1} : \mathbb{Z}_6$	$6(q-\sqrt{3q}+1)$
$(2^2 : D_{(q+1)/2}) : 3$	$6(q+1)$	$R(q_0), q = q_0^\alpha, \alpha$ prime	$q_0^3(q_0^3+1)(q_0+1)$

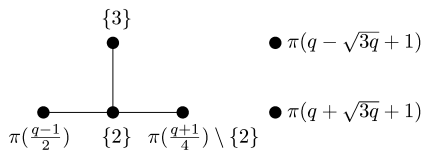

Figure 1. $\Gamma_c(R(q))$, $q = 3^{2m+1} > 3$.

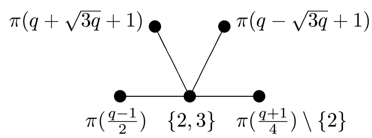

Figure 2. $\Gamma_s(R(q))$, $q = 3^{2m+1} > 3$.

- $\mathrm{Sz}(q)$: the Suzuki simple group defined over the field with $q = 2^{2m+1}$ elements. Again, we need information about the spectrum and the structure of maximal subgroups of $\mathrm{Sz}(q)$ in order to draw its cyclic and solvable graphs.

  The spectrum of $\mathrm{Sz}(q)$ is as follows (see [16, Theorem 2]):

$$\mathrm{Spec}(\mathrm{Sz}(q)) = \big\{2, 4, \text{all factors of } q-1, q-\sqrt{2q}+1 \text{ and } q+\sqrt{2q}+1\big\}.$$

Every maximal subgroup of $\mathrm{Sz}(q)$ is isomorphic to one of the following (Suzuki [17]):

$$\mathbb{Z}_{q^2} : \mathbb{Z}_{q-1},\ \ \mathbb{Z}_{q-1} : \mathbb{Z}_2,\ \ \mathbb{Z}_{q+\sqrt{2q}+1} : \mathbb{Z}_4,\ \ \mathbb{Z}_{q-\sqrt{2q}+1} : \mathbb{Z}_4, \\ \mathrm{Sz}(q_0),\ q = q_0^\alpha,\ \alpha \in \mathbb{Z}.$$

According to these information, we can draw the cyclic and solvable graph of the Suzuki groups $\mathrm{Sz}(q)$ as shown in Figures 3 and 4.

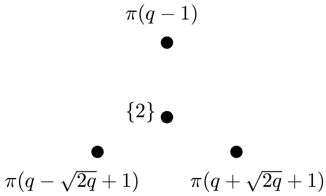

Figure 3. $\Gamma_{\mathrm{c}}(\mathrm{Sz}(q))$, $q = 2^{2m+1} > 2$.

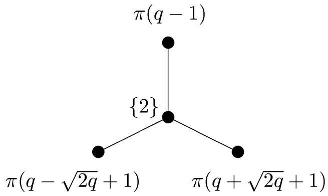

Figure 4. $\Gamma_{\mathrm{s}}(\mathrm{Sz}(q))$, $q = 2^{2m+1} > 2$.

The *degree* $d_{\mathrm{s}}(p)$ (resp. $d_{\mathrm{c}}(p)$) of a vertex $p \in \pi(G)$ is the number of adjacent vertices to p in $\Gamma_{\mathrm{s}}(G)$ (resp. $\Gamma_{\mathrm{c}}(G)$). Clearly, $d_{\mathrm{c}}(p) \leqslant d_{\mathrm{s}}(p)$ for every vertex $p \in \pi(G)$. In the case when $\pi(G) = \{p_1, p_2, \ldots, p_k\}$ with $p_1 < p_2 < \cdots < p_k$, we define

$$\mathrm{D}_{\mathrm{s}}(G) = \big(d_{\mathrm{s}}(p_1), d_{\mathrm{s}}(p_2), \ldots, d_{\mathrm{s}}(p_k)\big),$$

which is called the *degree pattern of the solvable graph of* G. For every non-negative integer $m \in \{0, 1, 2, \ldots, k-1\}$, we put

$$\Delta_m(G) := \{p \in \pi(G) |\ d_{\mathrm{s}}(p) = m\}.$$

Clearly,

$$\pi(G) = \bigcup_{m=0}^{k-1} \Delta_m(G).$$

When $\Delta_{k-1}(G) \neq \emptyset$, the prime p with $d_{\mathrm{s}}(p) = k-1$ is called a *complete prime*.

Given a finite group G, denote by $h_{\mathrm{OD_s}}(G)$ the number of isomorphism classes of finite groups H such that $|H| = |G|$ and $\mathrm{D}_{\mathrm{s}}(H) = \mathrm{D}_{\mathrm{s}}(G)$. In terms of the function $h_{\mathrm{OD_s}}(\cdot)$, we have the following definition.

Definition 1 A finite group G is said to be k-fold $\mathrm{OD_s}$-characterizabale if $h_{\mathrm{OD_s}}(G) = k$. The group G is $\mathrm{OD_s}$-characterizabale if $h_{\mathrm{OD_s}}(G) = 1$. Moreover, we will say that the $\mathrm{OD_s}$-characterization problem is solved for a group G, if the value of $h_{\mathrm{OD_s}}(G)$ is known.

One of the purposes of this paper is to characterize some simple groups by order and degree pattern of solvable graph. For instance, we will prove the following theorems.

Theorem A *All sporadic simple groups are* $\mathrm{OD_s}$*-characterizable.*

Theorem B *Let* $L = L_2(q)$*,* $q = p^n > 3$*, and one of the following conditions is fulfilled*:

(a) $p = 2$, $|\pi(q+1)| = 1$ *or* $|\pi(q-1)| = 1$,
(b) $q \equiv 1 \pmod 4$, $|\pi(q+1)| = 2$ *or* $|\pi(q-1)| \leqslant 2$,
(c) $q \equiv -1 \pmod 4$.

Then L *is* $\mathrm{OD_s}$*-characterizable.*

It is important to notice that there exist some groups which are not $\mathrm{OD_s}$-characterizable. For example, the following groups:

$$S_6(3),\ O_7(3),\ \ H \times \mathrm{Sz}(8),$$

where H is an arbitrary group of order $2^3 \cdot 3^9$, have the same order and degree pattern of solvable graph. In fact, we have

$$|S_6(3)| = |O_7(3)| = |H \times \mathrm{Sz}(8)| = 2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13,$$

and

$$\mathrm{D_s}(S_6(3)) = \mathrm{D_s}(O_7(3)) = \mathrm{D_s}(H \times \mathrm{Sz}(8)) = (4, 4, 2, 2, 2).$$

In [8], the authors proved that if a finite group G and a finite simple group S have the same sets of all orders of solvable subgroups, then G is isomorphic to S, or G and S are isomorphic to $O_{2n+1}(q)$, $S_{2n}(q)$, where $n \geqslant 3$ and q is odd. This immediately implies the following:

Corollary C *If* $G \in \{O_{2n+1}(q), S_{2n}(q)\}$*, where* $n \geqslant 3$ *and* q *is odd, then* $h_{\mathrm{OD_s}}(G) \geqslant 2$*.*

More Notation and Terminology. Given a graph Γ, Γ^c is said to be complementary graph if the set of vertices of Γ and Γ^c coincide with each other and two vertices u and v of Γ^c are joined in Γ^c if and only if u and v are not joined in Γ. An acyclic graph is one that contains no cycles. A connected acyclic graph is called a tree. In the case when $U \subseteq V$, the graph $\Gamma - U$ is defined to be a graph whose vertex set is $V - U$ and two vertices

u and v are joined if they are joined in Γ. In addition, $\Gamma[U]$ denotes the induced subgraph of Γ whose vertex set is U and whose edges are precisely the edges of Γ which have both ends in U. The union of graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is the graph $\Gamma_1 \cup \Gamma_2$ with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. If Γ_1 and Γ_2 are disjoint (we recall that two graphs are disjoint if they have no vertex in common), we refer to their union as a disjoint union, and generally denote it by $\Gamma_1 \oplus \Gamma_2$. Given a natural number m and a prime number p, we denote by m_p the p-part of m, that is the largest power of p dividing m. Let G be a finite group and p be a prime divisor of $|G|$. We denote by $O_p(G)$ the maximal normal p-subgroup of G, and by $O^p(G)$ the smallest normal subgroup of G for which $G/O^p(G)$ is a p-group.

2. Preliminary Results

In this section, we first state some fundamental results for our studies of solvable graphs of finite groups, and then we find the structure of a group that its solvable graph has certain properties. We begin with some fundamental lemmas.

