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Projection of generic 1 and 2-parameter families of space curves

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Abstract. The present paper deals with the study of the geometrical properties of generic 1 and 2-parameter families of space curves by using projections into planes. It presents directions of projection and conditions on the coefficients of these families such that the projection exhibits Morsifications of the A_4, A_6 and E_6 singularities and transitions between the Morsifications of the E_8 singularity.

Key words: projection of space curves; plane curve singularities, Morsification.

1. Introduction

Given a C^∞ embedded space curve $\gamma : I \to \mathbb{R}^3$ where $I \subset \mathbb{R}$ is an open interval, we can look at the orthogonal projections of γ into planes. David [4] showed that there is a residual subset $\Omega \subset C^\infty(I, \mathbb{R}^3)$ endowed with the Whitney C^∞-topology such that the only singularities that can appear in any orthogonal projection of γ are the following 10 $\mathcal{A}$-classes: $A_0, A_1, \ldots, A_5, D_4, D_5, D_6$ and $\tilde{E}_7$. Moreover, each one of these 10 singularities corresponds to a geometric phenomenon of the space curve γ (zero torsion point, cross tangent, trisecant line, etc.). The curves in the subset Ω are called *projection-generic* by David in [4] (see also [12] and more recently [13]). We say that γ is *generic* if it is projection-generic in the above sense.

In [5] we obtained a classification of the singularities of orthogonal projections of a generic space curve $\gamma : I \to \mathbb{R}^3$, which takes into account the flat geometry of the projected plane curve α. This classification is a refinement of the one in [4]. (See also [11]). With the additional information on the geometry of γ, we showed that the above classes split into several $\mathcal{A}_h$-classes, totalling 17 classes for generic space curves. A geometric characterization of each of the $\mathcal{A}_h$-classes is given in [5] in terms of the geometry of both space curve and direction of projection.

Given a projection-generic curve γ, it follows from genericity that it is an embedding and moreover, for any $t \in I$, $\gamma'(t)$ and $\gamma''(t)$ are not collinear.

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In particular, it has a well defined Frenet frame. A line ℓ in $\mathbb{R}^3$ which meets the curve γ in 2, 3 or 4 points is said to be a *secant, trisecant* or *quadrisecant line*, respectively. A *cross tangent* is a secant line which is tangent to one of the points. A plane π in $\mathbb{R}^3$ which is tangent to the curve in 2 or 3 points is called a *bitangent* or *tritangent plane*, respectively. Finally, we say that π has *contact of order* n with γ at t_0 provided that $\gamma^{(k)}(t_0)$ is parallel to π for all $1 \leq k \leq n$ and $\gamma^{(n+1)}(t_0)$ is not parallel to π. A brief review of the generic curves and projections is presented in Section 2.

In this paper we study the geometric properties of generic 1 and 2-parameter families of space curves by the transition analysis of the bifurcation set of projections. This study is presented in Sections 3 and 4, see for example: Figures 1, 2, 4, 5 and their comments. By observing the bifurcation set of projections we find regions where the germ of projection has a maximal number of real double points (this is called *Morsification*). It was proved in Gusein-Zade [7] and A'Campo [1] that in the case of germs of plane curves always exist such Morsifications and Mond [8] proved the maximum number m of double points that appear in a real deformation of a quasi-homogeneous singularity is given by

$$m = \mathcal{A}_{\mathrm{e}}\text{-cod}(f) + r - 1, \tag{1}$$

where r is the number of branches of the curve.

Here we find explicitly the directions of projections and conditions on the coefficients of a generic family of space curves such that the projection exhibits Morsifications of the A_4, A_6, E_6 singularities and transitions between the Morsifications of the E_8 singularity. This study is presented in Section 5.

2. Generic curves and projections

The geometry of a space curve $\gamma : I \to \mathbb{R}^3$ can be described by analyzing the contacts of curves with the straight lines and planes in $\mathbb{R}^3$. These contacts can be studied with the help of the following functions:

Definition 2.1 Let γ be a regular space curve. Then we define the functions

$$K(t) = \|\gamma'(t) \times \gamma''(t)\|^2 \quad \text{and} \quad T(t) = \det\big(\gamma'(t), \gamma''(t), \gamma'''(t)\big).$$

Note that these are the numerators of the curvature and torsion of γ, respectively.

The properties of these functions were studied in [9] and [10]. Observe that K vanishes at t_0 if and only if γ has contact of order at least 2 with its tangent line at t_0. Provided that $K(t_0) \neq 0$, we have $T(t_0) = 0$ if and only if the osculating plane has contact at least 3 with γ at t_0.

Definition 2.2 Let t_0 be an element of I. We say that γ has an *n-flattening* at t_0, $n \geq 1$, if $T^{(i)}(t_0) = 0$, $i = 0, \dots, n$, $T^{(n+1)}(t_0) \neq 0$.

Definition 2.3 We say that γ with an n-flattening at t_0 is *of type $\mathcal{A}$* when $K(t_0) \neq 0$ (i.e., the curvature is not zero) and *of type $\mathcal{B}$* when $K(t_0) = 0$ (i.e., when the curvature vanishes).

Remark 2.4 Consider

$$\gamma(t) = (t, f(t), g(t)) = (t, a_2t^2 + a_3t^3 + O(t^4), b_3t^3 + b_4t^4 + O(t^5)).$$

Suppose that γ has a 1-flattening at $t_0 = 0$. Then

$$T(0) = 0 \Leftrightarrow a_2b_3 = 0 \quad \text{and} \quad T'(0) = 0 \Leftrightarrow a_2b_4 = 0.$$

If γ has a 1-flattening of type $\mathcal{A}$, then $b_3 = b_4 = 0$ and

$$T''(0) \neq 0 \Leftrightarrow a_2b_5 \neq 0.$$

Thus, we can write $f(t) = a_2t^2 + O(t^3)$ and $g(t) = b_5t^5 + O(t^6)$ with $a_2b_5 \neq 0$. On the other hand, if γ has a 1-flattening of type $\mathcal{B}$, then $a_2 = 0$ and

$$T''(0) \neq 0 \Leftrightarrow a_3b_4 - a_4b_3 \neq 0.$$

Therefore, we can write $f(t) = a_3t^3 + a_4t^4 + O(t^5)$ and $g(t) = b_3t^3 + b_4t^4 + O(t^5)$ with $a_3b_4 - a_4b_3 \neq 0$. Analogously, if γ has a 2-flattening of type $\mathcal{A}$ we can write $f(t) = a_2t^2 + O(t^3)$ and $g(t) = b_6t^6 + O(t^7)$ with $a_2b_6 \neq 0$. If γ has a 2-flattening of type $\mathcal{B}$, by using an isometry in $\mathbb{R}^3$, we can write $f(t) = a_3t^3 + O(t^4)$ and $g(t) = b_5t^5 + O(t^6)$ with $a_3b_5 \neq 0$.

The family of orthogonal projections of $\gamma : I \to \mathbb{R}^3$ is given by

$$
\begin{aligned}
P : I \times S^2 &\to TS^2\\
(t, \mathbf{v}) &\mapsto \big(\mathbf{v}, \gamma(t) - \langle \gamma(t), \mathbf{v}\rangle \mathbf{v}\big)
\end{aligned}
$$

where TS^2 is the tangent bundle of the sphere S^2. We denote the projection of γ along the direction $\mathbf{v}$ to $T_{\mathbf{v}}S^2$ by $P_{\mathbf{v}}(t) = \gamma(t) - \langle \gamma(t), \mathbf{v}\rangle \mathbf{v}$. We consider a modified family of projections that is affine equivalent to P, so the flat geometry of the projected curve is preserved. If the singularity is a local one, we write

$$\gamma(t) = (t, f(t), g(t)) = (t, a_2t^2 + a_3t^3 + O(t^4), b_3t^3 + b_4t^4 + O(t^5)) \tag{2}$$

and project along directions $(\sqrt{1 - u^2 - v^2}, u, v)$ near $(1, 0, 0)$ to the fixed plane $(0, u, v)$. The modified family of projections is given by

$$
\begin{aligned}
\tilde{P} : I \times S^2 &\to TS^2\\
(t, (u, v)) &\mapsto (f(t) - ut, g(t) - vt).
\end{aligned} \tag{3}
$$

In particular, $\tilde{P}_{(0,0)}(t) = (f(t), g(t))$, which is a singular germ. As pointed out in the introduction, the $\mathcal{A}$-classes of the singularities of $\tilde{P}_{(0,0)}$ that can occur generically are those of $\mathcal{A}_e$-codimension≤ 2. The family $\tilde{P}_{(u,v)}$ is a versal unfolding of these singularities (see [6]). The bifurcation set $Bif(\tilde{P})$ of $\tilde{P}$ is the set of (u, v) for which this map has A_2, A_3 or D_4 singularities. We denote these sets by:

