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Boundedness of maximal operators and Sobolev's theorem for non-homogeneous central Morrey spaces of variable exponent

Yoshihiro Mizuta, Takao Ohno and Tetsu Shimomura

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Abstract. Our aim in this paper is to deal with the boundedness of the Hardy-Littlewood maximal operator in non-homogeneous central Morrey spaces of variable exponent. Further, we give Sobolev's inequality and Trudinger's exponential integrability for generalized Riesz potentials.

Key words: Maximal operator, non-homogeneous central Morrey spaces of variable exponent, Riesz potentials, Sobolev's theorem, Sobolev's inequality, Trudinger's exponential integrability.

1. Introduction

Let $\mathbf{R}^N$ be the Euclidean space. In [4], Beurling introduced the space $B^p(\mathbf{R}^N)$ to extend Wiener's ideas [21], [22] which describes the behavior of functions at infinity. Feichtinger [8] gave an equivalent norm on $B^p(\mathbf{R}^N)$, which is a special case of norms in Herz spaces $K_p^{\alpha,r}(\mathbf{R}^N)$ introduced by Herz [12]. Precisely speaking, $B^p(\mathbf{R}^N) = K_p^{-N/p,\infty}(\mathbf{R}^N)$ (see also [11]). In [10], García-Cuerva studied the boundedness of the maximal operator on the space $B^p(\mathbf{R}^N)$. As an extension of the space $B^p(\mathbf{R}^N)$, García-Cuerva and Herrero [11] introduced the central Morrey spaces $B^{p,\nu}(\mathbf{R}^N)$ (see also [3]). Alvarez, Guzmán-Partida and Lakey [3] obtained the boundedness of a class of singular integrals operators on the central Morrey spaces (see also Komori [13]), which are more singular than Calderón-Zygmund operators and include pseudo-differential operators.

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth condition. Our first aim in this paper is to introduce the non-homogeneous central Morrey spaces of variable exponent, and study the boundedness of the Hardy-Littlewood maximal operator (see Theorem 3.1), in a way

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different from Almeida and Drihem [2].

In classical Lebesgue spaces, we know Sobolev's inequality :

$$\|I_\alpha f\|_{L^{p^\sharp}(\mathbf{R}^N)} \le C\|f\|_{L^p(\mathbf{R}^N)}$$

for $f \in L^p(\mathbf{R}^N)$, $0 < \alpha < N$ and $1 < p < N/\alpha$, where I_α is the Riesz kernel of order α and $1/p^\sharp = 1/p - \alpha/N$ (see, e.g. [1, Theorem 3.1.4]). This result was extended to the central Morrey spaces by Fu, Lin and Lu [9, Proposition 1.1] (see also Matsuoka and Nakai [15]).

To obtain general results, for $0 < \alpha < N$ and an integer k, we define the generalized Riesz potential $I_{\alpha,k}f$ of order α of a locally integrable function f on $\mathbf{R}^N$ by

$$I_{\alpha,k}f(x) = \int_{\mathbf{R}^N\setminus B(0,1)} \left\{ I_\alpha(x-y) - \sum_{\{\mu:|\mu|\le k-1\}} \frac{x^\mu}{\mu!}(D^\mu I_\alpha)(-y) \right\} f(y)\,dy,$$

where $I_\alpha(x) = |x|^{\alpha-n}$ (see [16], [17]). Remark here that

$$I_{\alpha,k}f(x) = \int_{\mathbf{R}^N\setminus B(0,1)} I_\alpha(x-y)f(y)\,dy$$

when $k \le 0$.

In Section 4, when $p^+ < N/\alpha$ (see Section 2 for the definition of p^+), we shall give Sobolev's inequality for $I_{\alpha,k}f$ with functions in the non-homogeneous central Morrey spaces of variable exponent (see Theorem 4.5); for related result, we refer the reader to Fu, Lin and Lu [9, Theorem 2.1].

In the last section, when $p = N/\alpha$, we treat Trudinger's exponential integrability for $I_{\alpha,k}f$ (see Theorem 5.1).

2. Preliminaries

Consider a function $p(\cdot)$ on $\mathbf{R}^N$ such that

(P1) $1 < p^- := \inf_{x\in\mathbf{R}^N} p(x) \le \sup_{x\in\mathbf{R}^N} p(x) =: p^+ < \infty$;
(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{c_p}{\log(e + 1/|x-y|)} \quad \text{for } x, y \in \mathbf{R}^N$$

with a constant $c_p \ge 0$;

(P3) $p(\cdot)$ is log-Hölder continuous at ∞, namely

$$|p(x) - p(\infty)| \le \frac{c_\infty}{\log(e+|x|)} \quad \text{whenever } |x| > 0$$

with constants $p(\infty) > 1$ and $c_\infty \ge 0$;

$p(\cdot)$ is referred to as a variable exponent.