Lemma 1 ([2, Lemma 2]) *Let G be a finite group. Let H and N be two subgroups of G with $N \trianglelefteq G$. Then the following statements hold*:

(1) *If p and q are joined in $\Gamma_{\rm s}(H)$ for $p, q \in \pi(H)$, then p and q are joined in $\Gamma_{\rm s}(G)$, in other words, $\Gamma_{\rm s}(H)$ is a subgraph of $\Gamma_{\rm s}(G)$.*

(2) *If p and q are joined in $\Gamma_{\rm s}(G/N)$ for $p, q \in \pi(G/N)$, then p and q are joined in $\Gamma_{\rm s}(G)$, in other words, $\Gamma_{\rm s}(G/N)$ is a subgraph of $\Gamma_{\rm s}(G)$.*

(3) *For $p \in \pi(N)$ and $q \in \pi(G) \setminus \pi(N)$, p and q are joined in $\Gamma_{\rm s}(G)$.*

Lemma 2 *Let G be a finite group with $|\pi(G)| = k$. Then, the following statements hold*:

(1) *If $\Delta_{k-1}(G) = \emptyset$, then G is a non-abelian simple group.*

(2) *If G is not isomorphic to a non-abelian simple group, then $\Gamma_{\rm s}(G)$ is regular if and only if $\Gamma_{\rm s}(G)$ is complete.*

Proof. Part (1) is Lemma 3 in [1]. Part (2) is an easy consequence of part (1). $\square$

Lemma 3 *Let G be a finite group with $|\pi(G)| = k$. Then the following statements hold*:

(1) *If G is a solvable group, then $\Gamma_{\rm s}(G)$ is complete.*
(2) *$\Gamma_{\rm s}(G)$ is always connected. In particular, $\Delta_0(G) \neq \emptyset$ if and only if G is a p-group for some prime p.*
(3) *If G is a non-abelian simple group, then $\Gamma_{\rm s}(G)$ is not complete.*
(4) *If R is the solvable radical of G, then $\pi(R) \subseteq \Delta_{k-1}(G) \subseteq \pi(G)$.*

Proof. Parts (1)-(3) are Lemma 1 (2), Corollary 2 and Theorem 2 in [2], respectively. Part (4) follows immediately from part (1) and Lemma 1 (2). □

Corollary 1 *Let N be a normal subgroup of a finite group G. Then there hold*:

(1) *If $\{p, q\} \subseteq \pi(G) \setminus \pi(N)$, then $p \approx q$ in $\Gamma_{\rm s}(G/N)$ if and only if $p \approx q$ in $\Gamma_{\rm s}(G)$.*
(2) *If N is a normal Hall subgroup of G, then $\Gamma_{\rm s}(G)$ is complete if and only if $\Gamma_{\rm s}(N)$ and $\Gamma_{\rm s}(G/N)$ are complete too.*

Proof. (1) In view of Lemma 1 (2), it is enough to prove the sufficiency. Let $\{p, q\} \subseteq \pi(G) \setminus \pi(N)$ and $p \approx q$ in $\Gamma_{\rm s}(G)$. Then by the definition there exists a solvable subgroup H of G such that $|H|$ is divisible by pq. Let K be a $\{p, q\}$-Hall subgroup of H and put $\overline{K} := KN/N$. Clearly $\overline{K} = KN/N \cong K/(N \cap K) \cong K$ is a solvable subgroup of G/N such that its order is divisible by pq. This means that $p \approx q$ in $\Gamma_{\rm s}(G/N)$, as required.

(2) Sufficiency follows immediately from part (1), so we just need to prove the necessity. Let N be a normal Hall subgroup of G for which $\Gamma_{\rm s}(G)$ is complete. First of all, considering part (1), it is easy to see that $\Gamma_{\rm s}(G/N)$ is complete. Next, we show that $\Gamma_{\rm s}(N)$ is complete, too. Since $|N|$ and $|G/N|$ are relatively prime integers, a theorem of Schur [10, p. 224] asserts that in this case G must contain a subgroup K such that $G = KN$ and $K \cap N = 1$. Now, suppose p and q are two primes in $\pi(N)$. Since $p \approx q$ in $\Gamma_s(G)$, there exists a solvable subgroup H of G such that $|H|$ is divisible by pq. Let H_0 be a Hall $\{p, q\}$-subgroup of H. Obviously, $H_0 \leqslant N$. This forces $p \approx q$ in $\Gamma_s(N)$. Therefore, $\Gamma_s(N)$ is a complete graph. □

Remark It is not true in general that if N is a normal subgroup of G then $\Gamma_{\rm s}(N)$ and $\Gamma_{\rm s}(G/N)$ are complete if $\Gamma_{\rm s}(G)$ is complete. An example is provided by $G = \mathbb{Z}_3 \times \mathbb{A}_5$, $N = \mathbb{A}_5$ and $M = \mathbb{Z}_3$. In this case, $\Gamma_{\rm s}(G)$ is complete while $\Gamma_{\rm s}(N)$ and $\Gamma_{\rm s}(G/M)$ are not complete.

Corollary 2 *Let G be a finite group such that $\Gamma_s(G)$ is a complete graph. Moreover, let R be the solvable radical of G. Then, one of the following statements holds*:

(i) $\pi(R) = \pi(G)$,
(ii) $\pi(R) \subset \pi(G)$ *and G is an extension of R by a non-solvable group Q for which the induced subgraph $\Gamma_s(Q)[\pi(Q) \setminus \pi(R)]$ is a complete graph.*

Proof. If $\pi(R) = \pi(G)$, then there is nothing to prove. Suppose now that $\pi(R) \subset \pi(G)$. Clearly $Q := G/R$ is a non-solvable group and in view of Corollary 1 we conclude that $\Gamma_s(Q)[\pi(Q) \setminus \pi(R)]$ is a complete graph, as required. □

Lemma 4 ([2, Theorem 3]) *Let G be a finite group and $\{p, q\} \subseteq \pi(G)$. Then p and q are not joined in $\Gamma_{\rm s}(G)$ if and only if there exists a series of normal subgroups of G, say*

$$1 \trianglelefteq M \triangleleft N \trianglelefteq G,$$

such that M and G/N are $\{p, q\}'$-groups and N/M is a non-abelian simple group such that p and q are not joined in $\Gamma_{\rm s}(N/M)$.

Following the conventions of [1] and [2] for notation concerning solvable graphs, such a series as in Lemma 4, is called a *GKS-series of G* and we will say *p and q are expressed to be disjoint by this GKS-series.*

Lemma 5 ([1, Lemma 4]) *Let G be a finite group with $|\pi(G)| = k$ and $\widetilde{\Gamma}(G) = (\Gamma_{\rm s}(G) - \Delta_{k-1}(G))^c$. If the number of connected components of $\widetilde{\Gamma}(G)$ equals n, then at most n GKS-series of G is necessary to express any pair of vertices of $\Gamma_{\rm s}(G)$ to be disjoined.*

Lemma 6 *Let G be a finite group with $|\pi(G)| = k \geqslant 4$ and $\widetilde{\Gamma}(G) := (\Gamma_{\rm s}(G) - \Delta_{k-1}(G))^c$. If one of the following conditions holds, then any disjoined pair of vertices of $\Gamma_{\rm s}(G)$ can be expressed by only one GKS-series.*

(1) $\Delta_{k-1}(G) \neq \emptyset$ *and* $\Delta_1(G) \neq \emptyset$.
(2) $\Delta_{k-1}(G) \neq \emptyset$ *and* $\Delta_2(G) \neq \emptyset$.

Proof. We only prove (2), and (1) goes similarly. By Lemma 5, it is enough to show that $\widetilde{\Gamma}(G)$ is connected. Since $\Delta_2(G) \neq \emptyset$, we can consider some vertex in $\pi(G)$, say p, with $d_{\rm s}(p) = 2$. It is clear that $|\Delta_{k-1}(G)| \leqslant d_{\rm s}(p) = 2$. We now distinguish two cases.

Case 1. $|\Delta_{k-1}(G)| = 1$. In this case, $\Gamma_{\rm s}(G) - \Delta_{k-1}(G)$ is a graph with $|\pi(G) \setminus \Delta_{k-1}(G)| = k - 1$ vertices, which contains the vertex p as a vertex of degree 1 (Note $p \notin \Delta_{k-1}(G)$, because $k \geqslant 4$). Therefore, the vertex p in $\widetilde{\Gamma}(G)$ has degree $k-2$, which forces the graph $\widetilde{\Gamma}(G)$ is connected.

Case 2. $|\Delta_{k-1}(G)| = 2$. Here, $\Gamma_{\rm s}(G) - \Delta_{k-1}(G)$ is a graph with $|\pi(G) \setminus \Delta_{k-1}(G)| = k - 2$ vertices, and it contains the vertex p as an isolated vertex. Thus p is a vertex of $\widetilde{\Gamma}(G)$ of degree $k-3$, which shows that $\widetilde{\Gamma}(G)$ is connected. □

We state also the following well-known result due to Artin (see [3] and [4]).

Lemma 7 (Artin Theorem) *Two finite simple groups of the same order are isomorphic except for the pairs* $\{L_4(2) \cong \mathbb{A}_8, L_3(4)\}$ *and* $\{O_{2n+1}(q), S_{2n}(q) : n \geqslant 3$ *and* q *is odd* $\}$.

Remark Notice that by [7] one can draw the solvable graphs $\Gamma_{\rm s}(L_4(2))$ and $\Gamma_{\rm s}(L_3(4))$ and obtain their degree patterns as $\mathrm{D_s}(L_4(2)) = (2, 3, 2, 1)$ and $\mathrm{D_s}(L_3(4)) = (2, 2, 1, 1)$.