Cusp Curve (CC): projection of the curve in a direction containing a tangent line (A_2 singularity). This set consists of (u, v) for which

$$\frac{\partial \tilde{P}_{(u,v)}}{\partial t}(t) = 0. \tag{4}$$

Tacnode Curve (TC): projection of the curve in a direction contained in a bitangent plane and the direction given by the secant joining the tangency points (A_3 singularity). This set consists of (u, v) for which there are two distinct points t and s such that:

$$\frac{\tilde{P}_{(u,v)}(t) - \tilde{P}_{(u,v)}(s)}{t - s} = 0; \tag{5}$$

$$\frac{\partial \tilde{P}_{(u,v)}}{\partial t}(t) \; // \; \frac{\partial \tilde{P}_{(u,v)}}{\partial s}(s). \tag{6}$$

Triple Point Curve (TPC): projection of the curve in a trisecant direction (D_4 singularity). This set consists of (u, v) for which there are three distinct points t, s and r such that:

$$\frac{\tilde{P}_{(u,v)}(t) - \tilde{P}_{(u,v)}(s)}{t - s} = 0; \tag{7}$$

$$\frac{1}{s-r}\left(\frac{\tilde{P}_{(u,v)}(t) - \tilde{P}_{(u,v)}(s)}{t - s} - \frac{\tilde{P}_{(u,v)}(t) - \tilde{P}_{(u,v)}(r)}{t - r}\right) = 0. \tag{8}$$

When we consider a generic 1 and 2-parameter family of space curves, more degenerate singularities can occur in the projections. Bruce and Gaffney [2] classified the simple singularities of irreducible plane curves and their list is given in Table 1.

Table 1. Simple singularities of irreducible plane curves.

Name	Normal form	$\mathcal{A}_e$-cod
Immersion	$(t, 0)$	0
A_{2k}	(t^2, t^{2k+1})	k
E_{6k}	(t^3, t^{3k+1})	$3k$
E_{6k+2}	(t^3, t^{3k+2})	$3k+1$
$-$	$(t^3, t^{3k+1} + t^{3p+2}),\ k \le p < 2k,\ k, p > 1$	$k+p+1$
$-$	$(t^3, t^{3k+1} + t^{3p+2}),\ p \le k < 2p,\ k, p > 1$	$k+p+1$
W_{12}	$(t^4, t^5);\ \ (t^4, t^5 + t^7)$	$6; 5$
W_{18}	$(t^4, t^7);\ \ (t^4, t^7 + t^9);\ \ (t^4, t^7 + t^{10})$	$9; 7; 8$
$W^{\sharp}_{1,2k-5}$	$(t^4, t^6 + t^{2k+1}),\ k \ge 3.$	$k+3$

The following proposition shows the conditions on the coefficients of γ as in (2) so that projection $\tilde{P}_{(0,0)}$ given in (3) has relevant local singularities.

Proposition 2.5 *Consider γ as in (2). The projection $\tilde{P}_{(0,0)}$ has one of the following singularities:*

(i) $A_2 \Leftrightarrow a_2 b_3 \neq 0$,
(ii) $A_4 \Leftrightarrow a_2 \neq 0$, $b_3 = 0$ *and* $a_2 b_5 - 2a_3 b_4 \neq 0$,
(iii) $A_6 \Leftrightarrow a_2 \neq 0$, $b_3 = 0$, $a_2 b_5 - 2a_3 b_4 = 0$ *and* $a_2^2(a_2 b_7 - 3b_6 a_3) + b_4(3a_3^3 - 2a_2^2 a_5 + 4a_2 a_3 a_4) \neq 0$,
(iv) $E_6 \Leftrightarrow a_2 = 0$ *and* $b_3 a_4 - b_4 a_3 \neq 0$,
(v) $E_8 \Leftrightarrow a_2 = 0$, $b_3 a_4 - b_4 a_3 = 0$ *and* $b_3 a_5 - b_5 a_3 \neq 0$.

Proof. (i) The 3-jet of the projection $\tilde{P}_{(0,0)}(t)$ is given by

$$j^3 \tilde{P}_{(0,0)}(t) = (a_2 t^2 + a_3 t^3, b_3 t^3).$$

Thus, we have a A_2 singularity if and only if $a_2 b_3 \neq 0$.

(ii) Consider the 5-jet of the projection with $a_2 \neq 0$ and $b_3 = 0$. Otherwise, we have a A_2 singularity. Thus

$$j^5 \tilde{P}_{(0,0)}(t) = (a_2 t^2 + \cdots + a_5 t^5, b_4 t^4 + b_5 t^5).$$

With changes of coordinates in the source and target, we obtain that $j^5 \tilde{P}_{(0,0)}(t)$ is $\mathcal{A}$-equivalent to

$$\big(t^2, (a_2 b_5 - 2a_3 b_4) t^5\big).$$

Therefore the projection has a A_4 singularity if and only if $a_2 \neq 0$, $b_3 = 0$ and $a_2 b_5 - 2a_3 b_4 \neq 0$.

(iii) The proof of the A_6 singularity follows similarly.

(iv) Consider $j^4 \tilde{P}_{(0,0)}(t)$ with $a_2 = 0$. Otherwise, we have a A_{2k} singularity. Thus

$$j^4 \tilde{P}_{(0,0)}(t) = (a_3 t^3 + a_4 t^4, b_3 t^3 + b_4 t^4).$$

If $\tilde{P}_{(0,0)}$ has E_6 singularity then $b_3 a_4 - b_4 a_3 \neq 0$. Now, if $b_3 a_4 - b_4 a_3 \neq 0$, without loss of generality we can assume that $a_3 \neq 0$ or $a_4 \neq 0$. Consider $a_3 \neq 0$. So with changes of coordinates in the source and target we obtain

$$j^4 \tilde{P}_{(0,0)}(t) = (t^3, (b_3 a_4 - b_4 a_3) t^4).$$

Therefore, we obtain the E_6 singularity if and only if $a_2 = 0$ and $b_3a_4 - b_4a_3 \neq 0$.

(v) The proof of the E_8 singularity follows similarly. $\square$

3. Projection of generic 1-parameter family of space curves

Consider a 1-parameter family of space curves

$$\begin{aligned} \gamma : \mathbb{R} \times \mathbb{R} &\to \mathbb{R}^3 \\ (t, a) &\mapsto \gamma_a(t). \end{aligned}$$

Correspondingly we have a 1-parameter families of functions:

$$\begin{aligned} K : \mathbb{R} \times \mathbb{R} &\to \mathbb{R} & \qquad T : \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (t, a) &\mapsto K(t, a) = K_a(t). & (t, a) &\mapsto T(t, a) = T_a(t). \end{aligned}$$

For simplicity, we denote $T_0(0)$, $\partial T_0(0)/\partial t$, $\partial^2 T_0(0)/\partial t^2, \ldots$ by $T(0)$, $T'(0)$, $T''(0), \ldots$ respectively. We will use the same notations for K. By using the standard transversality techniques, see [10] or [12] for example, it is possible to show the following results:

Proposition 3.1 *The subset of families $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ such that the function $T : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ has 0 as a regular value is residual in $C^\infty(\mathbb{R}\times\mathbb{R}, \mathbb{R}^3)$ with the Whitney C^∞-topology.*

Proposition 3.2 *There is a residual subset among the family of curves satisfying the hypothesis of Proposition* 3.1 *for which if $T(t) = T'(t) = 0$ then $T''(t) \neq 0$, for any t.*

Definition 3.3 The family of curves described by Proposition 3.2 will be called a *generic* 1-*parameter family of curves.*

It follows from Propositions 3.1 and 3.2 that the genericity condition of the family of curves γ_a is given by

$$T_a(t) = T'_a(t) = 0 \Longrightarrow \frac{\partial T}{\partial a}(t, a) \neq 0. \tag{9}$$

Given a generic 1-parameter family, when taking a plane projection we have a 3-parameter deformation of the plane curve singularity: one parameter is the parameter of the space curve family and the other two parameters correspond to the direction of projection.

Henceforth we will consider the generic 1-parameter family given by

$$\gamma_a(t) = (t, f(t) + ap(a,t), g(t) + aq(a,t)) \tag{10}$$

where $p, q \in C^\infty(\mathbb{R}^2)$ and $\gamma_0(t) = (t, f(t), g(t)) = (t, a_2t^2 + O(t^3), b_3t^3 + O(t^4))$ has a 1-flattening at the origin. Therefore, similarly to equation (3), the family of projections of γ_a is given by

$$\tilde{P}_{(u,v)}(t) = \big(f(t) + ap(a,t) - ut, g(t) + aq(a,t) - vt\big). \tag{11}$$

Proposition 3.4 *Let γ_a as in* (10) *and $\tilde{P}_{(u,v)}$ as in* (11).