For $\nu \ge 0$, we denote by $\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)$ the class of locally integrable functions f on $\mathbf{R}^N$ satisfying

$$\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} = \sup_{R\ge 1} R^{-\nu/p(\infty)} \|f\|_{L^{p(\cdot)}(B(0,R))} < \infty,$$

where

$$\|f\|_{L^{p(\cdot)}(B(0,R))} = \inf\left\{\lambda > 0 : \int_{B(0,R)} \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} dy \le 1\right\}.$$

The space $\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)$ is referred to as a non-homogeneous central Morrey spaces of variable exponent. If $p(\cdot)$ is a constant and $\nu = N$, then $\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N) = B^p(\mathbf{R}^N)$.

Throughout this paper, let C denote various constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1}h \le g \le Ch$ for some constant $C > 0$.

Lemma 2.1 *Set*

$$\|f\|_{\tilde{\mathcal{B}}^{p(\cdot),\nu}(\mathbf{R}^N)} = \inf\left\{\lambda > 0 : \sup_{R\ge 1} R^{-\nu} \int_{B(0,R)} \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} dy \le 1\right\}.$$

Then

$$\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} \sim \|f\|_{\tilde{\mathcal{B}}^{p(\cdot),\nu}(\mathbf{R}^N)}$$

for all $f \in L^1_{loc}(\mathbf{R}^N)$.

Proof. We may assume that $\nu > 0$. First we find a constant $C > 0$ such that

$$\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} \le C\|f\|_{\tilde{\mathcal{B}}^{p(\cdot),\nu}(\mathbf{R}^N)}$$

for all $f \in L^1_{loc}(\mathbf{R}^N)$. Let f be a nonnegative function on $\mathbf{R}^N$ with $\|f\|_{\tilde{\mathcal{B}}^{p(\cdot),\nu}(\mathbf{R}^N)} \le 1$. Then note that

$$R^{-\nu} \int_{B(0,R)} f(y)^{p(y)}\, dy \le 1$$

for all $R \ge 1$. To end the proof, it is sufficient to find a constant $C > 0$ such that

$$\int_{B(0,R)\setminus B(0,1)} \left(R^{-\nu/p(\infty)} f(y)\right)^{p(y)} dy \le C$$

for all $R \ge 1$. For this purpose, let $R \ge 1$ and take an integer $j_0 \ge 1$ such that $2^{-j_0}R \le 1 < 2^{-j_0+1}R$. We have

$$\begin{aligned}
&\int_{B(0,R)\setminus B(0,1)} \left(R^{-\nu/p(\infty)} f(y)\right)^{p(y)} dy \\
&\quad \le \sum_{j=0}^{j_0} \int_{B(0,2^{-j+1}R)\setminus B(0,2^{-j}R)} \left(R^{-\nu/p(\infty)} f(y)\right)^{p(y)} dy \\
&\quad \le \sum_{j=0}^{j_0} (2^{-j})^{\nu/p(\infty)} \int_{B(0,2^{-j+1}R)\setminus B(0,2^{-j}R)} \left\{(2^{-j}R)^{-\nu/p(\infty)} f(y)\right\}^{p(y)} dy \\
&\quad \le \sum_{j=0}^{j_0} (2^{-j})^{\nu/p(\infty)} (2^{-j}R)^{\nu} \int_{B(0,2^{-j+1}R)} f(y)^{p(y)}\, dy \\
&\quad \le C \sum_{j=0}^{j_0} (2^{-j})^{\nu/p(\infty)} \le C
\end{aligned}$$

since $|y|^{-p(y)} \le C|y|^{-p(\infty)}$ for $y \in B(0,2^{-j+1}R)\setminus B(0,2^{-j}R)$ and $0 \le j \le j_0$ by (P3).