The following lemma is due to K. Zsigmondy (See [21]).

Lemma 8 (Zsigmondy Theorem) *Let* q *and* f *be integers greater than* 1. *There exists a prime divisor* r *of* $q^f - 1$ *such that* r *does not divide* $q^e - 1$ *for all* $0 < e < f$, *except in the following cases*:

(a) $f = 6$ *and* $q = 2$;
(b) $f = 2$ *and* $q = 2^l - 1$ *for some natural number* l.

Such a prime r is called a primitive prime divisor of $q^f - 1$. When $q > 1$ is fixed, we denote by $\mathrm{ppd}(q^f - 1)$ any primitive prime divisor of $q^f - 1$. Of course, there may be more than one primitive prime divisor of $q^f - 1$, however the symbol $\mathrm{ppd}(q^f - 1)$ denotes any one of these primes. For example, the primitive prime divisors of $53^5 - 1$ are 11, 131, 5581 and thus $\mathrm{ppd}(53^5 - 1)$ denotes any one of these primes.

As an immediate consequence of Lemma 8, we have the following corollary.

Corollary 3 *Let* p *and* q *be two primes and* m, n *be natural numbers such that* $p^m - q^n = 1$. *Then one of the following holds*:

(a) $(p, n) = (2, 1)$, *and* $q = 2^m - 1$ *is a Mersenne prime*;
(b) $(q, m) = (2, 1)$, *and* $p = 2^n + 1$ *is a Fermat prime*;
(c) $(p, n) = (3, 3)$ *and* $(q, m) = (2, 2)$.

A finite group G is called a *Frobenius group* with kernel N and complement M, if $G = NM$ where N is a normal subgroup of G and $M \leqslant G$, and for all $1 \neq g \in N$, $C_G(g) \subseteq N$. Also, a finite group G is called a 2-*Frobenius group* if it has a normal series $1 \trianglelefteq M \trianglelefteq N \trianglelefteq G$ such that N is a Frobenius group with kernel M and G/M is a Frobenius group with kernel N/M.

Lemma 9 *Let G be a Frobenius group. Then, one of the following statements holds*:

(1) *G is solvable and* $\Gamma_{\mathrm{s}}(G) = K_{|\pi(G)|}$.
(2) *G is non-solvable and* $\Gamma_{\mathrm{s}}(G)$ *can be obtained from the complete graph on* $\pi(G)$ *by deleting the edge* $\{3, 5\}$.

Proof. (1) This is a special case of Lemma 3 (1).

(2) Suppose $G = NM$ is a Frobenius group with kernel N and complement M. Note that $\Gamma_{\mathrm{c}}(N)$ and $\Gamma_{\mathrm{c}}(M)$ are connected components of $\Gamma_{\mathrm{c}}(G)$, and in fact

$$\Gamma_{\mathrm{c}}(G) = \Gamma_{\mathrm{c}}(N) \oplus \Gamma_{\mathrm{c}}(M).$$

In addition, $\Gamma_{\mathrm{c}}(N)$ and so $\Gamma_{\mathrm{s}}(N)$ is complete, because N is a nilpotent group. Note that, by Lemma 1 (3), any prime of $\pi(N)$ is joint to any prime of $\pi(G/N) = \pi(M)$ in $\Gamma_{\mathrm{s}}(G)$. On the other hand, M is non-solvable, as G is non-solvable. Thus, by the structure of non-solvable complement, M has a normal subgroup M_0 with $|M : M_0| \leqslant 2$ such that $M_0 = \mathrm{SL}(2, 5) \times Z$, where every Sylow subgroup of Z is cyclic and $\pi(Z) \cap \pi(30) = \emptyset$ (see Theorem 18.6 in [15]). Moreover, $\Gamma_{\mathrm{c}}(M)$ and so $\Gamma_{\mathrm{s}}(M)$ can be obtained from the complete graph on $\pi(M)$ by deleting the edge $\{3, 5\}$ (see Lemma 5 in [14]). Finally, it is easy to see that the group $\mathrm{SL}(2, 5)$ and so G has no solvable subgroup whose order is divisible by 15, hence 3 and 5 are not joint in $\Gamma_{\mathrm{s}}(G)$. This completes the proof. □

Lemma 10 *The solvable graph of a 2-Frobenius group is always complete.*

Proof. The conclusion follows immediately from Lemma 3 (1), because 2-Frobenius groups are always solvable. □

3. Solvable Graphs with Certain Properties

We begin with recalling some definitions in Graph Theory. A graph is *bipartite* if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y, such a partition (X, Y) is called a *bipartition* of the graph, and X and Y its *parts*. We recall that a graph is bipartite if and only if it contains no odd cycle (a cycle of odd length). A bipartite graph with bipartition (X, Y) in which every two vertices from X and Y are adjacent is called a *complete bipartite graph* and denoted by $K_{|X|,|Y|}$. A *star graph* is a complete bipartite graph of the form $K_{1,n}$ which consists of one central vertex having edges to other vertices in it.

Proposition 1 *Let G be a finite group. Let R stand for the solvable radical of G and $\overline{G} = G/R$. Let $\overline{M}$ be the smallest normal subgroup of $\overline{G}$ among subgroups $\overline{L}$ such that $\overline{G}/\overline{L}$ is solvable. The solvable graph of G is a star graph if and only if*

(a) *G is solvable with $|\pi(G)| \leqslant 2$ or*
(b) *there exists a prime $r \in \pi(G)$ such that $R = O_r(G)$, $\overline{M} = O^r(\overline{G})$ is a simple group and $(\overline{M}, r)$ is one of the following pairs*:

$$\begin{array}{lllll} (\mathbb{A}_5, 2), & (\mathbb{A}_6, 2), & (U_4(2), 2), & (L_2(7), 3), & (L_2(8), 2), \\ (L_2(17), 2), & (L_3(3), 3), & (U_3(3), 3), & or & (\mathrm{Sz}(8), 2). \end{array} \tag{1}$$

Proof. Let G be a finite group such that $\Gamma_s(G)$ is a star graph. If G is a solvable group, then it follows from Lemma 3 (1) that $\Gamma_s(G) = K_{|\pi(G)|}$ is a complete graph, which forces $|\pi(G)| \leqslant 2$.

Thus, we may assume that G is a non-solvable group. Clearly $|\pi(G)| \geqslant 3$. Let R be the solvable radical of G. Since $\Gamma_s(R) = K_{|\pi(R)|}$ is a subgraph of $\Gamma_s(G)$, as before, it concludes that $|\pi(R)| \leqslant 2$. We distinguish three cases separately: $|\pi(R)| = 2$, $|\pi(R)| = 1$ or $|\pi(R)| = 0$ (i.e., $R = 1$).

Assume first that $|\pi(R)| = 2$. If p, q are two distinct primes that divide $|R|$, then $p \approx q$ in $\Gamma_s(R)$ and so in $\Gamma_s(G)$, by Lemma 1 (1). Since $|\pi(G)| \geqslant 3$, we can consider the prime $r \in \pi(G) \setminus \{p, q\}$. Now, by Lemma 1 (3), it follows that $p \approx r \approx q \approx p$ in $\Gamma_s(G)$, which contradicts the fact that $\Gamma_s(G)$ is a star graph.

Assume next that $|\pi(R)| = 1$. Clearly, $R = O_r(G)$ for some prime $r \in \pi(G)$. Put $\overline{G} := G/R$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times \cdots \times P_k$, where P_i

are non-abelian simple groups and $S \leqslant \overline{G} \leqslant \mathrm{Aut}(S)$. It is clear that $k = 1$, otherwise $\Gamma_{\mathrm{c}}(G)$ and so $\Gamma_{\mathrm{s}}(G)$ contains a cycle, which is a contradiction. Therefore, we have $P \leqslant \overline{G} \leqslant \mathrm{Aut}(P)$, for a non-abelian simple group P.

If $\overline{G}/P \neq 1$, then there exist primes $p \in \pi(\overline{G}/P)$, $q_i \in \pi(P) - \{r\}$ $(i = 1, 2)$. Let $\overline{Q_i}$ be a Sylow q_i-subgroup of P. Since $N_{\overline{G}}(\overline{Q_i})P = \overline{G}$, there exists an element $\overline{x_i} \in N_{\overline{G}}(\overline{Q_i})$ of order p for each i. Let $L_i(< G)$ be the inverse image of $\langle\overline{x_i}\rangle\overline{Q_i}$. Then L_i is a solvable group with $\pi(L_i) = \{p, r, q_i\}$ for $i = 1, 2$. Since $\Gamma_{\mathrm{s}}(G)$ is a star graph, we have $p = r$ and $\Gamma_{\mathrm{s}}(G) = \Gamma_{\mathrm{s}}(M)$, where $M(< G)$ is the inverse image of P. Hence we may assume $G = M$, i.e., $\overline{G} = P$.