(i) *If γ_0 is of type $\mathcal{A}$, then the projection $\tilde{P}_{(0,0)}$ has an A_4 singularity.*
(ii) *If γ_0 is of type $\mathcal{B}$, then the projection $\tilde{P}_{(0,0)}$ has an E_6 singularity.*

These singularities are versally unfolded by the family of projections $P_{(u,v)}$.

Proof. The first part of the proof follows from Remark 2.4 and Proposition 2.5. In the second part, consider the generic 1-parameter family as in equation (10). Therefore,

$$\begin{aligned} T_a(t) &= \det(\gamma_a'(t), \gamma_a''(t), \gamma_a'''(t)) \\ &= \begin{vmatrix} 1 & f'(t) + a\dfrac{\partial p}{\partial t}(a,t) & g'(t) + a\dfrac{\partial q}{\partial t}(a,t) \\ 0 & f''(t) + a\dfrac{\partial^2 p}{\partial t^2}(a,t) & g''(t) + a\dfrac{\partial^2 q}{\partial t^2}(a,t) \\ 0 & f'''(t) + a\dfrac{\partial^3 p}{\partial t^3}(a,t) & g'''(t) + a\dfrac{\partial^3 q}{\partial t^3}(a,t) \end{vmatrix}. \end{aligned}$$

By calculating the above determinant we have

$$\begin{aligned} T_a(t) = &\left(f''(t) + a\frac{\partial^2 p}{\partial t^2}(a,t)\right)\left(g'''(t) + a\frac{\partial^3 q}{\partial t^3}(a,t)\right) \\ &- \left(f'''(t) + a\frac{\partial^3 p}{\partial t^3}(a,t)\right)\left(g''(t) + a\frac{\partial^2 q}{\partial t^2}(a,t)\right). \end{aligned}$$

The genericity condition (9) of family γ_a is given by

$$\begin{aligned}\frac{\partial T}{\partial a}(0,0) &= f''(0)\frac{\partial^3 q}{\partial t^3}(0,0) + g'''(0)\frac{\partial^2 p}{\partial t^2}(0,0)\\ &\quad - f'''(0)\frac{\partial^2 q}{\partial t^2}(0,0) - g''(0)\frac{\partial^3 p}{\partial t^3}(0,0) \neq 0.\end{aligned}$$

When γ_0 has a 1-flattening of type $\mathcal{A}$, the projection of γ_a is given by

$$\tilde{P}_{(u,v)}(t) = (a_2t^2 + a_3t^3 + O(t^4) + ap(a,t) - ut, b_5t^5 + O(t^6) + aq(a,t) - vt)$$

and the genericity condition is given by

$$\frac{\partial T}{\partial a}(0,0) = 2a_2\frac{\partial^3 q}{\partial t^3}(0,0) - 6a_3\frac{\partial^2 q}{\partial t^2}(0,0) \neq 0.$$

Since $\tilde{P}_{(0,0)}$ has an A_4 singularity at the origin, it follows from Proposition 2.5 that

$$j^5\Big(T\mathcal{A}_e(\tilde{P}_{(0,0)}) + \mathbb{R}\{\dot{\tilde{P}}_a, \dot{\tilde{P}}_u, \dot{\tilde{P}}_v\}\Big) \supseteq J^5(1,2),$$

where $\dot{\tilde{P}}_a = \partial\tilde{P}_{(0,0)}(t)/\partial a$, except for terms $(0,t^2)$ and $(0,t^3)$. It is straightforward to check that $\dot{\tilde{P}}_a$ and the genericity conditions span these terms, so that the standard criterion for versality is satisfied (for details see [3]). When γ_0 has a 1-flattening of type $\mathcal{B}$, the projection of γ_a is given by

$$\tilde{P}_{(u,v)}(t) = (a_3t^3+a_4t^4+O(t^5)+ap(a,t)-ut, b_3t^3+b_4t^4+O(t^5)+aq(a,t)-vt),$$

with $a_3b_4 - a_4b_3 \neq 0$. Thus, the genericity condition of γ_a is given by

$$\frac{\partial T}{\partial a}(0,0) = 6b_3\frac{\partial^2 p}{\partial t^2}(0,0) - 6a_3\frac{\partial^2 q}{\partial t^2}(0,0) \neq 0.$$

It is easy to show that the criterion for versality is satisfied if and only if $\partial T(0,0)/\partial a \neq 0$ and the proposition follows. □

Remark 3.5 Geometrically Proposition 3.4 can be reinterpreted by stating that the projection onto the normal plane at a degenerate torsion zero (zero curvature) point has a singularity of type A_4 (E_6).

In the following two subsections we study the geometrical properties of a generic 1-parameter family such that γ_0 is of types $\mathcal{A}$ and $\mathcal{B}$. For simplicity, a generic 1-parameter family such that γ_0 is of type $\mathcal{A}$ (respectively, $\mathcal{B}$) will be denoted by 1-flattening of type $\mathcal{A}$ (respectively, $\mathcal{B}$).

3.1. 1-flattening of type $\mathcal{A}$

From Remark 2.4 and Proposition 3.4 we can write a 1-flattening of type $\mathcal{A}$ in the form

$$\gamma_a(t) = (t, t^2 + O(t^3), t^5 + at^3 + O(t^6)) \tag{12}$$

and its projection is given by

$$\tilde{P}_{(u,v)}(t) = (t^2 - ut + O(t^3), t^5 + at^3 - vt + O(t^6)). \tag{13}$$

The bifurcation set of projection $\tilde{P}_{(u,v)}$ in the plane (u, v) is given by the following curves:

(i) **CC**: $(u, v) = (2t + O(t^2), 5t^4 + 3at^2 + O(t^5))$.
It follows directly from the equation (4).

(ii) **TC**: $(u, v) = (t+O(t^2), -5t^4/4 - at^2/2 - a^2/4 + O(t^5))$, where $(5u^2 + 2a) < 0$.
We will consider only the initial terms of the projection given in (13). In this case equation (5) is given by

$$\frac{(t^2 - s^2) - u(t - s)}{t - s} = 0 \quad \text{and} \quad \frac{(t^5 - s^5) + a(t^3 - s^3) - v(t - s)}{t - s} = 0.$$

So $u = (t + s)$ and $v = (t^4 + t^3s + t^2s^2 + ts^3 + s^4) + a(t^2 + ts + s^2)$. Then by changes of coordinates $\sigma_1 = t + s$ and $\sigma_2 = ts$ we obtain

$$u = \sigma_1 \quad \text{and} \quad v = \sigma_1^4 - 3\sigma_1^2\sigma_2 + \sigma_2^2 + a(\sigma_1^2 - \sigma_2). \tag{14}$$

Note that the condition $\sigma_1^2 - 4\sigma_2 > 0$ ensures that $t \neq s$. The equation (6) with these values of u and v is given by $(4\sigma_2 - \sigma_1^2)(2\sigma_2 - 3\sigma_1^2 - a) = 0$. As $\sigma_1^2 - 4\sigma_2 > 0$ we obtain $\sigma_2 = (1/2)(3\sigma_1^2 + a)$. Substituting the value of σ_2 into equation (14) we obtain

$$v = -\frac{5}{4}\sigma_1^4 - \frac{1}{2}a\sigma_1^2 - \frac{1}{4}a^2.$$

Therefore $u = \sigma_1$ and $v = -5\sigma_1^4/4 - a\sigma_1^2/2 - a^2/4$. The condition $(5u^2 + 2a) < 0$ follows from condition $\sigma_1^2 - 4\sigma_2 > 0$ substituting $\sigma_1 = u$ and $\sigma_2 = (1/2)(3\sigma_1^2 + a)$.

Note that the **TPC** does not appear in this case. When $a \geq 0$ the **TC** does not occur, thus the bifurcation set is simply given by the **CC** in Figure 1. When $a = 0$ the **CC** has a degenerate inflection at the origin corresponding to a 1-flattening of type $\mathcal{A}$ ($T = T' = 0$).

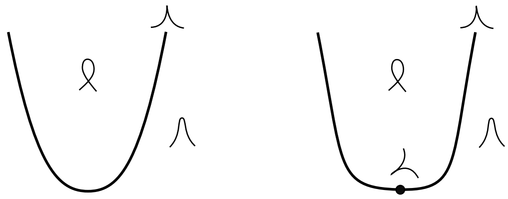

Figure 1. Bifurcation set of P when $a > 0$ and $a = 0$.

