

Absence of zero resonances of massless Dirac operators

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(Received January 31, 2014; Revised April 15, 2014)

Abstract. We consider the massless Dirac operator $H = \alpha \cdot D + Q(x)$ on the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^4)$, where $Q(x)$ is a 4×4 Hermitian matrix valued function which decays suitably at infinity. We show that the zero resonance is absent for H , extending recent results of Saitō-Umeda [6] and Zhong-Gao [7].

Key words: Dirac operators, zero resonances.

1. Introduction and Result

We consider the massless Dirac operator

$$H = \alpha \cdot D + Q(x), \quad D = -i\nabla_x, \quad x \in \mathbb{R}^3, \quad (1.1)$$

acting on \mathbb{C}^4 -valued functions on \mathbb{R}^3 . Here $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the triple of 4×4 Dirac matrices:

$$\alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} \quad j = 1, 2, 3,$$

with the 2×2 zero matrix $\mathbf{0}$ and the triple of 2×2 Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and we use the vector notation that $(\alpha \cdot D)u = \sum_{j=1}^3 \alpha_j (-i\partial_{x_j})u$. We assume that $Q(x)$ is a 4×4 Hermitian matrix valued function satisfying the following assumption:

Assumption 1.1 *There exists a positive constant C and $\rho > 1$ such that, for each component $q_{jk}(x)$ ($j, k = 1, \dots, 4$) of $Q(x)$,*

$$|q_{jk}(x)| \leq C\langle x \rangle^{-\rho}, \quad x \in \mathbb{R}^3.$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$.

We remark that the Dirac operator for a Dirac particle minimally coupled to the electromagnetic field described by the potential (q, A) is given by

$$\alpha \cdot (D - A(x)) + q(x)I_4, \tag{1.2}$$

where I_4 is the 4×4 identity matrix, and is a special case of (1.1). And, if $q(x) = 0$, (1.2) reduces to

$$\alpha \cdot (D - A(x)) = \begin{pmatrix} \mathbf{0} & \sigma \cdot (D - A(x)) \\ \sigma \cdot (D - A(x)) & \mathbf{0} \end{pmatrix},$$

where $\sigma \cdot (D - A(x))$ is the Weyl-Dirac operator.

To state the result of the paper, we introduce some notations and terminologies. \mathcal{F} is the Fourier transform:

$$(\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(\xi)e^{-ix \cdot \xi} d\xi.$$

We often write $\hat{f}(\xi) = (\mathcal{F}f)(\xi)$ and $\check{f}(\xi) = (\mathcal{F}^{-1}f)(\xi)$. $\mathcal{L}^2(\mathbb{R}^3) = L^2(\mathbb{R}^3, \mathbb{C}^4)$ is the Hilbert space of all \mathbb{C}^4 -valued square integrable functions. For $s \in \mathbb{R}$, $\mathcal{L}^{2,s}(\mathbb{R}^3) = L^{2,s}(\mathbb{R}^3, \mathbb{C}^4) := \langle x \rangle^{-s} L^2(\mathbb{R}^3, \mathbb{C}^4)$ is the weighted $\mathcal{L}^2(\mathbb{R}^3)$ space. $\mathcal{S}'(\mathbb{R}^3) = \mathcal{S}'(\mathbb{R}^3, \mathbb{C}^4)$ is the space of \mathbb{C}^4 -valued tempered distributions. $\mathcal{H}^s(\mathbb{R}^3) = H^s(\mathbb{R}^3, \mathbb{C}^4)$ is the Sobolev space of order s :

$$\mathcal{H}^s(\mathbb{R}^3) = \{f \in \mathcal{S}'(\mathbb{R}^3) \mid \hat{f} \in \mathcal{L}^{2,s}(\mathbb{R}^3)\}$$

with the inner product $(f, g)_{\mathcal{H}^s} = \sum_{j=1}^4 (\hat{f}_j, \hat{g}_j)_{L^{2,s}}$. The spaces $\mathcal{H}^{-s}(\mathbb{R}^3)$ and $\mathcal{H}^s(\mathbb{R}^3)$ are duals of each other with respect to the coupling

$$\langle f, g \rangle := \sum_{j=1}^4 \int_{\mathbb{R}^3} (\mathcal{F}f_j)(\xi) \overline{(\mathcal{F}g_j)(\xi)} d\xi, \quad f \in \mathcal{H}^{-s}(\mathbb{R}^3), \quad g \in \mathcal{H}^s(\mathbb{R}^3).$$

For Hilbert spaces \mathcal{X} and \mathcal{Y} , $B(\mathcal{X}, \mathcal{Y})$ stands for the Banach space of bounded operators from \mathcal{X} to \mathcal{Y} , $B(\mathcal{X}) = B(\mathcal{X}, \mathcal{X})$.