If $r \notin \pi(\overline{G})$, then any prime in $\pi(G) \setminus \pi(R)$ is joined to r in $\Gamma_{\mathrm{s}}(G)$, and since $\Gamma_{\mathrm{s}}(\overline{G})$ is connected with more than two vertices, we conclude that $\Gamma_{\mathrm{s}}(G)$ has a 3-cycle containing r, a contradiction. Therefore, $r \in \pi(\overline{G})$ and $\Gamma_{\mathrm{s}}(\overline{G}) \subseteq \Gamma_{\mathrm{s}}(G)$ which is a star graph with central vertex r. Since $\Gamma_{\mathrm{c}}(P) \subseteq \Gamma_{\mathrm{s}}(P) \subseteq \Gamma_{\mathrm{s}}(\overline{G})$, therefore $\Gamma_{\mathrm{s}}(P)$ is also star with central vertex r, while $\Gamma_{\mathrm{c}}(P)$ is a forest and its connected components consist of the following possibilities: $\{r$ and its neighbours$\}$ and $\{q$, a single prime$\}$ (see Fig. 5).

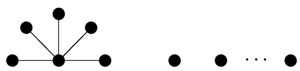

Figure 5. The cyclic graph $\Gamma_{\mathrm{c}}(P)$.

Note that, from the structures of the solvable graph $\Gamma_{\mathrm{s}}(P)$ and the cyclic graph $\Gamma_{\mathrm{c}}(P)$, it is easily seen that *they do not contain two vertices with degrees greater than or equal* 2. Since $\Gamma_{\mathrm{c}}(P)$ is a forest, P is isomorphic to one of the following simple groups ([13, Proposition 4])[*1]:

(1) $\mathbb{A}_5$, $\mathbb{A}_6$, $\mathbb{A}_7$, $\mathbb{A}_8$; M_{11}, M_{12}, M_{22}, M_{23};
(2) $L_4(3)$, $B_2(3)$, $G_2(3)$, $U_4(3)$, $U_5(2)$, ${}^2F_4(2)'$;
(3) $L_2(q)$ with $q \geqslant 4$, $|\pi\big((q-1)/(2, q-1)\big)| \leqslant 2$ and $|\pi\big((q+1)/(2, q-1)\big)| \leqslant 2$;
(4) $L_3(q)$ with $|\pi\big((q^2+q+1)/(3, q-1)\big)| \leqslant 2$ and $|\pi\big((q^2-1)/(3, q-1)\big)| \leqslant 2$;
(5) $U_3(q)$ with $|\pi\big((q^2-q+1)/(3, q+1)\big)| \leqslant 2$ and $|\pi\big((q^2-1)/(3, q+1)\big)| \leqslant 2$;
(6) $\mathrm{Sz}(q)$, with $|\pi(q \pm \sqrt{2q}+1)| \leqslant 2$ and $|\pi(q-1)| \leqslant 2$, where $q = 2^{2m+1} > 2$ and $2m + 1$ is an odd prime;
(7) $R(q)$ with $|\pi(q \pm \sqrt{3q}+1)| \leqslant 2$ and $|\pi(q \pm 1)| \leqslant 2$, where $q = 3^{2m+1} > 3$ and $2m + 1$ is an odd prime.

[*1] Notice that there are two misprints in [13, List A], that is: M_{12} and M_{23}.

Case (1). $P \cong \mathbb{A}_5$, $\mathbb{A}_6$, $\mathbb{A}_7$, $\mathbb{A}_8$, M_{11}, M_{12}, M_{22} or M_{23}. In this case, the alternating groups $\mathbb{A}_5$ and $\mathbb{A}_6$ are the only simple groups among others whose solvable graphs are stars (with central vertex 2 for both of them). Note that, from [7], the solvable graphs of these groups as follows:

$$\Gamma_s(\mathbb{A}_5) = \Gamma_s(\mathbb{A}_6) : 3 \approx 2 \approx 5; \quad \Gamma_s(\mathbb{A}_7) : 5 \approx 2 \approx 3 \approx 7;$$

$$\Gamma_s(\mathbb{A}_8) : 3 \approx 2 \approx 5 \approx 3 \approx 7;$$

$$\Gamma_s(M_{11}) = \Gamma_s(M_{12}) : 3 \approx 2 \approx 5 \approx 11; \quad \Gamma_s(M_{22}) : 11 \approx 5 \approx 2 \approx 3 \approx 7 \approx 2;$$

$$\Gamma_s(M_{23}) : 23 \approx 11 \approx 5 \approx 2 \approx 3 \approx 7 \approx 2 \bigcup 3 \approx 5.$$

Case (2). $P \cong L_4(3)$, $U_4(2)$, $G_2(3)$, $U_4(3)$, $U_5(2)$ or ${}^2F_4(2)'$. Again, from [7], we can easily determine the solvable graphs of these groups, as shown below:

$$\Gamma_s(L_4(3)) : 5 \approx 2 \approx 3 \approx 13; \qquad \Gamma_s(U_4(2)) : 5 \approx 2 \approx 3;$$

$$\Gamma_s(G_2(3)) : 2 \approx 7 \approx 3 \approx 13 \approx 2 \approx 3; \qquad \Gamma_s(U_4(3)) : 5 \approx 2 \approx 3 \approx 7;$$

$$\Gamma_s(U_5(2)) : 11 \approx 5 \approx 2 \approx 3 \approx 5; \qquad \Gamma_s({}^2F_4(2)') : 5 \approx 2 \approx 3 \approx 13 \approx 2.$$

Clearly, $U_4(2)$ is the only simple group for which the solvable graph is star. Therefore, P can only be isomorphic to $U_4(2)$.

Case (3). $P \cong L_2(q)$ with $q = p^f \geqslant 4$ and $|\pi\big((q \pm 1)/(2, q-1)\big)| \leqslant 2$. First of all, we recall that

$$\mu(L_2(q)) = \left\{p, \frac{q-1}{d}, \frac{q+1}{d}\right\},$$

where q is a power of the prime p and $d = (q-1, 2)$. Moreover, in order to draw $\Gamma_s(P)$ we need some information about the structure of subgroups of P. We state here a result [18, Theorem 6.25] which determines the structure of all subgroups of $L_2(q)$: *Let q be a power of the prime p and let $d = (q-1, 2)$. Then, a subgroup of $L_2(q)$ is isomorphic to one of the following groups*:

- *The dihedral groups of order $2(q \pm 1)/d$ and their subgroups.*
- *The group $(\mathbb{Z}_p)^f \rtimes \mathbb{Z}_{(q-1)/d}$ of order $q(q-1)/d$ and its subgroups.*
- *$L_2(r)$ or $\mathrm{PGL}(2, r)$, where r is a power of p such that $r^m = q$.*
- *$\mathbb{A}_4$, $\mathbb{S}_4$ or $\mathbb{A}_5$.*

We deal with odd and even q cases separately.

(3.1) $q \geqslant 4$ *is even.* In this case, we get the compact form of $\Gamma_s(P)$ as follows:

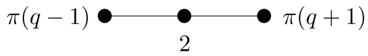

Figure 6. $\Gamma_s(L_2(q))$, $q \geqslant 2$ is even.

Since $\Gamma_s(P)$ is a star graph, this forces $|\pi(q-1)| = |\pi(q+1)| = 1$. From Corollary 3, we conclude that $q = 4, 8$, and so P is isomorphic to $L_2(4) \cong \mathbb{A}_5$ or $L_2(8)$.

(3.2) $q \geqslant 5$ *is odd.* Here, we get the compact form of $\Gamma_s(P)$ as follows:

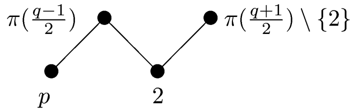

Figure 7. $\Gamma_s(L_2(q))$, $5 < q \equiv -1 \pmod 4$.

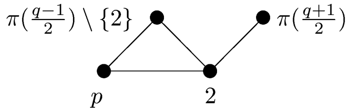

Figure 8. $\Gamma_s(L_2(q))$, $5 \leqslant q \equiv 1 \pmod 4$.

If $q \equiv -1 \pmod 4$, then $\Gamma_s(P)$ is a star graph if and only if $|\pi(q+1)| = |\pi(\frac{q-1}{2})| = 1$. Since $|\pi(q+1)| = 1$, Corollary 3 implies that q is a Mersenne prime, say $q := 2^r - 1$ for some odd prime r. But then, we obtain $|\pi((q-1)/2)| = |\pi(2^{r-1} - 1)| = 1$. In view of Corollary 3 this is possible only for $r = 3$. Therefore $q = 7$ and $P \cong L_2(7)$.

If $q \equiv 1 \pmod 4$, then $\Gamma_s(P)$ is a star graph if and only if $|\pi(q-1)| = |\pi(\frac{q+1}{2})| = 1$. Since $|\pi(q-1)| = 1$, in view of Corollary 3 it follows that $q = 9$ or q is a Fermat prime. Let $q := 2^{2^t} + 1$. Now, easy calculations show that $|\pi(\frac{q+1}{2})| = |\pi(2^{2^t - 1} + 1)| = 1$. Again, by Corollary 3 this is possible only for $t = 2$, and so $q = 17$. Therefore, P is isomorphic to $L_2(9) \cong \mathbb{A}_6$ or $L_2(17)$.