When $a < 0$ the **TC** (dotted line in Figure 2) contains two inflection points corresponding to the two bitangent osculating planes of γ_a. This property is not detected by stratum A_3. (See [5] for more details). The **CC** has two inflection points corresponding to the two zero torsion points of γ_a (see Figure 2). Similar results were obtained in [10] considering the associated bitangency surfaces and the singularities of their projections onto $\mathbb{R}$.

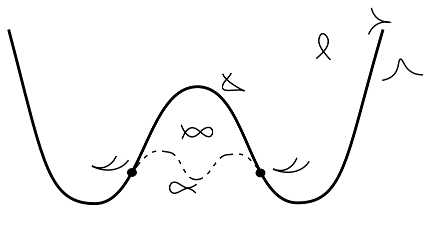

Figure 2. Bifurcation set of P when $a < 0$.

When $a < 0$ we obtain a region in the plane (u, v) so that the projection of γ_a has the maximum number of double points (Morsification) of the A_4 singularity. In Section 5 presents the directions of projections where this Morsification occurs.

3.2. 1-flattening of type $\mathcal{B}$

From Remark 2.4 and Proposition 3.4 we can write a 1-flattening of type $\mathcal{B}$ in the form

$$\gamma_a(t) = (t, t^3 + O(t^4), t^4 + at^2 + O(t^5)) \tag{15}$$

and its projection is given by

$$\tilde{P}_{(u,v)}(t) = (t^3 - ut + O(t^4), t^4 + at^2 - vt + O(t^5)). \tag{16}$$

A calculation shows that the bifurcation set of $\tilde{P}_{(u,v)}$ in the plane (u, v) is given by the following curves:

(i) **CC**: $(u, v) = (3t^2 + O(t^3), 4t^3 + 2at + O(t^4))$,

(ii) **TC**: $(u, v) = ((3t^2 - a)/2 + O(t^3), 2t^3 + O(t^4))$, with $(3t^2 - 2a) > 0$, The calculations for the curves **CC** and **TC** are analogous to the calculations in the previous cases and and will be omit here.

(iii) **TPC**: $u = -a + O(u, v)$ with $(4a^3 + 27v^2) < 0$.
We will consider only the initial terms of the projection given in (16). Here the equations (7) and (8) are given by

$$\begin{aligned}
(t^2 + ts + s^2) - u &= 0,\\
(t^3 + s^2t + t^2s + s^3) + a(t + s) - v &= 0,\\
(sr + s^2 + ts + tr + t^2 + r^2) + a &= 0,\\
(s + t + r) &= 0.
\end{aligned}$$

Then by changes of coordinates $\sigma_1 = t + s$ and $\sigma_2 = ts$ we obtain

$$\begin{aligned}
(\sigma_1^2 - \sigma_2) - u &= 0,\\
(\sigma_1^3 - 2\sigma_1\sigma_2) + a\sigma_1 - v &= 0,\\
(r^2 + r\sigma_1 + \sigma_1^2 - \sigma_2) + a &= 0, \qquad (17)\\
(r + \sigma_1) &= 0. \qquad (18)
\end{aligned}$$

From the equation (18), $r = -\sigma_1$. Substituting into equation (17) we obtain $\sigma_1^2 - \sigma_2 + a = 0$. Isolating σ_2 and substituting into the other equations we have

$$u = -a,$$
$$\sigma_1^3 + a\sigma_1 + v = 0. \tag{19}$$

So we have a triple point when $u = -a$ and the equation (3) has three real roots. This is equivalent to $(4a^3 + 27v^2) < 0$. Therefore, $(u, v) = (-a, v)$ with v satisfying the inequality $(4a^3 + 27v^2) < 0$.

When $a \geq 0$ the **TPC** does not appear. In the inflection points of **CC** (solid line in Figure 3) we have the points where the torsion of the curve γ_a is zero. When $a = 0$, the two zero torsion points coalesce at the origin, corresponding to the zero curvature points of γ_0. (Figure 3).

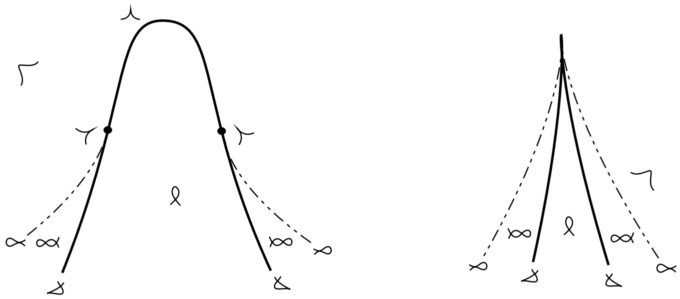

Figure 3. Bifurcation set of P when $a > 0$ and $a = 0$.

When $a < 0$, the two zero torsion points disappear. On the other hand, two new cross tangents appear at the intersection of three curves. The **TC** (dotted line in Figure 4) has a cusp where an A_5 singularity occurs. The geometrical interpretation of the A_5 singularity is given in [5]. Again, when $a < 0$ we obtain a region in the plane (u, v) so that the projection of γ_a has the maximum number of double points of the E_6 singularity (see Figure 4).

4. Projection of a generic 2-parameter family of space curves

Consider a 2-parameter family of space curves

$$\gamma : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^3$$
$$(t, a, b) \mapsto \gamma_{(a,b)}(t).$$

Correspondingly we have a 2-parameter families of functions:$K : \mathbb{R} \times \mathbb{R}^2 \to$

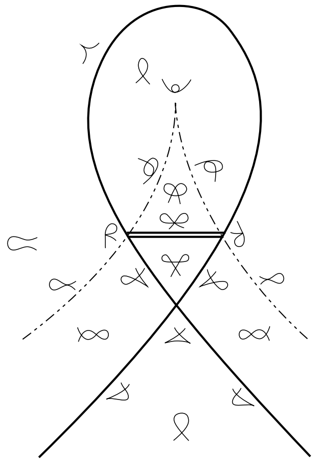

Figure 4. Bifurcation set of P when $a < 0$.

$\mathbb{R}$, $K(t,a,b) = K_{(a,b)}(t)$ and $T : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, $T(t,a,b) = T_{(a,b)}(t)$. Similarly to Section 3 we have the following proposition.

Proposition 4.1

(i) *The subset of families $\gamma : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^3$ such that the function $T : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ has 0 as a regular value is residual in $C^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^3)$ with the Whitney C^∞-topology.*

(ii) *There is a residual subset among the families of curves satisfying the hypothesis of item* (i) *for which if $T(t) = T'(t) = T''(t) = 0$ then $T'''(t) \neq 0$, for any t.*

Definition 4.2 The family of curves described by Proposition 4.1 (ii) will be called a *generic 2-parameter family of curves.*

We consider the generic 2-parameter families given by

$$\gamma_{(a,b)}(t) = \big(t, f(t) + ap_1(a,b,t) + bp_2(a,b,t), \\ g(t) + aq_1(a,b,t) + bq_2(a,b,t)\big) \tag{20}$$

where $p_i, q_i \in C^\infty(\mathbb{R}^3)$, $i = 1, 2$ and $\gamma_{(0,0)}(t) = (t, f(t), g(t)) = (t, a_2t^2 + O(t^3), b_3t^3 + O(t^4))$ has a 2-flattening at the origin. The family of projections of $\gamma_{(a,b)}$ as in (20) is given by

$$\tilde{P}_{(u,v)}(t) = \big(f(t) + ap_1(a,b,t) + bp_2(a,b,t) - ut,$$
$$g(t) + aq_1(a,b,t) + bq_2(a,b,t) - vt\big). \qquad (21)$$

Proposition 4.3 *Consider $\gamma_{(a,b)}$ as in* (20) *and $\tilde{P}_{(u,v)}$ as in* (21)*.*

(i) *If $\gamma_{(0,0)}$ is of type $\mathcal{A}$ and $T^{(4)}(0)K(0) - 5T'''(0)K'(0) \neq 0$, then the projection $\tilde{P}_{(0,0)}$ has an A_6 singularity.*

(ii) *If $\gamma_{(0,0)}$ is of type $\mathcal{B}$, then the projection $\tilde{P}_{(0,0)}$ has an E_8 singularity.*

These singularities are versally unfolded by the family of projection $\tilde{P}_{(u,v)}$ if and only if

(iii) A_6 : *always.*

(iv) E_8 :

$$K''(0)\bigg(\frac{\partial T_{(0,0)}}{\partial b}(0)\frac{\partial^3 T_{(0,0)}}{\partial t^2 \partial a}(0) - \frac{\partial T_{(0,0)}}{\partial a}(0)\frac{\partial^3 T_{(0,0)}}{\partial t^2 \partial b}(0)\bigg)$$
$$+ T^{(3)}(0)\bigg(\frac{\partial T_{(0,0)}}{\partial a}(0)\frac{\partial^2 K_{(0,0)}}{\partial t \partial b}(0) - \frac{\partial T_{(0,0)}}{\partial b}(0)\frac{\partial^2 K_{(0,0)}}{\partial t \partial a}(0)\bigg) \neq 0.$$

Proof. The proof of cases (i) and (ii) is analogous to the proof of Proposition 3.4.