It is well known that the free Dirac operator $H_0 := \alpha \cdot D$ is self-adjoint in

$\mathcal{L}^2(\mathbb{R}^3)$ with domain $\mathcal{D}(H_0) = \mathcal{H}^1(\mathbb{R}^3)$. Hence by the Kato-Rellich theorem, H is also self-adjoint in $\mathcal{L}^2(\mathbb{R}^3)$ with domain $\mathcal{D}(H) = \mathcal{D}(H_0)$. We denote their self-adjoint realizations again by H_0 and H respectively. In what follows, we write $H_0 f$ also for $(\alpha \cdot D)f$ when $f \in \mathcal{S}'(\mathbb{R}^3)$.

Definition 1.2 If $f \in \mathcal{L}^2(\mathbb{R}^3)$ satisfies $Hf = 0$, we say f is a zero mode of H . If $f \in \mathcal{L}^{2,-3/2}(\mathbb{R}^3)$ satisfies $Hf = 0$ in the sense of distributions, but $f \notin \mathcal{L}^2(\mathbb{R}^3)$, then f is said to be a zero resonance state and zero to be a resonance of it.

The following is the main result of this paper.

Theorem 1.3 Let $Q(x)$ satisfy Assumption 1.1. Suppose $f \in \mathcal{L}^{2,-3/2}(\mathbb{R}^3)$ satisfies $Hf = 0$ in the sense of distributions, then for any $\mu < 1/2$, we have $\langle x \rangle^\mu f \in \mathcal{H}^1(\mathbb{R}^3)$. In particular, there is no resonance for H .

Remark 1.4 The decay result $\langle x \rangle^\mu f \in \mathcal{H}^1(\mathbb{R}^3)$, $\mu < 1/2$ cannot be improved. This can be seen from the example of zero mode of the Weyl-Dirac operator which was constructed by Loss-Yau [3]. Loss and Yau have constructed a vector potential $A_{LY}(x)$ and a zero mode $\phi_{LY}(x)$ satisfying $\sigma \cdot (D - A_{LY}(x))\phi_{LY} = 0$, where A_{LY} and ϕ_{LY} satisfy $A_{LY}(x) = \mathcal{O}(\langle x \rangle^{-2})$, $|\phi_{LY}(x)| = \langle x \rangle^{-2}$. Define $f_{LY} = {}^t(0, \phi_{LY})$ and $Q(x) = -\alpha \cdot A_{LY}(x)$, then

$$Hf_{LY} = (H_0 + Q)f_{LY} = \begin{pmatrix} \mathbf{0} & \sigma \cdot (D - A_{LY}(x)) \\ \sigma \cdot (D - A_{LY}(x)) & \mathbf{0} \end{pmatrix} f_{LY} = 0,$$

and $f_{LY} \in \mathcal{L}^{2,\mu}(\mathbb{R}^3)$ for any $\mu < 1/2$. However, $f_{LY} \notin \mathcal{L}^{2,1/2}(\mathbb{R}^3)$.

We remark that Saitō-Umeda [6] have proven the following result under the assumption $|Q(x)| \leq C\langle x \rangle^{-\rho}$, $\rho > 3/2$. If f satisfies $f \in \mathcal{L}^{2,-s}(\mathbb{R}^3)$ for some $0 < s \leq \min\{3/2, \rho - 1\}$ and $Hf = 0$ in the sense of distributions, then $f \in \mathcal{H}^1(\mathbb{R}^3)$. In [7], Zhong and Gao have pointed out that arguments of [6] go through under the assumption $\rho > 1$. Our theorem improves over results of [6] and [7] by weakening the assumption $f \in \mathcal{L}^{2,-s}(\mathbb{R}^3)$ to $\mathcal{L}^{2,-3/2}(\mathbb{R}^3)$, which is independent of $\rho > 1$, and by strengthening the result $f \in \mathcal{H}^1(\mathbb{R}^3)$ to a sharp decay estimate $\langle x \rangle^\mu f \in \mathcal{H}^1(\mathbb{R}^3)$, $\mu < 1/2$.