Before proceeding to other cases, it seems appropriate to point out that the spectra of the simple groups $L_3(q)$ and $U_3(q)$. We will study together these groups, and, in order to unify our treatment, we introduce the following useful notation. For $\epsilon \in \{+, -\}$ we let $L_3^\epsilon(q) = L_3(q)$ if $\epsilon = +$; and $L_3^\epsilon(q) = U_3(q)$ if $\epsilon = -$. For simplicity, we always identify $q - \epsilon$ with $q - \epsilon 1$. Now, the set of maximal elements in the spectrum of $L_3^\epsilon(q)$, $\epsilon = \pm$, is as follows:

$$\mu(L_3^{\epsilon}(q)) = \begin{cases} \left\{q-\epsilon, \dfrac{p(q-\epsilon)}{3}, \dfrac{q^2-1}{3}, \dfrac{q^2+\epsilon q+1}{3}\right\} & \text{if } d=3; \\ \{p(q-\epsilon), q^2-1, q^2+\epsilon q+1\} & \text{if } d=1, \end{cases}$$

where $q = p^n$ is odd and $d = (3, q-\epsilon)$, and

$$\mu(L_3^{\epsilon}(2^n)) = \begin{cases} \left\{4, 2^n-\epsilon, \dfrac{2(2^n-\epsilon)}{3}, \dfrac{2^{2n}-1}{3}, \dfrac{2^{2n}+\epsilon 2^n+1}{3}\right\} & \text{if } d=3; \\ \{4, 2(2^n-\epsilon), 2^{2n}-1, 2^{2n}+\epsilon 2^n+1\} & \text{if } d=1, \end{cases}$$

where $d = (3, 2^n-\epsilon)$, except $(\epsilon, n) \in \{(+,1), (+,2)\}$. It can be checked in the Atlas [7], that if $(\epsilon, n) = (+,1)$, then $L_3(2) \cong L_2(7)$ and $\mu(L_3(2)) = \{3,4,7\}$, while if $(\epsilon, n) = (+,2)$, then $\mu(L_3(4)) = \{3,4,5,7\}$.

Case (4). $P \cong L_3(q)$ with $|\pi\big((q^2+q+1)/(3,q-1)\big)| \leqslant 2$ and $|\pi\big((q^2-1)/(3,q-1)\big)| \leqslant 2$. First of all, the latter inequality forces (see [13, Lemma 2]): q = 2, 3, 4, 5, 7, 8, 9, 16, 17, 25, 49, 97 or q is a prime number satisfies the conditions $q-1 = 3\cdot 2^{\alpha}$ and $q+1 = 2t$, where $\alpha \geqslant 2$ and t is an odd prime. Moreover, note that the simple group $L_3(q)$ has a maximal subgroup of order $3(q^2+q+1)/(3,q-1)$ (see for example [11, Theorems 2.4 and 2.5]). Now, from this fact and the spectra of these groups, it is easy to check that:

- if $q = 5, 7, 9, 17, 25, 49, 97$ or q is a prime number satisfies the conditions $q-1 = 3\cdot 2^{\alpha}$ and $q+1 = 2t$, where $\alpha \geqslant 2$ and t is an odd prime, then $d_{\rm s}(2), d_{\rm s}(3) \geqslant 2$; while
- if $q = 8$ or 16, then $d_{\rm s}(3), d_{\rm s}(7) \geqslant 2$,

which show that $\Gamma_s(P)$ can not be a star graph. If $q = 4$, then $\Gamma_{\rm s}(L_3(4))$: $2 \approx 3 \approx 5 \approx 7$, which is not a star graph. Therefore, the simple groups $L_3(2) \cong L_2(7)$ and $L_3(3)$ are the only simple groups among others whose solvable graphs are stars (with central vertex 3 for both of them).

Case (5). $P \cong U_3(q)$ with $|\pi\big((q^2-q+1)/(3,q+1)\big)| \leqslant 2$ and $|\pi\big((q^2-1)/(3,q+1)\big)| \leqslant 2$. We conclude from the latter inequality that (see [13, Lemma 2]): $q = 2^f$, f a prime, $q = 3$ or 9 or q is a prime number such that $q+1 = 3\cdot 2^{\alpha}$. Again, we recall that the simple group $U_3(q)$ has a maximal subgroup of order $3(q^2-q+1)/(3,q+1)$ (see for example [11, Theorems 2.6 and 2.7]). As previous case, it is easy to verify that $d_s(2)$, $d_s(3) \geqslant 2$ in all cases except $q \neq 3$. On the other hand, the solvable graph $\Gamma_s(U_3(3))$ is a star graph (with central vertex 3), that is: $2 \approx 3 \approx 7$.

Case (6). $P \cong \mathrm{Sz}(q)$ with $|\pi(q \pm \sqrt{2q}+1)| \leqslant 2$ and $|\pi(q-1)| \leqslant 2$, where $q = 2^{2m+1} > 2$ and $2m+1$ is an odd prime. Since $\Gamma_{\mathrm{s}}(P)$ is a star graph with central vertex 2 (see Fig. 4), it follows that $|\pi(2^{2m+1}-1)| = 1$, and from Corollary 3 we conclude that $m = 1$ and $q = 8$. Thus, $p = 2$ and P is isomorphic to $\mathrm{Sz}(8)$.

Case (7). $P \cong R(q)$, with $|\pi(q \pm \sqrt{3q}+1)| \leqslant 2$ and $|\pi(q \pm 1)| \leqslant 2$, where $q = 3^{2m+1} > 3$ and $2m+1$ is an odd prime. In this case we have a 3-cycle in $\Gamma_{\mathrm{s}}(R(q))$ (see Fig. 2).

Finally, we assume that $|\pi(R)| = 0$, i.e., $R = 1$. Since $\Gamma_{\mathrm{s}}(G)$ is a star graph, $S = \mathrm{Soc}(G)$ is a non-abelian simple group. Let $p, q, r \in \pi(S)$ be distinct three primes such that $p \approx q$. If $\pi(G/S)$ contain two distinct primes u and v, then $\Gamma_{\mathrm{s}}(G)$ contains a 3-cycle. Therefore we have $|\pi(G/S)| \leqslant 1$. Since $\Gamma_{\mathrm{s}}(G)$ is a star graph and $\Gamma_{\mathrm{s}}(S)$ is connected and $\Gamma_{\mathrm{s}}(S) \leqslant \Gamma_{\mathrm{s}}(G)$, it follows that $\Gamma_{\mathrm{s}}(S) = \Gamma_{\mathrm{s}}(G)$. Hence we may assume $G = S$. The rest of proof is similar to the proof of previous case. $\square$

Proposition 2 *Let G be a finite group such that its solvable graph is bipartite, and let R be the solvable radical of G. Then, either G is solvable with $|\pi(G)| = 2$ or G is non-solvable and one of the following statements holds*:

(a) *$R = O_p(G) \neq 1$ for some prime $p \in \pi(G)$ and $\Gamma_{\mathrm{s}}(G/R)$ is star with central vertex p. Furthermore, $S \leqslant G/R \leqslant \mathrm{Aut}(S)$ where S is one of the non-abelian simple groups as in* (1).

(b) *$R = 1$ and G contains a normal subgroup N which is isomorphic to one of the following non-abelian simple groups*:
  - (b.1) $\mathbb{A}_5 \cong L_2(4) \cong L_2(5)$, $\mathbb{A}_6 \cong L_2(9) \cong S_4(2)'$, $\mathbb{A}_7$, M_{11}, M_{12}, $U_4(2)$, $U_4(3)$, $L_4(3)$, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $L_3(4)$, $U_3(3)$ *or* $\mathrm{Sz}(8)$.
  - (b.2) $L_2(q)$, $q \equiv -1 \pmod 4$, $|\pi(q-1)| = |\pi(q+1)| = 2$.

Proof. First of all, if G is a solvable group, then $\Gamma_{\mathrm{s}}(G) = K_{|\pi(G)|}$ which is bipartite if and only if $|\pi(G)| = 2$. Therefore, we may assume that G is a non-solvable group and so $|\pi(G)| \geqslant 3$. With a similar reason, we observe that $|\pi(R)| \leqslant 2$, and hence we can consider three cases separately.

Case 1. $|\pi(R)| = 2$. Let $\pi(R) = \{p_1, p_2\}$. Evidently $p_1 \approx p_2$ in $\Gamma_{\mathrm{s}}(R)$ and so in $\Gamma_{\mathrm{s}}(G)$. On the other hand, since $|\pi(G)| \geqslant 3$, there exists a prime $q \in \pi(G) \setminus \pi(R)$. Now it follows from Lemma 1 (3) that $p_1 \approx q \approx p_2$ in

$\Gamma_{\rm s}(G)$, and so $\Gamma_{\rm s}(G)$ has a 3-cycle $p_1 \approx q \approx p_2 \approx p_1$. But this contradicts our hypothesis that $\Gamma_{\rm s}(G)$ is a bipartite graph.