(iii) When $\gamma_{(0,0)}$ has a 2-flattening of type $\mathcal{A}$, the projection $\tilde{P}_{(u,v)}$ is given by

$$\tilde{P}_{(u,v)}(t) = \big(a_2t^2 + a_3t^3 + O(t^4) + ap_1(t,a,b) + bp_2(t,a,b) - ut,$$
$$b_6t^6 + b_7t^7 + O(t^8) + aq_1(t,a,b) + bq_2(t,a,b) - vt\big),$$

with $a_2b_6 \neq 0$. Since $\tilde{P}_{(0,0)}$ has an A_6 singularity at the origin, it follows from Proposition 2.5 that $a_2b_7 - 3a_3b_6 \neq 0$. So

$$W = j^7\Big(T\mathcal{A}_e(\tilde{P}_{(0,0)}) + \mathbb{R}\{\dot{\tilde{P}}_a, \dot{\tilde{P}}_b, \dot{\tilde{P}}_u, \dot{\tilde{P}}_v\}\Big) \supseteq J^7(1,2),$$

where $\dot{\tilde{P}}_a = \partial\tilde{P}_{(0,0)}(t)/\partial a$, except for terms $(0,t^2)$ and $(0,t^3)$. We can write W by

$$\begin{pmatrix} 2a_2t + 3a_3t^2 + \cdots + 8a_8t^7 \\ 6b_6t^5 + 7b_7t^6 + 8b_8t^7 \end{pmatrix} \xi + \eta(a_2t^2 + \cdots + a_7t^7, b_6t^6 + b_7t^7)$$
$$+ \mathbb{R}\{(p_1(t,0,0), q_1(t,0,0)), (p_2(t,0,0), q_2(t,0,0)), (t,0), (0,t)\},$$

where $\xi \in C^\infty(\mathbb{R})$ and $\eta(x,y) = (\eta_1(x,y), \eta_2(x,y))$ with $\eta_i \in C^\infty(\mathbb{R}^2)$.

We obtain $(0, a_2t^2 + a_3t^3)$ taking $\eta_2(x,y) = x$. On the other hand, choosing $p_1(t,0,0) = p_2(t,0,0) = q_2(t,0,0) \equiv 0$ and $q_1(t,0,0) = \tilde{a_1}t + \tilde{a_2}t^2 + O(t^3)$, we have the term $(0, \tilde{a_2}t^2 + \tilde{a_3}t^3)$. Thus, the remaining terms are obtained when $a_2\tilde{a_3} - \tilde{a_2}a_3 \neq 0$, which is precisely the genericity condition

$$\frac{\partial T}{\partial a}(0) = 2a_2 \frac{\partial^3 q_1}{\partial t^3}(0) - 6a_3 \frac{\partial^2 q_1}{\partial t^2}(0) \neq 0.$$

(iv) The proof is similar to the proof of item (iii), therefore we have omitted it here. □

Remark 4.4

- (i) The condition of $\gamma_{(0,0)}$ being of types $\mathcal{A}$ and $\mathcal{B}$ in Proposition 4.3 means that the curve has degenerate zero of torsion ($T(0) = T'(0) = T''(0) = 0$) and degenerate zero of curvature, respectively, but the remaining conditions are a bit of a mystery.
- (ii) It follows from Proposition 4.3 (i) that the condition for $\tilde{P}_{(u,v)}$ to be a versal unfolding of the A_6 singularity depends only on the parameters a, u and v. Thus, we take

  $$\gamma_{(a,0)}(t) = (t, t^2 + O(t^3), t^7 + t^6 + at^3 + O(t^8))$$

  as a generic 2-parameter family of type $\mathcal{A}$.
- (iii) A generic 2-parameter family of space curves with the hypotheses of Proposition 4.3 is also residual in $C^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^3)$ with the Whitney C^∞-topology.

4.1. 2-flattening of type $\mathcal{A}$

From Remark 4.4 we can write a generic family of space curves with a 2-flattening of type $\mathcal{A}$ in the form

$$\gamma_{(a,0)}(t) = (t, t^2 + O(t^3), t^7 + t^6 + at^3 + O(t^8)) \tag{22}$$

and its projection is given by

$$\tilde{P}_{(u,v)}(t) = (t^2 - ut + O(t^3), t^7 + t^6 + at^3 - vt + O(t^8)). \tag{23}$$

The bifurcation set of projection $\tilde{P}_{(u,v)}$ in the plane (u, v) is given by the following curves:

(i) **CC**: $(u, v) = (2t + O(t^2), 7t^6 + 6t^5 + 3at^2 + O(t^7))$.
(ii) The tacnode curve, **TC**, is given by the implicit equation:

$$\begin{aligned} &- 49u^{12} - 168u^{11} - 228u^{10} - 140u^9 + (210a - 36)u^8 + 318au^7 \\ &+ (192a - 378v)u^6 + (36a - 648v)u^5 - (432v + 57a^2)u^4 \\ &- (108v + 42a^2)u^3 + (54av - 9a^2)u^2 + (54au + 27v)v \\ &+ 4a^3 + O(u, v)^{13} = 0, \end{aligned}$$

with the condition

$$(7u^6 + 12u^5 + 5u^4 - au^2 + 2au + 6v)(-7u^4 - 8u^3 - 3u^2 + a) > 0.$$

Note that the **TPC** does not appear in this case.

Remark 4.5 The calculations for the curves **CC** and **TC** are analogous to previous calculations and will be omit here. We observe that the curve **TC** was found with the aid of the software Maple.

When $a < 0$ the **TC** (dotted line in Figure 5) contains two inflection points corresponding to the two bitangent osculating planes of γ_a. The **CC** (solid line in Figure 5) has two inflection points corresponding to the two zero torsion points of γ_a. When $a = 0$ the **CC** (solid line) has a degenerate inflection ($\kappa(0) = \kappa'(0) = \kappa''(0) = 0$) corresponding to a degenerate zero torsion point ($T = T' = T'' = 0$) of curve γ_0. (Figure 5).

When $0 < a < 80/343$ we obtain a region in the plane (u, v) so that the projection of γ_a has the maximum number of double points of the A_6 singularity. The **TC** has a cusp where an A_5 singularity occurs. (Figure 6 left). The number 80/343 is obtained by calculating the contact between the curves **CC** and **TC**.

When $80/343 \leq a < 135/343$ the region of the Morsification disappears. (Figure 6 right). The **CC** has a degenerate inflection ($\kappa(0) = \kappa'(0) = 0$)

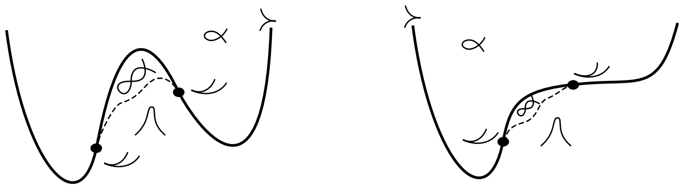

Figure 5. Bifurcation set of P when $a < 0$ and $a = 0$.

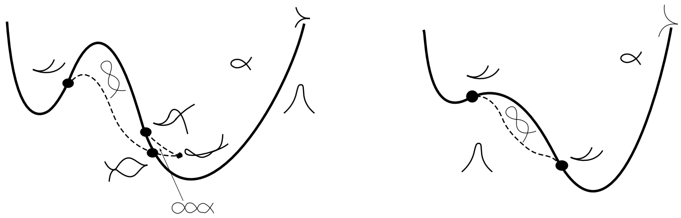

Figure 6. Bifurcation set of P when $0 < a < 80/343$ and $80/343 \leq a < 135/343$.

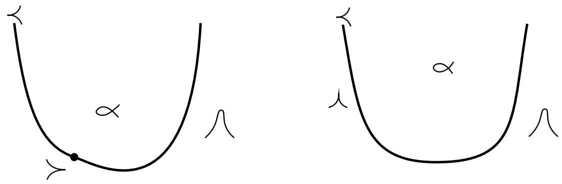

Figure 7. Bifurcation set of P when $a = 135/343$ and $a > 135/343$.

when $a = 135/343$ corresponding to a degenerate zero torsion point ($T = T' = 0$) of curve γ_0. When $a \geq 135/343$ the **TC** disappears. (Figure 7).