Next we prove the converse inequality. Then it is sufficient to find a constant $C > 0$ such that

$$R^{-\nu} \int_{B(0,R)\setminus B(0,1)} f(y)^{p(y)}\, dy \le C$$

for all $R \geq 1$ and $f \geq 0$ on $\mathbf{R}^N$ with

$$\sup_{R>1} \int_{B(0,R)} \left(R^{-\nu/p(\infty)} f(y)\right)^{p(y)} dy \leq 1.$$

For this purpose, let $R > 1$ and take an integer $j_0 \geq 1$ such that $2^{-j_0}R \leq 1 < 2^{-j_0+1}R$ as before. We find

$$\begin{aligned}
&\int_{B(0,R)\setminus B(0,1)} \left(R^{-\nu/p(y)} f(y)\right)^{p(y)} dy \\
&\leq \sum_{j=0}^{j_0} (2^{-j})^{\nu} \int_{B(0,2^{-j+1}R)\setminus B(0,2^{-j}R)} \left\{(2^{-j}R)^{-\nu/p(y)} f(y)\right\}^{p(y)} dy \\
&\leq \sum_{j=0}^{j_0} (2^{-j})^{\nu} \int_{B(0,2^{-j+1}R)} \left\{(2^{-j}R)^{-\nu/p(\infty)} f(y)\right\}^{p(y)} dy \\
&\leq C \sum_{j=0}^{j_0} (2^{-j})^{\nu} \leq C
\end{aligned}$$

since $|y|^{-1/p(y)} \leq C|y|^{-1/p(\infty)}$ for $y \in B(0, 2^{-j+1}R) \setminus B(0, 2^{-j}R)$ and $0 \leq j \leq j_0$ by (P3). Thus the proof is completed. $\square$

3. Boundedness of maximal operators

For a locally integrable function f on $\mathbf{R}^N$, the Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

where $B(x,r)$ is the ball in $\mathbf{R}^N$ with center x and of radius $r > 0$, and $|B(x,r)|$ denotes its Lebesgue measure. The mapping $f \mapsto Mf$ is called the maximal operator.

The maximal operator is a classical tool in harmonic analysis and studying Sobolev functions and partial differential equations, and it plays a central role in the study of differentiation, singular integrals, smoothness of functions and so on (see [5], [13], [14], [20], etc.).

It is well known that the maximal operator is bounded in the Lebesgue space $L^p(\mathbf{R}^N)$ when $p > 1$ (see [20]). We present the boundedness of maximal operator in the central Morrey spaces of variable exponent.

Theorem 3.1 *Let $0 \le \nu \le N$. Then the maximal operator: $f \to Mf$ is bounded from $\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)$ to $\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)$, that is,*

$$\|Mf\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} \le C\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} \qquad \text{for all } f \in \mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N).$$

When $0 \le \nu < N$, this theorem is essentially proved by Almeida and Drihem [2, Corollary 4.7]. But, for the readers' convenience, we give a proof of Theorem 3.1 different from [2].

Before doing this, we prepare the following results.

Lemma 3.2 ([7, Corollary 4.5.9]) *For all $R \ge 1$,*

$$\|1\|_{L^{p(\cdot)}(B(0,R))} \sim R^{N/p(\infty)},$$

that is, $1 \in \mathcal{B}^{p(\cdot),N}(\mathbf{R}^N)$.

Lemma 3.3 *There exists a constant $C > 0$ such that*

$$\frac{1}{|B(0,R)|}\int_{B(0,R)\setminus B(0,R/2)} f(y)\,dy \le CR^{-(N-\nu)/p(\infty)}$$

for all $R \ge 1$ and $f \ge 0$ such that $\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} \le 1$.

Proof. Let f be a nonnegative function on $\mathbf{R}^N$ such that $\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} \le 1$. Then we see from Lemma 2.1 that

$$R^{-\nu}\int_{B(0,R)\setminus B(0,R/2)} f(y)^{p(y)}\,dy \le C$$

for all $R \ge 1$. Hence we find by (P3)

$$\frac{1}{|B(0,R)|}\int_{B(0,R)\setminus B(0,R/2)} f(y)\,dy$$
$$\le R^{-(N-\nu)/p(\infty)} + \frac{1}{|B(0,R)|}\int_{B(0,R)\setminus B(0,R/2)} f(y)\left(\frac{f(y)}{R^{-(N-\nu)/p(\infty)}}\right)^{p(y)-1} dy$$

$$
\begin{aligned}
&\le R^{-(N-\nu)/p(\infty)} \\
&\quad + CR^{(N-\nu)(p(\infty)-1)/p(\infty)} \frac{1}{|B(0,R)|} \int_{B(0,R)\setminus B(0,R/2)} f(y)^{p(y)}\, dy \\
&\le CR^{-(N-\nu)/p(\infty)}
\end{aligned}
$$

for all $R \ge 1$, as required. $\square$

We denote by χ_E the characteristic function of E.

Lemma 3.4 *Let $0 \le \nu \le N$. Then there exists a constant $C > 0$ such that*

$$
M(f\chi_{\mathbf{R}^N \setminus B(0,2R)})(x) \le CR^{-(N-\nu)/p(\infty)}
$$

for all $x \in B(0,R)$ with $R \ge 1$ and $f \ge 0$ with $\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} \le 1$.