Case 2. $|\pi(R)| = 1$. In this case, $R = O_p(G)$ for some prime $p \in \pi(G)$ and $d_{\rm s}(p) = |\pi(G)| - 1$. If p does not divide the order of G/R, then since $\Gamma_{\rm s}(G/R)$ is connected with at least three vertices and the fact that every prime in $\pi(G) \setminus \pi(R)$ is adjacent to p, we obtain a 3-cycle in $\Gamma_{\rm s}(G)$, which is a contradiction. Thus $p \in \pi(G/R)$ and from Lemma 1 (2) we conclude that the induced graph $\Gamma_{\rm s}(G/R)[\pi(G) \setminus \{p\}]$ is an empty graph. This means that $\Gamma_{\rm s}(G/R)$ is a star graph with central vertex p. The rest of the proof follows immediately from Proposition 1 and the fact that G/R has trivial solvable radical.

Case 3. $|\pi(R)| = 0$. In this case, $R = 1$. Obviously, for every non-trivial normal subgroup N of G, $|\pi(N)| \geqslant 3$. Now, if $\pi(N) \neq \pi(G)$, then from the connectivity of solvable graph $\Gamma_{\rm s}(N)$ and part (3) of Lemma 1, one can easily obtain a 3-cycle in $\Gamma_{\rm s}(G)$, which is a contradiction. Finally, $\pi(N) = \pi(G)$ for every non-trivial normal subgroup N of G. On the other hand, since $\Gamma_{\rm s}(G)$ is not complete, there exist at least two primes, say $p, q \in \pi(G)$, such that they are not joined in $\Gamma_{\rm s}(G)$, and hence by Lemma 4, there exists a series of normal subgroups of G, say $1 \trianglelefteq M \triangleleft N \trianglelefteq G$, such that M and G/N are $\{p, q\}'$-groups and N/M is a non-abelian simple group such that p and q are not joined in $\Gamma_{\rm s}(N/M)$. By what observed above we deduce that $M = 1$, so N is a non-abelian simple group for which $\Gamma_{\rm s}(N)$ is a bipartite graph while $\Gamma_{\rm c}(N)$ is a forest (using the maximal tori when N is a simple group of Lie type). In a similar way as in the proof of Proposition 1, it follows that N is isomorphic to one of the simple groups in (b.1) and (b.2). $\square$

We notice that a star graph is a tree consisting of one vertex adjacent to all the others. Since a tree has no cycle, every nontrivial tree is always bipartite. Therefore, from these facts and Lemma 2 we have the following corollary.

Corollary 4 *Let G be a finite group which is not a non-abelian simple group. Then the following statements are equivalent*:

(a) $\Gamma_{\rm s}(G)$ *is a tree.*
(b) $\Gamma_{\rm s}(G)$ *is a bipartite graph.*
(c) $\Gamma_{\rm s}(G)$ *is a star graph.*

Proof. We will illustrate only the proof of (b) $\Longrightarrow$ (c). The remaining proofs are obvious. Let $\Gamma_s(G)$ be a bipartite graph. Since G is not a non-abelian simple group, Lemma 2 yields that $\Gamma_s(G)$ contains a complete prime (a vertex with full degree), which forces $\Gamma_s(G)$ is a star graph. $\square$

Note that, Corollary 4 is not true for non-abelian simple groups. For example, we consider the non-abelian simple group $L_2(11)$. Obviously, the solvable graph associated with $L_2(11)$ has the form $3 \approx 2 \approx 5 \approx 11$ which is a tree, while it is not a star graph.

4. $\mathbf{OD_s}$-Characterization of Some Simple Groups

We begin this section with general results on $\mathrm{OD_s}$-characterizability of some finite groups.

Theorem 1 *Suppose H is a finite group and $|\pi(H)| = k \geqslant 3$. If $\Delta_{k-1}(H) = \emptyset$ and*

$$H \notin \{O_{2n+1}(q), S_{2n}(q) : n \geqslant 3 \textit{ and } q \textit{ is odd}\},$$

then H is $\mathrm{OD_s}$-characterizable.

Proof. First of all, since $\Delta_{k-1}(H) = \emptyset$, Lemma 2 asserts that H is a non-abelian simple group. Now, we assume that G is a finite group satisfying the conditions $|G| = |H|$ and $\mathrm{D_s}(G) = \mathrm{D_s}(H)$. From these conditions we can easily deduce that $\Delta_{k-1}(G) = \Delta_{k-1}(H) = \emptyset$, and again by Lemma 2, G is a non-abelian simple group. Actually, G and H are two non-abelian simple groups with the same order, and thus the conclusion follows from Lemma 7. $\square$

In what follows, we introduce a new terminology. Let m be a positive integer with the following factorization into distinct prime power factors $m = q_1^{e_1} q_2^{e_2} \cdots q_k^{e_k}$ for some positive integers e_i and k. We put (see [1])

$$\mathrm{mpf}(m) := \max\{q_i^{e_i} \mid 1 \leqslant i \leqslant k\}.$$

For convenience, in Tables 1, we tabulate $|S|$ and $\mathrm{mpf}(|S|)$ for sporadic simple groups S using Atlas [7]. Moreover, in a similar way as in the proof of [1, Proposition 1], we can compute the value of $\mathrm{mpf}(|S|)$ for all simple groups S of Lie type. Our results are summarized in Table 2.

Given a prime $p \geqslant 5$, we denote by $\mathcal{S}_p$ the set of all finite non-abelian

Table 1. The order and degree pattern of solvable graph and mpf of a sporadic simple group.

S	$\lvert S\rvert$	$\mathrm{D_s}(S)$	$\mathrm{mpf}(\lvert S\rvert)$
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	$(3, 3, 2, 2)$	2^7
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	$(2, 1, 2, 1)$	2^4
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	$(2, 1, 2, 1)$	2^6
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$(2, 2, 2, 1, 1)$	2^7
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	$(2, 3, 3, 1, 1)$	2^9
M^cL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	$(3, 3, 3, 2, 1)$	3^6
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	$(4, 4, 3, 2, 1, 2)$	2^{13}
Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	$(5, 4, 3, 2, 2, 2)$	2^{17}
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	$(4, 3, 2, 2, 1)$	2^{10}
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	$(5, 4, 3, 2, 2, 2)$	19
J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	$(3, 3, 2, 1, 1)$	3^5
HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	$(4, 4, 4, 3, 2, 1)$	2^{14}
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$(3, 3, 3, 2, 2, 1)$	2^7
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$(4, 3, 3, 2, 3, 1)$	2^{10}
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	$(4, 3, 3, 2, 3, 1)$	3^7
Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	$(3, 3, 3, 2, 2, 1)$	2^{18}
Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	$(6, 4, 4, 3, 3, 2, 1, 1)$	3^{13}
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	$(5, 5, 4, 3, 4, 2, 1)$	2^{21}
Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	$(5, 4, 2, 3, 2, 2)$	2^{14}
Fi'_{24}	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$	$(7, 5, 4, 4, 4, 2, 1, 1, 2)$	3^{16}
$O'N$	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	$(5, 5, 4, 2, 2, 2, 2)$	2^9
Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	$(4, 6, 3, 2, 2, 1, 2)$	3^{10}
J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$	$(8, 5, 5, 5, 4, 1, 2, 2, 2, 2)$	2^{21}
B	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$	$(8, 7, 5, 3, 4, 2, 1, 2, 3, 2, 1)$	2^{41}
Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$	$(6, 6, 3, 2, 4, 1, 2, 2)$	5^6
M	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$	$(12, 10, 8, 6, 4, 2, 3, 3, 4, 4, 3, 2, 2, 1, 2)$	2^{46}

simple groups with prime divisors at most p. Clearly, if $q \leqslant p$, then $\mathcal{S}_q \subseteq \mathcal{S}_p$. In the next theorem, we deal with the finite non-abelian simple groups in class $\mathcal{S}_{71}$. Note that, the full list of all groups in $\mathcal{S}_{71}$ has been determined in [20]. Indeed, we will show that every sporadic simple group is characterized by order and degree pattern of its solvable graph.

Theorem 2 *Let G be a finite group and S one of the* 26 *sporadic simple groups. Then G is isomorphic to S if and only if $|G| = |S|$ and $\mathrm{D_s}(G) = \mathrm{D_s}(S)$.*

Proof. We need only prove the sufficiency. Let G be a finite group satisfying the conditions $|G| = |S| = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ $(p_1 < p_2 < \cdots < p_k)$ and

Table 2. The order and mpf of a simple group of Lie type.