4.2. 2-flattening of type $\mathcal{B}$

From Remark 2.4 and Proposition 4.3 we can write a generic family of space curves with a 2-flattening of type $\mathcal{B}$ in the form

$$\gamma_{(a,b)}(t) = (t, t^3 + O(t^4), t^5 + at^2 + bt^4 + O(t^6)) \tag{24}$$

and its projection is given by $\tilde{P}_{(u,v)}(t) = \big(t^3 - ut + O(t^4), t^5 + at^2 + bt^4 - vt + O(t^6)\big)$.

In this case, the bifurcation set has many transitions, therefore it is impractical to study the geometric properties of such families from the point of view previously used. Thus we present only the curves that compose the bifurcation set of projection in the plane (u, v).

(i) **CC**: $(u,v) = (3t^2 + O(t^3), 5t^4 + 4bt^3 + 2at + O(t^5))$,

(ii) The tacnode curve is given by the implicit equation

$$\begin{aligned}
&400u^6 - 668b^2u^5 + (323b^4 - 1040v - 704ba)u^4 \\
&\quad + 4\left(223b^2v - 37a^2 - 8b^6 + \frac{319}{2}b^3a\right)u^3 \\
&\quad + (896v^2 - 48b^5a + 533b^2a^2 - 66b^4v + 880abv)u^2 \\
&\quad + 6(24a^2v + b^3av - 4a^2b^4 + 33a^3b - 40v^2b^2)u + 27v^2b^4 + 6vb^2a^2 \\
&\quad - 192v^2ab - 4a^3b^3 - 256v^3 + 27a^4 + O(u,v)^7 = 0, \quad \text{with}
\end{aligned}$$

$$\begin{aligned}
&\Bigg(100u^7 - 517b^2u^6 + (-811ba - 160v + 523b^4)u^5 \\
&\quad + \left(\frac{10129}{8}ab^3 + 572b^2v - \frac{8691}{64}b^6 - \frac{1083}{4}a^2\right)u^4 \\
&\quad + \left(\frac{9947}{8}b^2a^2 - \frac{489}{4}vb^4 - \frac{8241}{32}b^5a + 980vba + 64v^2 + 9b^8\right)u^3 \\
&\quad + \Bigg(\frac{45}{32}b^6v - \frac{9945}{64}b^4a^2 + \frac{4275}{8}ba^3 - 144v^2b^2 + 342va^2 \\
&\qquad - \frac{549}{8}vb^3a + 9b^7a\Bigg)u^2 \\
&\quad + \Bigg(81a^4 + \frac{63}{32}b^5va - 288v^2ba - \frac{45}{2}vb^2a^2 + 9/4a^2b^6 \\
&\qquad - \frac{981}{32}b^3a^3 + 63b^4v^2\Bigg)u \\
&\quad + \frac{81}{32}vb^4a^2 - \frac{27}{64}b^2a^4 - 108v^2a^2 - \frac{27}{2}va^3b - \frac{243}{64}v^2b^6 + \frac{81}{2}v^2ab^3\Bigg) \\
&> 0.
\end{aligned}$$

(iii) **TPC**: $v = u^2 - b^2u - ab + O(u,v)^3$, with the condition $(4u^3 - 27(a + ub)^2) > 0$.

5. Morsifications

This section describes the directions of projection of some space curves (parametrized in a simple form) where the Morsifications for the singularities A_4, A_6 and E_6 occur and transitions between the Morsifications of the E_8 singularity. More specifically, we will consider only the initial terms of the generic families of the space curves given in (12), (15), (22) and (24). For A_4 and A_6 singularities we have the following results:

Theorem 5.1 *Let $\gamma_a = (t, t^2, t^5 + at^3)$. The projection of γ_a as in (13) presents a Morsification of the A_4 singularity if and only if the direction of projection is given by $\mathbf{v} = (u, u^4 + au^2)$ with $a < 0$ and*

$$-\sqrt{\frac{-4a}{11}} < u < \sqrt{\frac{-4a}{11}}.$$

Proof. If follows from equation (13) that the projection of γ_a is given by

$$\tilde{P}_{(u,v)}(t) = (t^2 - ut, t^5 + at^3 - vt).$$

Since an A_4 singularity has an $\mathcal{A}_e$-cod $= 2$, it follows from equation (1) that $\tilde{P}_{(u,v)}$ has at most two double points. Without loss of generality, we can assume that one double point occurs at the origin. Thus, the projection $\tilde{P}_{(u,v)}$ presents a double point at the origin, given by $t_1 = 0$ and $s_1 = u$, if and only if the direction is given by $\mathbf{v} = (u, u^4 + au^2)$. The other double point of $\tilde{P}_{(u,v)}$ can be obtained by the system $\tilde{P}_{(u,v)}(t_2) = \tilde{P}_{(u,v)}(s_2)$ with $t_2 \neq s_2$. So

$$\begin{cases} s_2 + t_2 - u = 0, \\ s_2^4 + s_2^3 t_2 + s_2^2 t_2^2 + s_2 t_2^3 + t_2^4 + a(s_2^2 + s_2 t_2 + t_2^2) - v = 0. \end{cases}$$

Thus, $s_2 = u - t_2$ according to the first equation. Substituting the value of s_2 into the second equation and using the condition $v = u^4 + au^2$, we have

$$-t_2(u - t_2)(t_2^2 - ut_2 + 3u^2 + a) = 0.$$

Therefore, $\tilde{P}_{(u,v)}$ has two double points if and only if the equation $t_2^2 - ut_2 + 3u^2 + a$ has two real roots, which is equivalent to $11u^2 + 4a < 0$, and the theorem follows. $\square$

Theorem 5.2 *Let* $\gamma_{(a,0)} = (t, t^2, t^7 + t^6 + at^3)$. *The projection of* $\gamma_{(a,0)}$ *as in* (23) *presents a Morsification of the* A_6 *singularity if and only if the direction of projection is given by* $\mathbf{v} = (u, u^6 + u^5 + au^2)$ *with*

$$-\frac{6}{11} < u < 0 \text{ and}$$
$$-u^3\left(\frac{57}{16}u + \frac{13}{4}\right) < a < u^2\left(4u^2 + 5u + \frac{9}{4}\right).$$

Proof. The proof is similar to the proof of Theorem 5.1, therefore we have omitted it here. □

Unfortunately, the approach used in Theorems 5.1 and 5.2 is not satisfactory to find Morsifications of the singularities E_6 and E_8, thus a different one is required.

5.1. Morsification of the E_6 singularity

The E_6 singularity has two Morsifications as in Figure 8 which can be viewed as a Gauss word as follows: Consider t_i and s_i with $i = 1, 2, 3$, the parameters for which a plane curve α has double points, that is $\alpha(t_1) = \alpha(s_1) = a$, $\alpha(t_2) = \alpha(s_2) = b$ and $\alpha(t_3) = \alpha(s_3) = c$.

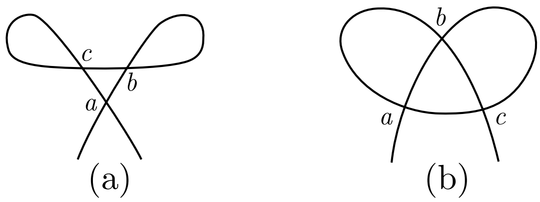

Figure 8. (a) Morsification $\mathcal{M}_1$ and (b) Morsification $\mathcal{M}_2$.

Definition 5.3 We say that the Morsification of the E_6 singularity is *of type* $\mathcal{M}_1$ when we have the Gauss word *abbcca*. We say that the Morsification is *of type* $\mathcal{M}_2$ when we have the Gauss word *abcabc*.

Definition 5.3 yields a way to differentiate the two Morsifications of the E_6 singularity, as illustrated in Figure 8. In the next theorem we present the directions of projection in which the Morsifications of the E_6 singularity are realized as projections of a generic family of space curves.

Theorem 5.4 *Let* $\gamma_a = (t, t^3, t^4 + at^2)$ *and* $\tilde{P}_{(u,v)}$ *as in* (16). *The pro-*

jection $\tilde{P}_{(u,v)}$ *has a Morsification of the* E_6 *singularity if and only if the direction of projection is given by* $\mathbf{v} = \big(u, \sqrt{u}(u+a)\big)$ *with* $a < 0, u \neq -a$ *and* $-4a/5 < u < (-3-6\sqrt{3})a/11$. *Moreover, the projection* $\tilde{P}_{(u,v)}$ *has a Morsification*:

(i) $\mathcal{M}_1$ *when* $-a < u < ((-3-6\sqrt{3})/11)a$.
(ii) $\mathcal{M}_2$ *when* $-4a/5 < u < -a$.