We briefly explain the significance of the theorem. The solution of the time-dependent Dirac equation

$$i \frac{\partial u}{\partial t} = Hu, \quad u(0) = \phi$$

is given by $e^{-itH}\phi$. Under Assumption 1.1, it has been proven that the spectrum $\sigma(H) = \mathbb{R}$, the limiting absorption principle is satisfied and that $\sigma_p(H) \setminus \{0\}$ is discrete. To make the argument simple, we assume $\sigma_p(H) \subset \{0\}$. Then for $\phi \in \mathcal{L}_{ac}^2(H)$, the absolutely continuous spectral subspace of \mathcal{L}^2 for H , $e^{-itH}\phi$ may be represented in terms of the boundary values of the resolvent $(H - \lambda \pm i0)^{-1}$:

$$e^{-itH}\phi = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} e^{-it\lambda} \{ (H - \lambda - i0)^{-1} - (H - \lambda + i0)^{-1} \} \phi d\lambda, \quad t > 0,$$

and the asymptotic behavior as $t \rightarrow \pm\infty$ of $e^{-itH}\phi$ depends on whether (1) $\lambda = 0$ is a regular point, viz, $(H - (\lambda \pm i0))^{-1}$ is smooth up to $\lambda = 0$, (2) $\lambda = 0$ is a resonance of it, (3) $\lambda = 0$ is an eigenvalue of H or (4) $\lambda = 0$ is an eigenvalue at the same time is a resonance. Thus, Theorem 1.3 eliminates the possibility (2) and (4). We should recall that if $m \neq 0$, then all four cases mentioned above appear at the threshold points $\pm m$. It is well-known that $\lambda = 0$ is not a regular point if $f + (H_0 \pm i0)^{-1}Qf = 0$ has a non-trivial solution $f \in \mathcal{L}^{2, -\rho/2}$ and this f satisfies $Hf = 0$. Note that, to conclude that $f \in \mathcal{H}^1$ by using results of [6] or [7], we need assume $0 < \rho/2 \leq \min\{3/2, \rho - 1\}$ or $2 \leq \rho \leq 3$, which is a severe restriction for this application, whereas Theorem 1.3 does not impose only such restriction.

The rest of the paper is devoted to the proof of Theorem 1.3. In Section 2, we prepare some lemmas for proving the main theorem. In Section 3, we prove the main theorem 1.3.

2. Preliminaries

In this section, we prepare some lemmas which are necessary for proving the theorem. We use the following well-known lemma:

Theorem 2.1 (Nirenberg-Walker [2]) *Let $1 < p < \infty$ and let $a, b \in \mathbb{R}$ be such that $a + b > 0$. Define*

$$k(x, y) = \frac{1}{|x|^a |x - y|^{d-(a+b)} |y|^b}, \quad x, y \in \mathbb{R}^d, \quad x \neq y.$$

Then, integral operator

$$(K\phi)(x) = \int_{\mathbb{R}^d} k(x, y)\phi(y)dy$$

is bounded in $L^p(\mathbb{R}^d)$ if and only if $a < d/p$ and $b < d/q$, where $q = p/(p-1)$ is the dual exponent of p .

For $f = {}^t(f_1, f_2, f_3, f_4)$, we define the integral operator T by

$$(Tf)(x) = \frac{i}{4\pi} \int_{\mathbb{R}^3} \frac{\alpha \cdot (x - y)}{|x - y|^3} f(y)dy.$$

Since

$$\frac{i}{4\pi} \mathcal{F}^{-1} \left(\frac{\xi}{|\xi|^3} \right) (x) = \frac{1}{(2\pi)^{3/2}} \frac{x}{|x|^2},$$

it is obvious that

$$\mathcal{F}^{-1}(Tf)(x) = \frac{\alpha \cdot x}{|x|^2} (\mathcal{F}^{-1}f)(x) = (\alpha \cdot x)^{-1} (\mathcal{F}^{-1}f)(x) \tag{2.1}$$

Lemma 2.2 For any $t \in (-3/2, 1/2)$, $T \in B(\mathcal{L}^{2,-t}(\mathbb{R}^3), \mathcal{L}^{2,-t-1}(\mathbb{R}^3))$.