S	Restrictions on S	$\|S\|$	mpf($\|S\|$)
$A_n(q)$	$n \geqslant 2$	$(n+1, q-1)^{-1} q^{n(n+1)/2} \prod_{i=2}^{n+1}(q^i - 1)$	$q^{n(n+1)/2}$
$A_1(q)$	$\|\pi(q+1)\| = 1$	$(2, q-1)^{-1} q(q-1)(q+1)$	$q+1$
$A_1(q)$	$\|\pi(q+1)\| \geqslant 2$	$(2, q-1)^{-1} q(q-1)(q+1)$	q
$B_n(q)$	$n \geqslant 2$	$(2, q-1)^{-1} q^{n^2} \prod_{i=1}^{n}(q^{2i} - 1)$	q^{n^2}
$C_n(q)$	$n \geqslant 3$	$(2, q-1)^{-1} q^{n^2} \prod_{i=1}^{n}(q^{2i} - 1)$	q^{n^2}
$D_n(q)$	$n \geqslant 4$	$(4, q^n - 1)^{-1} q^{n(n-1)} (q^n - 1) \prod_{i=1}^{n-1}(q^{2i} - 1)$	$q^{n(n-1)}$
$G_2(q)$		$q^6(q^6 - 1)(q^2 - 1)$	q^6
$F_4(q)$		$q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$	q^{24}
$E_6(q)$		$(3, q-1)^{-1} q^{12}(q^9 - 1)(q^5 - 1)\|F_4(q)\|$	q^{36}
$E_7(q)$		$(2, q-1)^{-1} q^{39}(q^{18} - 1)(q^{14} - 1)(q^{10} - 1)\|F_4(q)\|$	q^{63}
$E_8(q)$		$q^{96}(q^{30} - 1)(q^{12} + 1)(q^{20} - 1)(q^{18} - 1)(q^{14} - 1)$ $(q^6 + 1)\|F_4(q)\|$	q^{120}
${}^2A_n(q)$	$n \geqslant 2$, $(n, q) \neq (2, 3), (3, 2)$	$(n+1, q+1)^{-1} q^{n(n+1)/2} \prod_{i=2}^{n+1}(q^i - (-1)^i)$	$q^{n(n+1)/2}$
${}^2A_3(2)$		$2^6 \cdot 3^4 \cdot 5$	3^4
${}^2A_2(3)$		$2^5 \cdot 3^3 \cdot 7$	2^5
${}^2B_2(q)$	$q = 2^{2m+1}$, $\|\pi(q^2+1)\| \geqslant 2$	$q^2(q^2 + 1)(q - 1)$	q^2
${}^2B_2(q)$	$q = 2^{2m+1}$, $\|\pi(q^2+1)\| = 1$	$q^2(q^2 + 1)(q - 1)$	$q^2 + 1$
${}^2D_n(q)$	$n \geqslant 4$	$(4, q^n + 1)^{-1} q^{n(n-1)} (q^n + 1) \prod_{i=1}^{n-1}(q^{2i} - 1)$	$q^{n(n-1)}$
${}^3D_4(q)$		$q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$	q^{12}
${}^2G_2(q)$	$q = 3^{2m+1}$	$q^3(q^3 + 1)(q - 1)$	q^3
${}^2F_4(q)$	$q = 2^{2m+1}$	$q^{12}(q^6 + 1)(q^4 - 1)(q^3 + 1)(q - 1)$	q^{12}
${}^2E_6(q)$		$(3, q+1)^{-1} q^{12}(q^9 + 1)(q^5 + 1)\|F_4(q)\|$	q^{36}

$\mathrm{D_s}(G) = \mathrm{D_s}(S)$, where S is one of the 26 sporadic finite simple groups. It will be convenient to consider two cases separately:

Case 1. *Let S be one of the following sporadic simple groups: M_{11}, M_{12}, M_{22}, HS, M^cL, Suz, J_3, HN, M_{23}, M_{24}, Co_3, Co_2, Fi_{23}, Co_1, Fi_{24}', $O'N$, J_4, Ly, B, M.* According to Table 1, it is easy to see that in these cases $\Delta_{k-1}(G) = \emptyset$. Therefore, it follows from Theorem 1 that G is $\mathrm{OD_s}$-characterizable, that is $G \cong S$.

Case 2. *Let S be one of the following sporadic simple groups: J_2, Fi_{22}, He, J_1, Ru, Th.* According to Table 1, in all cases we have $\Delta_{k-1}(G) \neq \emptyset \neq \Delta_2(G)$. Thus, by Lemma 6, any disjoined pair of vertices of $\Gamma_s(G)$ can be expressed by only one GKS-series, say

$$1 \trianglelefteq M \triangleleft N \trianglelefteq G. \qquad (2)$$

If we show that $N/M \cong S$, then it follows that $M = 1$ and $G = N \cong S$, as required. Clearly, N/M is a non-abelian simple group in $\mathcal{S}_{p_k}$. Moreover, if $\{p_i, p_j\}$ is a pair of vertices of $\Gamma_{\mathrm{s}}(G)$ which is expressed to be disjoined by this GKS-series, then $|N/M|$ is divisible by $p_i^{\alpha_i} p_j^{\alpha_j}$. On the other hand, all non-abelian simple groups whose order prime divisors not exceeding 100 are determined in [20, Table 1]. Comparing the order of N/M with orders of non-abelian simple groups in [20, Table 1], we obtain $N/M \cong S$. This is illustrated here for two simple groups J_2 and He, other simple groups Fi_{22}, J_1, Ru and Th may be verified similarly.

- $S = J_2$. In this case, we have $|G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ and $\mathrm{D}_{\mathrm{s}}(G) = (3, 3, 2, 2)$. Thus, $\{5, 7\}$ is a pair of vertices of $\Gamma_{\mathrm{s}}(G)$ which is expressed to be disjoint by GKS-series (2), and so N/M is a non-abelian simple group in $\mathcal{S}_7$ whose order is divisible by $5^2 \cdot 7$. Using [20, Table 1], we conclude that $N/M \cong J_2$.
- $S = He$. Here, we have $|G| = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$ and $\mathrm{D}_{\mathrm{s}}(G) = (4, 3, 2, 2, 1)$. Clearly, the pairs $\{3, 17\}$, $\{5, 17\}$ and $\{7, 17\}$ are expressed to be disjoint by GKS-series (2), and so N/M is a non-abelian simple group in $\mathcal{S}_{17}$ whose order is divisible by $3^3 \cdot 5^2 \cdot 7^3 \cdot 17$. Again by [20, Table 1], it follows that $N/M \cong He$.

This completes the proof. $\square$

Theorem 3 *All simple groups $L_2(2^f)$ $(f \geqslant 2)$, such that $|\pi(2^f + 1)| = 1$ or $|\pi(2^f - 1)| = 1$, are OD_{s}-characterizable.*

Proof. Let G be a finite group such that $|G| = |L_2(q)| = q(q^2 - 1)$ and $\mathrm{D}_{\mathrm{s}}(G) = \mathrm{D}_{\mathrm{s}}(L_2(q))$ where $q = 2^f$, $f \geqslant 2$, and $|\pi(q+1)| = 1$ or $|\pi(q-1)| = 1$. We are going to show that $G \cong L_2(q)$. Generally, the solvable graph of $L = L_2(q)$, when $q = 2^f$, is shown in Fig. 6. In addition, we have

- $d_{\mathrm{s}}(2) = |\pi(L)| - 1$,
- $d_{\mathrm{s}}(s) = |\pi(q - 1)|$ for every prime $s \in \pi(q - 1)$,
- $d_{\mathrm{s}}(r) = |\pi(q + 1)|$ for every prime $r \in \pi(q + 1)$.

Under our assumptions, we may assume that $\pi(q-1) = \{p\}$ or $\pi(q+1) = \{p\}$, where p is a prime number. Now, it follows from Corollary 3 that $q - 1 = p$ or $q+1 = p$, where p is a prime. Clearly, $\tilde{\Gamma}(G) = (\Gamma_{\mathrm{s}}(G) - \{2\})^c$ is connected, and hence, any disjoint pair of vertices of $\Gamma_{\mathrm{s}}(G)$ can be expressed by only one GKS-series, say $1 \trianglelefteq M \triangleleft N \trianglelefteq G$, such that M and G/N are 2-groups.

Note that, 2 is the only vertex which is adjacent to all other vertices and $d_s(p) = 1$ (i.e. $p \approx 2$). Let $|M| = 2^m$ and $|G/N| = 2^k$. Thus, $q^2 - 1$ divides the order of N/M and since N/M is a non-abelian simple group, it follows that $|N/M|$ is also divisible by 4. In more details, we have

$$|N/M| = 2^{f'}(q^2 - 1) = 2^{f'}(2^{2f} - 1),$$

where $f' = f - (m + k)$. On the other hand, according to the classification of finite simple groups, the possibilities for N/M are: an alternating group $\mathbb{A}_m$ on $m \geqslant 5$ letters, one of the 26 sporadic simple groups, and a simple group of Lie type.

If $N/M \cong L_2(q)$, then $M = 1$, $N = G$ and so $G \cong L_2(q)$, as required. Therefore, from now on, we assume that N/M is isomorphic to non-abeilan simple group $S \ncong L_2(q)$, and we will try to get a contradiction. First of all, we notice that $m \neq 0$ or $k \neq 0$. In fact, if $m = k = 0$, then $M = 1$, $N = G$, and $G = N = N/1 = N/M \cong S$. Thus S and $L_2(q)$ are non-isomorphic simple groups with the same order, which is a contradiction by Artin Theorem.