Proof. It is easy to see that the projection $\tilde{P}_{(u,v)}$ presents a double point at the origin, given by $t_3 = 0$ and $s_3 = \sqrt{u}$ with $u > 0$, if and only if the direction of projection is given by $\mathbf{v} = (u, \sqrt{u}(u+a))$. Therefore, after some calculations, the other double points are given by

$$t_2 = -\frac{1}{4}\Big(\sqrt{5u+4a} + \sqrt{u} + \sqrt{-2u-12a-6\sqrt{u}\sqrt{5u+4a}}\Big),$$
$$s_2 = -\frac{1}{4}\Big(\sqrt{5u+4a} + \sqrt{u} - \sqrt{-2u-12a-6\sqrt{u}\sqrt{5u+4a}}\Big),$$
$$t_1 = \frac{1}{4}\Big(\sqrt{5u+4a} - \sqrt{u} - \sqrt{-2u-12a+6\sqrt{u}\sqrt{5u+4a}}\Big),$$
$$s_1 = \frac{1}{4}\Big(\sqrt{5u+4a} - \sqrt{u} + \sqrt{-2u-12a+6\sqrt{u}\sqrt{5u+4a}}\Big).$$

These points are real, i.e., the projection has a Morsification of the E_6 singularity if and only if $-4a/5 < u < (-3-6\sqrt{3})a/11$. Note that when $u = -a$ the projection has a triple point at the origin. Now we prove (i) and (ii).

(i) The increasing sequence of parameters $t_1 < t_2 < s_2 < t_3 < s_3 < s_1$ implies in the Gauss word *abbcca*. Inequalities $t_2 < s_2$ and $t_3 < s_3$ are obvious. By hypothesis, $(a+u) > 0$, therefore

$$0 < 4(5u+4a)(a+u) \Rightarrow$$
$$0 < -11u^2 - 6ua + 9a^2 < 9u^2 + 30ua + 25a^2 \Rightarrow$$
$$0 < -11u^2 - 6ua + 9a^2 < (3u+5a)^2 \Rightarrow$$
$$\frac{(-2u-12a+6\sqrt{u}\sqrt{5u+4a})(-2u-12a-6\sqrt{u}\sqrt{5u+4a})}{16} < (3u+5a)^2.$$

Since $u < (-3-6\sqrt{3})a/11$, then $(3u+5a) < 0$, hence

$$\sqrt{(-2u-12a+6\sqrt{u}\sqrt{5u+4a})(-2u-12a-6\sqrt{u}\sqrt{5u+4a})} < 4(-3u-5a) \Rightarrow$$

$$-\frac{\left(\sqrt{-2u-12a+6\sqrt{u}\sqrt{5u+4a}}-\sqrt{-2u-12a-6\sqrt{u}\sqrt{5u+4a}}\right)^2}{2}+$$

$$-2u-12a < -12u-20a \Rightarrow$$

$$20u+16a < \left(\sqrt{-2u-12a+6\sqrt{u}\sqrt{5u+4a}}-\sqrt{-2u-12a-6\sqrt{u}\sqrt{5u+4a}}\right)^2 \Rightarrow$$

$$2\sqrt{5u+4a} < \sqrt{-2u-12a+6\sqrt{u}\sqrt{5u+4a}}-\sqrt{-2u-12a-6\sqrt{u}\sqrt{5u+4a}} \Rightarrow$$

$$\sqrt{5u+4a}-\sqrt{-2u-12a+6\sqrt{u}\sqrt{5u+4a}}$$
$$< -\sqrt{5u+4a}-\sqrt{-2u-12a-6\sqrt{u}\sqrt{5u+4a}} \Rightarrow$$

$$t_1 < t_2.$$

We will prove now the inequality $s_2 < t_3 = 0$. By hypothesis, $-u^2+a^2 < 0$, therefore $4(-u^2+a^2) < 0$ implies $(u+2a)^2 < u(5u+4a)$. Since $u < (-3-6\sqrt{3})a/11$, then $(u+2a) < 0$, hence

$$(-u-2a) < \sqrt{u(5u+4a)} \Rightarrow$$

$$0 < \frac{u}{2}+a+\frac{\sqrt{u}\sqrt{5u+4a}}{2} \Rightarrow$$

$$0 < \left(-\frac{\sqrt{5u+4a}-\sqrt{u}}{4}\right)^2-\left(\frac{\sqrt{-2u-12a-6\sqrt{u}\sqrt{5u+4a}}}{4}\right)^2 \Rightarrow$$

$$\left(\frac{\sqrt{-2u-12a-6\sqrt{u}\sqrt{5u+4a}}}{4}\right)^2 < \left(-\frac{\sqrt{5u+4a}-\sqrt{u}}{4}\right)^2 \Rightarrow$$

$$\frac{\sqrt{-2u-12a-6\sqrt{u}\sqrt{5u+4a}}}{4} < \frac{\sqrt{5u+4a}+\sqrt{u}}{4} \Rightarrow$$

$$s_2 < 0.$$

The proof of inequality $s_3 < s_1$ is analogous to the proof of the above case. Thus, we have proved the existence of an increasing sequence of parameters of type $t_1 < t_2 < s_2 < t_3 < s_3 < s_1$. According to Definition 5.3, the Morsification $\mathcal{M}_1$ of the E_6 singularity occurs.

(ii) Analogously to the previous case we can show that $t_1 < t_2 < t_3 <$

$s_1 < s_2 < s_3$. Again according to Definition 5.3, the Morsification $\mathcal{M}_2$ of the E_6 singularity occurs. $\square$

5.2. Morsification of the E_8 singularity

There are three types of Morsifications of the E_8 singularity (see [1] or [7] for more details), which are denoted by $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ as in Figure 9.

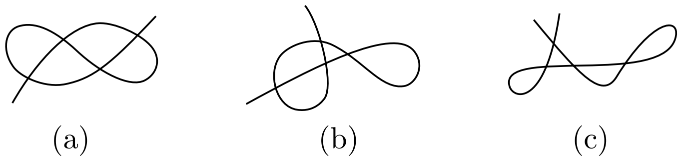

Figure 9. (a) Morsification $\mathcal{M}_1$, (b) Morsification $\mathcal{M}_2$ and (c) Morsification $\mathcal{M}_3$.

The transitions between these Morsifications have a triple point as in Figure 10.

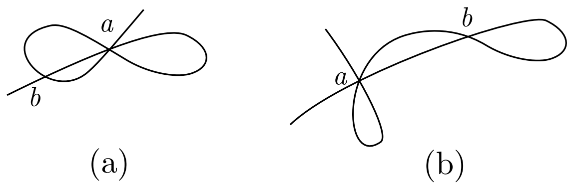

Figure 10. (a) Transition $\mathcal{T}_{12}$ and (b) Transition $\mathcal{T}_{23}$.

Definition 5.5 The transition between the Morsifications $\mathcal{M}_1$ and $\mathcal{M}_2$ (respectively, $\mathcal{M}_2$ and $\mathcal{M}_3$) will be denoted by $\mathcal{T}_{12}$ (respectively, $\mathcal{T}_{23}$).

Consider a generic family of space curves given by

$$\gamma_{(a,b)}(t) = (t, t^3, t^5 + at^2 + bt^4) \tag{25}$$

whose projection is given by

$$\tilde{P}_{(u,v)}(t) = \big(t^3 - ut, t^5 + at^2 + bt^4 - vt\big). \tag{26}$$

We are seeking conditions on the coefficients of the family (25) such that the projection (26) presents these transitions. Note that we can obtain the Morsifications $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ by small deformations in these directions of projection. Consider t_1, t_2 and t_3 the parameters for which the curve $\tilde{P}_{(u,v)}$

given by (26) has a triple point, that is $\tilde{P}_{(u,v)}(t_1) = \tilde{P}_{(u,v)}(t_2) = \tilde{P}_{(u,v)}(t_3) = a$ and t_4 and t_5 are the parameters for which $\tilde{P}_{(u,v)}$ has a double point, that is $\tilde{P}_{(u,v)}(t_4) = \tilde{P}_{(u,v)}(t_5) = b$. See Figure 10.

Definition 5.6 We say that the transition is *of type* $\mathcal{T}_{12}$ when we have the Gauss word *abaab*. We say that the transition is *of type* $\mathcal{T}_{23}$ when we have the Gauss word *abbaa*.