Proof. The multiplication by $\langle x \rangle^t$ is isomorphism from $\mathcal{L}^2(\mathbb{R}^3)$ onto $\mathcal{L}^{2,-t}(\mathbb{R}^3)$. It follows that $T \in B(\mathcal{L}^{2,-t}(\mathbb{R}^3), \mathcal{L}^{2,-t-1}(\mathbb{R}^3))$ if and only if $\langle x \rangle^{-t-1} T \langle x \rangle^t \in B(\mathcal{L}^2(\mathbb{R}^3))$. The absolute value of the integral kernel of $\langle x \rangle^{-t-1} T \langle x \rangle^t$ is bounded above by

$$\frac{1}{4\pi \langle x \rangle^{t+1} |x - y|^2 \langle y \rangle^{-t}}.$$

Lemma 2.2 follows by applying Lemma 2.1 with $a = t + 1$, $b = -t$, $d = 3$, $p = q = 2$. □

Lemma 2.3 Let $-3/2 < s < 1/2$. Then for any $g \in \mathcal{L}^{2,-s}(\mathbb{R}^3)$ and $\phi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}^4)$, we have the identity;

$$\langle \mathcal{F}^{-1}(Tg), \phi \rangle = \left\langle \mathcal{F}^{-1}g, \frac{\alpha \cdot x}{|x|^2} \phi \right\rangle. \tag{2.2}$$

Proof. We note that both $\mathcal{F}^{-1}g \in \mathcal{H}^{-s-1}(\mathbb{R}^3)$ and $\mathcal{F}^{-1}(Tg) \in \mathcal{H}^{-s-1}(\mathbb{R}^3)$. Indeed, the former is obvious by $g \in \mathcal{L}^{2,-s}(\mathbb{R}^3) \subset \mathcal{L}^{2,-s-1}(\mathbb{R}^3)$ and the latter follows since $Tg \in \mathcal{L}^{2,-s-1}(\mathbb{R}^3)$ by virtue of the assumption $g \in \mathcal{L}^{2,-s}(\mathbb{R}^3)$, $-3/2 < s < 1/2$ and Lemma 2.2. Let $\phi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}^4)$. Take a sequence $g_n \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ such that $\lim_{n \rightarrow \infty} \|g_n - g\|_{\mathcal{L}^{2,-s}} = 0$. Since T is continuous from $\mathcal{L}^{2,-s}(\mathbb{R}^3)$ to $\mathcal{L}^{2,-s-1}(\mathbb{R}^3)$ by virtue of Lemma 2.2, it follows that

$$\begin{aligned} \langle \mathcal{F}^{-1}(Tg), \phi \rangle &= \lim_{n \rightarrow \infty} \langle \mathcal{F}^{-1}(Tg_n), \phi \rangle \\ &= \lim_{n \rightarrow \infty} \left\langle \frac{\alpha \cdot x}{|x|^2} \mathcal{F}^{-1}g_n, \phi \right\rangle \\ &= \lim_{n \rightarrow \infty} \left\langle \mathcal{F}^{-1}g_n, \frac{\alpha \cdot x}{|x|^2} \phi \right\rangle \\ &= \left\langle \mathcal{F}^{-1}g, \frac{\alpha \cdot x}{|x|^2} \phi \right\rangle. \end{aligned}$$

Here we used (2.1) in the second step and that $((\alpha \cdot x)/|x|^2)\phi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}^4)$ in the final step. This completes the proof. \square

The following is an extension of Theorem 4.1 of [6] and plays an important role in the proof of theorem.