In the rest of proof we will try to get a contradiction from the following equality:[*2]

$$\mathrm{mpf}(|S|) = \mathrm{mpf}(|N/M|) = \mathrm{mpf}(2^{f'}(2^{2f} - 1)).$$

First, we compute the value $\mathrm{mpf}(2^{f'}(2^{2f} - 1))$. In the case when $q + 1 = p$, it is easy to see that

$$\mathrm{mpf}(2^{f'}(2^{2f} - 1)) = \mathrm{mpf}(2^{f'}(2^f - 1)(2^f + 1)) = \mathrm{mpf}(2^{f'}(2^f - 1)p) = p,$$

because $2^f - 1 < 2^f < 2^f + 1 = p$. Note that, the numbers $2^f - 1$, 2^f and $2^f + 1$ are pairwise coprime. Similarly, in the case when $q - 1 = p$, we obtain

$$\begin{aligned}\mathrm{mpf}(2^{f'}(2^{2f} - 1)) &= \mathrm{mpf}(2^{f'}(2^f - 1)(2^f + 1)) \\ &= \mathrm{mpf}(2^{f'}p(2^f + 1)) = \begin{cases} 5 & \text{if } f = 2, \\ p & \text{if } f \neq 2. \end{cases}\end{aligned}$$

[*2]The idea of proof was borrowed from [1].

(1) *S is not isomorphic to an alternating group $\mathbb{A}_m$, $m \geqslant 5$.*

Assume that S is isomorphic to an alternating group $\mathbb{A}_m$, $m \geqslant 5$. From the equality

$$\mathrm{mpf}(|\mathbb{A}_m|) = \mathrm{mpf}(|S|) = \mathrm{mpf}(|N/M|) = \mathrm{mpf}(2^{f'}(2^{2f} - 1)) = p,$$

we deduce that $p = \max \pi(\mathbb{A}_m)$, and so $m \geqslant p$. On the other hand, we have

$$\frac{m!}{2} = |\mathbb{A}_m| = |S| = |N/M| = 2^{f'}(2^{2f} - 1) = 2^{f'}p(p \pm 2),$$

which is a contradiction.

(2) *S is not isomorphic to one of the* 26 *sporadic simple groups.*

Suppose that S is isomorphic to one of the 26 sporadic simple groups. An argument similar to that in the previous case shows that $\mathrm{mpf}(|S|) = \mathrm{mpf}(|N/M|) = p$ (a prime number), which forces $S \cong J_1$ (see [7]). But then, $\mathrm{mpf}(|J_1|) = 19 = 2^f \pm 1$, which is a contradiction.

(3) *S is not isomorphic to a simple group of Lie type, except $L_2(q)$.*

We only discuss on some of these cases, for example, we consider the cases $A_n(q_0)$, ${}^3D_4(q_0)$, ${}^2E_6(q_0)$, other cases are similar, so we omit them.

- Suppose that S is isomorphic to $A_n(q_0)$ for some integer $n \geqslant 2$ and for a power q_0 of a prime p_0. Then, we have

$$|S| = |A_n(q_0)| = (n+1, q_0 - 1)^{-1} \cdot q_0^{n(n+1)/2} \prod_{i=2}^{n+1} (q_0^i - 1).$$

By [1], we have

$$\mathrm{mpf}(|A_n(q_0)|) = q_0^{n(n+1)/2} \quad (n \geqslant 2),$$

and hence

$$\begin{aligned} q_0^{n(n+1)/2} &= \mathrm{mpf}(|A_n(q_0)|) = \mathrm{mpf}(|S|) = \mathrm{mpf}(|N/M|) \\ &= \mathrm{mpf}(2^{f'}(2^{2f} - 1)) = p. \end{aligned}$$

This shows that $q_0 = p_0 = p$ and $n(n+1)/2 = 1$, which is a contradiction.

- Suppose that S is isomorphic to ${}^3D_4(q_0)$. Then, we have

$$|{}^3D_4(q_0)| = q_0^{12}(q_0^8 + q_0^4 + 1)(q_0^6 - 1)(q_0^2 - 1).$$

One can easily obtain that $\mathrm{mpf}(|{}^3D_4(q_0)|) = q_0^{12}$. But then, we observe that

$$\begin{aligned} q_0^{12} &= \mathrm{mpf}(|{}^3D_4(q_0)|) = \mathrm{mpf}(|S|) = \mathrm{mpf}(|N/M|) \\ &= \mathrm{mpf}(2^{f'}(2^{2f} - 1)) = p, \end{aligned}$$

which is a contradiction.

- Suppose that S is isomorphic to ${}^2E_6(q_0)$. Then, we have

$$|S| = |{}^2E_6(q_0)| = q_0^{36}(q_0^{12} - 1)(q_0^9 + 1)(q_0^8 - 1)(q_0^6 - 1)(q_0^5 + 1)(q_0^2 - 1).$$

It is obvious that $\mathrm{mpf}(|{}^2E_6(q_0)|) = q_0^{36}$, and so we deduce that $q_0^{36} = p$, which is a contradiction.

This completes the proof of theorem. $\square$

Theorem 4 *Let G be a finite group satisfying $|G| = |L_2(q)|$ and $\mathrm{D_s}(G) = \mathrm{D_s}(L_2(q))$, where $q = p^f > 3$. Furthermore, assume one of the following conditions is fulfilled*:

(a) $q \equiv 1 \pmod 4$, *and* $|\pi(q+1)| = 2$ *or* $|\pi(q-1)| \leqslant 2$;
(b) $q \equiv -1 \pmod 4$.

Then $G \cong L_2(q)$.

Proof. (a) The solvable graph of $L_2(q)$, where $q \equiv 1 \pmod 4$, is shown in Fig. 8. If $|\pi(\frac{q+1}{2})| = 1$ or $|\pi(q-1)| \leqslant 2$, then $\tilde{\Gamma}(G)$ is connected by Lemma 6 (1). Therefore, any disjoint pair of vertices of $\Gamma_{\mathrm{s}}(G)$ can be expressed by only one GKS-series, say $1 \trianglelefteq M \vartriangleleft N \trianglelefteq G$. Note that M and G/N are 2-groups because 2 is the only prime whose degree is complete and N/M is a non-abelian simple group such that $\pi(G) = \pi(N/M)$. Let $|M| = 2^m$ and $|G/N| = 2^k$. Then, we have

$$|N/M| = q(q^2 - 1)/2^{k+m+1}.$$

We need first to compute $\mathrm{mpf}(|N/M|)$. If $|\pi(q+1)| = 2$, then $(q+1)/2 < q$, $q-1 < q$ and $|N/M|_2 \leqslant |G|_2 \leqslant q-1 < q$, which shows that $\mathrm{mpf}(|N/M|) = q$. Similarly, if $|\pi(q-1)| \leqslant 2$, then it is easy to see that $\mathrm{mpf}(|N/M|) = q$.

If $N/M = L_2(q)$, then $M = 1$, $N = G$ and $G = L_2(q)$, as desired. Therefore, from now on, we assume that $N/M \neq L_2(q)$. Now, we will compare the values $\mathrm{mpf}(|N/M|)$ and $\mathrm{mpf}(|S|)$ for all other non-abelian simple groups to get a contradiction.

Suppose first that N/M is a simple group of Lie type. If N/M is isomorphic to $A_n(q_0)$ for some integer $n \geqslant 2$ and for a power q_0 of a prime p_0, then we have

$$\begin{aligned} q(q^2-1)/2^{k+m+1} &= |N/M| = |A_n(q_0)| \\ &= (n+1, q_0-1)^{-1} q_0^{n(n+1)/2} \prod_{i=2}^{n+1} (q_0^i - 1), \end{aligned}$$

and also (see Table 2)

$$q = \mathrm{mpf}(|N/M|) = \mathrm{mpf}(|A_n(q_0)|) = q_0^{n(n+1)/2}.$$

We now observe that

$$|N/M| = q(q^2-1)/2^{k+m+1} = q_0^{n(n+1)/2}(q_0^{n(n+1)} - 1)/2^{k+m+1},$$

which forces $\mathrm{ppd}(q_0^{n(n+1)} - 1) \in \pi(N/M) = \pi(A_n(q_0))$, a contradiction.

If N/M is isomorphic to ${}^2E_6(q_0)$, then we have

$$|N/M| = q_0^{36}(q_0^{12}-1)(q_0^9+1)(q_0^8-1)(q_0^6-1)(q_0^5+1)(q_0^2-1),$$

and also (see Table 2)

$$q = \mathrm{mpf}(|N/M|) = \mathrm{mpf}(|{}^2E_6(q_0)|) = q_0^{36}.$$

But then, we obtain

$$|N/M| = q(q^2-1)/2^{k+m+1} = q_0^{36}(q_0^{72}-1),$$

and it follows that $\mathrm{ppd}(q_0^{72}-1) \in \pi(N/M) = \pi({}^2E_6(q_0))$, a contradiction.

The possibility for N/M to be isomorphic to another simple group of Lie type would be terminated in the same way. Similarly, when N/M is isomorphic to an alternating or a sporadic simple group we can also derive a contradiction.

(b) The solvable graph of $L = L_2(q)$, where $q \equiv -1 \pmod 4$, is shown in Fig. 7. Since $\Delta_{|\pi(L)|-1}(L) = \emptyset$, L is $\mathrm{OD_s}$-characterizable by Theorem 1. $\square$

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