Lemma 5.7 *Let* $\gamma_{(a,b)}$ *be as in* (25) *and* $P_{(u,v)}$ *as in* (26). *Then the projection* $\tilde{P}_{(u,v)}$ *has a transition* $\mathcal{T}_{12}$ *or* $\mathcal{T}_{23}$ *if and only if the direction of projection is given by*

$$\mathbf{v} = (u, v) = \left(-\frac{a}{b}, \frac{a^2}{b^2}\right) \tag{27}$$

with $u > 3b^2/4$ *and* $b \neq 0$.

Proof. Without loss of generality, we can assume that the triple point occurs at the origin. Therefore

$$\tilde{P}_{(u,v)}(t_i) = (t_i^3 - ut_i, t_i^5 + at_i^2 + bt_i^4 - vt_i) = (0, 0) \quad \text{for } i = 1, 2, 3.$$

The triple point occurs at $t_1 = -\sqrt{u}$, $t_2 = 0$ and $t_3 = \sqrt{u}$ if and only if $u > 0$. Substituting these values in the second coordinate of $\tilde{P}_{(u,v)}$, we obtain

$$\begin{cases} u^2\sqrt{u} + bu^2 + au - v\sqrt{u} = 0, \\ -u^2\sqrt{u} + bu^2 + au + v\sqrt{u} = 0. \end{cases}$$

By solving the above system we have

$$\begin{cases} v = u^2, \\ a = -ub. \end{cases}$$

Therefore we obtain the equality given in (27) with $b \neq 0$. Now, the projection has a transition $\mathcal{T}_{12}$ or $\mathcal{T}_{23}$ if and only if there exists a double point distinct from the origin. Consider

$$\tilde{P}_{(u,v)}(t_i) = (t_i^3 - ut_i, t_i^5 + at_i^2 + bt_i^4 - vt_i) = (x, y), \tag{28}$$

with $i = 4, 5$, $(x, y) \neq (0, 0)$ and $t_4 \neq t_5$. Equation (28) and condition (27) are equivalent to system

$$\begin{cases} t_i^3 - ut_i = x, \\ (t_i^3 - ut_i)(t_i^2 + bt_i + u) = y, \quad i = 4, 5. \end{cases}$$

Thus, we obtain the equations $t_i^2 + bt_i + u - y/x = 0$ which have roots

$$t_4 = \frac{-bx + \sqrt{b^2x^2 - 4ux^2 + 4xy}}{2x} \quad \text{and} \quad t_5 = \frac{-bx - \sqrt{b^2x^2 - 4ux^2 + 4xy}}{2x}.$$

The condition $\Delta = b^2x^2 - 4ux^2 + 4xy > 0$ ensures the existence of the double point out of the origin. On the other hand, from equations (28), it follows

$$\begin{cases} t_4^3 - ut_4 = t_5^3 - ut_5 = x, \\ (t_4^3 - ut_4) - (t_5^3 - ut_5) = 0. \end{cases}$$

Therefore, $x = b(b^2 - u)$ and $y = (-b^2 + 2u)x$. Substituting the values of x and y into Δ, we have

$$\Delta = -3b^2 + 4u > 0.$$

The lemma has been proved. □

In the proof of Lemma 5.7 we found that a triple point occurs at $t_1 = -\sqrt{u}$, $t_2 = 0$ and $t_3 = \sqrt{u}$. Substituting the values of x and y in t_4 and t_5 in the proof of Lemma 5.7 we have

$$t_4 = \frac{-b}{2} - \frac{|b||u - b^2|\sqrt{4u - 3b^2}}{2b(u - b^2)} \quad \text{and} \quad t_5 = \frac{-b}{2} + \frac{|b||u - b^2|\sqrt{4u - 3b^2}}{2b(u - b^2)}.$$

Lemma 5.8 *Consider $u > 3b^2/4$ and the points t_i, $i = 1, \ldots, 5$ as above.*

(i) *$b < 0$ and $u > b^2$ if and only if $t_1 < t_5 < t_2 < t_3 < t_4$.*
(ii) *$b > 0$ and $u > b^2$ if and only if $t_4 < t_1 < t_2 < t_5 < t_3$.*
(iii) *$b < 0$ and $u < b^2$ if and only if $t_4, t_5 \in (t_1, t_2)$.*
(iv) *$b > 0$ and $u < b^2$ if and only if $t_4, t_5 \in (t_2, t_3)$.*

Proof. We will prove only case (i). The remaining cases follow by using a similar argument and will be omitted here. Suppose $b < 0$ and $u - b^2 > 0$. Then we can write

$$t_4 = \frac{1}{2}\big(-b + \sqrt{4u - 3b^2}\big) \quad \text{and} \quad t_5 = \frac{1}{2}\big(-b - \sqrt{4u - 3b^2}\big).$$

We first show that $t_5 \in (t_1, t_2)$. Note that

$$t_5 = \frac{-b - \sqrt{4u - 3b^2}}{2} > \frac{-b - \sqrt{4u}}{2} = \frac{-b - 2\sqrt{u}}{2} > -\sqrt{u} = t_1.$$

On the other hand,

$$u > b^2 \Rightarrow 4u - 3b^2 > b^2 \Rightarrow \sqrt{4u - 3b^2} > |b|.$$

Since $b < 0$, then $\sqrt{4u - 3b^2} > -b$. Thus,

$$t_5 = \frac{-b - \sqrt{4u - 3b^2}}{2} < \frac{-b}{2} + \frac{b}{2} = 0 = t_2.$$

The inequality $t_2 < t_3$ is obvious. In what follows we prove that $t_4 > t_3 = \sqrt{u}$. It is easy to see that $4b(b + \sqrt{u}) < 0$, therefore

$$\begin{aligned} -3b^2 + 4u + 4b(b + \sqrt{u}) &< -3b^2 + 4u \\ b^2 + 4b\sqrt{u} + 4u &< -3b^2 + 4u \\ (b + 2\sqrt{u})^2 &< -3b^2 + 4u \quad \text{(by hypothesis, } b + 2\sqrt{u} > 0) \\ b + 2\sqrt{u} &< \sqrt{4u - 3b^2} \\ t_3 = \sqrt{u} &< \frac{-b + \sqrt{4u - 3b^2}}{2} = t_4. \end{aligned}$$

Therefore, $t_4 > t_3$. Now, we prove the converse. Suppose $t_1 < t_5 < t_2 < t_3 < t_4$. In particular, $t_5 - t_4 < 0$ so

$$\frac{|b||u - b^2|\sqrt{4u - 3b^2}}{b(u - b^2)} < 0$$

and conclude that $b(u - b^2) < 0$. Therefore either $b < 0$ and $u > b^2$ or $b > 0$

and $u < b^2$. We affirm that the second case does not occur, since otherwise we would have

$$t_4 = -\frac{b}{2} - \frac{|b||u-b^2|\sqrt{4u-3b^2}}{2b(u-b^2)} = \frac{-b+\sqrt{4u-3b^2}}{2} < -\frac{b}{2} + \sqrt{u} < \sqrt{u} = t_3.$$

The lemma has been proved. □

Finally, we present the directions of projection in which the transitions between the Morsifications of the E_8 singularity occur.

Theorem 5.9 *Let $\gamma_{(a,b)}$ be as in (25) and $\tilde{P}_{(u,v)}$ as in (26). The projection $\tilde{P}_{(u,v)}$ in the direction*

$$\mathbf{v} = (u, v) = \left(-\frac{a}{b}, \frac{a^2}{b^2}\right)$$

with $u > 3b^2/4$ and $b \neq 0$, presents a transition:

(i) *$\mathcal{T}_{12}$ when $u > b^2$.*
(ii) *$\mathcal{T}_{23}$ when $3b^2/4 < u < b^2$.*

Proof. Since the direction of projection is given by

$$\mathbf{v} = \left(-\frac{a}{b}, \frac{a^2}{b^2}\right),$$

it follows by Lemma 5.7 that a triple point occurs at the origin. Moreover, the triple point occurs when $t_1 = -\sqrt{u}$, $t_2 = 0$ and $t_3 = \sqrt{u}$.

(i) When $u > b^2$ it follows by Lemma 5.8 that either $t_4 > t_3$ and $t_5 \in (t_1, t_2)$ or $t_4 < t_1$ and $t_5 \in (t_2, t_3)$. Thus we have one of two increasing sequences of parameters

$$t_1 < t_5 < t_2 < t_3 < t_4 \quad \text{or} \quad t_4 < t_1 < t_2 < t_5 < t_3$$

and both sequences yield the Gauss word *abaab*. Therefore we obtain a transition $\mathcal{T}_{12}$ according to Definition 5.6.

(ii) The proof of this item follows similarly to the previous one. □

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