Lemma 2.4 *Suppose that $f \in \mathcal{L}^{2,-3/2}(\mathbb{R}^3)$ and $H_0f \in \mathcal{L}^{2,-s}(\mathbb{R}^3)$ for some $s \in (-3/2, 1/2)$. Then, f satisfies $TH_0f = f$.*

Proof. Since $f \in \mathcal{L}^{2,-3/2}(\mathbb{R}^3)$ and $TH_0f \in \mathcal{L}^{-s-1}(\mathbb{R}^3) \subset \mathcal{L}^{2,-3/2}(\mathbb{R}^3)$ by virtue of Lemma 2.2, it follows that $\mathcal{F}^{-1}f, \mathcal{F}^{-1}(TH_0f) \in \mathcal{H}^{-3/2}(\mathbb{R}^3)$. Thus, it suffice to show that

$$\langle \mathcal{F}^{-1}(TH_0f), \phi \rangle = \langle \mathcal{F}^{-1}f, \phi \rangle, \text{ for any } \phi \in \mathcal{H}^{3/2}(\mathbb{R}^3). \tag{2.3}$$

Since $C_0^\infty(\mathbb{R}^d \setminus \{0\}, \mathbb{C}^4)$ is dense in $\mathcal{H}^s(\mathbb{R}^d)$ for $s \leq d/2$, we need only prove (2.3) for $\phi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}^4)$. By setting $g = H_0f$ in (2.2) and using $\mathcal{F}^{-1}(H_0f)(x) = (\alpha \cdot x)(\mathcal{F}^{-1}f)(x)$ for $f \in \mathcal{L}^{2,-3/2}(\mathbb{R}^3)$, we have

$$\langle \mathcal{F}^{-1}(TH_0f), \phi \rangle = \left\langle (\alpha \cdot x)\mathcal{F}^{-1}f, \frac{\alpha \cdot x}{|x|^2} \phi \right\rangle$$

$$= \left\langle \mathcal{F}^{-1}f, \frac{(\alpha \cdot x)^2}{|x|^2} \phi \right\rangle = \langle \mathcal{F}^{-1}f, \phi \rangle.$$

This completes the proof. □

3. Proof of Theorem 1.3

We may assume $1 < \rho < 3$ without losing generality. We apply well-known Agmon’s bootstrap argument. Let $f \in \mathcal{L}^{2,-3/2}(\mathbb{R}^3)$ and $Hf = 0$ in the sense of distributions. Then $H_0f = -Qf \in \mathcal{L}^{2,-3/2+\rho}(\mathbb{R}^3)$ by the assumption 1.1. Since $-1/2 < \rho - 3/2 < 3/2$, we have $TQf \in \mathcal{L}^{2,-\frac{3}{2}+\rho-1}(\mathbb{R}^3)$ by virtue of Lemma 2.2. Then Lemma 2.4 implies $f = TH_0f = -TQf \in \mathcal{L}^{2,-3/2+\rho-1}(\mathbb{R}^3)$. Thus we may repeat the argument several times and obtain $f \in \mathcal{L}^{2,-3/2+n(\rho-1)}(\mathbb{R}^3)$ as long as $-3/2 + n(\rho - 1) + 1 < 3/2$. Let n_0 be the largest integer such that $-3/2 + n_0(\rho - 1) + 1 < 3/2$ so that $f \in \mathcal{L}^{2,-3/2+n_0(\rho-1)}(\mathbb{R}^3)$ and $Qf \in \mathcal{L}^{2,-3/2+n_0(\rho-1)+\rho}(\mathbb{R}^3)$, however $-3/2 + n_0(\rho - 1) + \rho > 3/2$. Then for $\mu < 1/2$ arbitrary close to $1/2$, $H_0f = -Qf \in \mathcal{L}^{2,\mu+1}(\mathbb{R}^3)$. Thus, $f \in \mathcal{L}^{2,\mu}(\mathbb{R}^3)$ by virtue of Lemma 2.2 and Lemma 2.4. By differentiating, we have

$$\begin{aligned} H_0\langle x \rangle^\mu f &= -i\mu(\alpha \cdot x)\langle x \rangle^{\mu-2}f + \langle x \rangle^\mu H_0f \\ &= -i\mu(\alpha \cdot x)\langle x \rangle^{\mu-2}f - \langle x \rangle^\mu Qf \in \mathcal{L}^2(\mathbb{R}^3). \end{aligned}$$

It follows that $\mathcal{F}(\langle x \rangle^\mu f) \in \mathcal{L}^{2,1}(\mathbb{R}^3)$ which is equivalent to $\langle x \rangle^\mu f \in \mathcal{H}^1(\mathbb{R}^3)$. This completes the proof of Theorem 1.3.

Acknowledgment I would like to express my sincere thanks to Professor Kenji Yajima for his unceasing encouragement and valuable advice.

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Partially supported by Fûjû-kai Foundation