Proof. Let f be a nonnegative function on $\mathbf{R}^N$ such that $\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} \le 1$. Let $R \ge 1$ and $x \in B(0,R)$. We have by Lemma 3.3

$$
\begin{aligned}
M(f\chi_{\mathbf{R}^N \setminus B(0,2R)})(x) &= \sup_{r>R} \frac{1}{|B(x,r)|} \int_{B(x,r)\setminus B(0,2R)} f(y)\, dy \\
&\le \sup_{r>R} \frac{1}{|B(0,r)|} \sum_{\{j \ge 1 : 2^j R < 2r\}} \int_{B(0,2^{j+1}R)\setminus B(0,2^j R)} f(y)\, dy \\
&\le C \sup_{r>R} \frac{1}{|B(0,r)|} \sum_{\{j \ge 1 : 2^j R < 2r\}} (2^{j+1}R)^{N-(N-\nu)/p(\infty)} \\
&\le C \sup_{r>R} \frac{1}{|B(0,r)|} r^{N-(N-\nu)/p(\infty)} \\
&\le CR^{-(N-\nu)/p(\infty)},
\end{aligned}
$$

as required. $\square$

We know the following result.

Lemma 3.5 ([6, Theorem 1.5]) *There exists a constant $c_0 > 0$ such that*

$$
\|Mf\|_{L^{p(\cdot)}(\mathbf{R}^N)} \le c_0 \|f\|_{L^{p(\cdot)}(\mathbf{R}^N)}
$$

for all $f \in L^{p(\cdot)}(\mathbf{R}^N)$.

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let f be a nonnegative function on $\mathbf{R}^N$ such that $\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} \le 1$. For $R \ge 1$, set

$$f = f\chi_{B(0,2R)} + f\chi_{\mathbf{R}^N \setminus B(0,2R)} = f_1 + f_2.$$

First we find from Lemmas 3.2 and 3.4

$$\begin{aligned}\|Mf_2\|_{L^{p(\cdot)}(B(0,R))} &\le CR^{-(N-\nu)/p(\infty)}\|1\|_{L^{p(\cdot)}(B(0,R))} \\ &\le CR^{-(N-\nu)/p(\infty)}R^{N/p(\infty)} = CR^{\nu/p(\infty)}.\end{aligned}$$

Next we obtain by Lemma 3.5

$$\begin{aligned}\|Mf\|_{L^{p(\cdot)}(B(0,R))} &\le \|Mf_1\|_{L^{p(\cdot)}(B(0,R))} + \|Mf_2\|_{L^{p(\cdot)}(B(0,R))} \\ &\le C\big\{\|f\|_{L^{p(\cdot)}(B(0,2R))} + R^{\nu/p(\infty)}\big\} \\ &\le C\big\{(2R)^{\nu/p(\infty)} + R^{\nu/p(\infty)}\big\} \le CR^{\nu/p(\infty)},\end{aligned}$$

so that

$$\sup_{R\ge 1} R^{-\nu/p(\infty)}\|Mf\|_{L^{p(\cdot)}(B(0,R))} \le C.$$

Thus we establish the required result. □

Remark 3.6 If $\nu > N$, then, as in the proof of Theorem 3.1, we find

$$\sup_{R\ge 1} R^{-\nu/p(\infty)}\|M(f\chi_{B(0,R)})\|_{L^{p(\cdot)}(B(0,R))} \le C.$$

4. Sobolev's inequality

For $\nu \ge 0$, take the integer $k \ge 0$ such that

$$k - 1 \le \alpha - (N-\nu)/p(\infty) < k \tag{4.1}$$

and consider the generalized Riesz potential

$$I_{\alpha,k}f(x) = \int_{\mathbf{R}^N\setminus B(0,1)} \left\{ I_\alpha(x-y) - \sum_{\{\mu:|\mu|\le k-1\}} \frac{x^\mu}{\mu!}(D^\mu I_\alpha)(-y) \right\} f(y)\,dy$$

for a locally integrable function f on $\mathbf{R}^N$.

The following estimates are fundamental (see [17] and [19]).

Lemma 4.1 *Let $k \ge 1$ be an integer.*

(1) *If $2|x| < |y|$, then*

$$\left| I_\alpha(x-y) - \sum_{\{\mu:|\mu|\le k-1\}} \frac{x^\mu}{\mu!}(D^\mu I_\alpha)(-y) \right| \le C|x|^k|y|^{\alpha-N-k};$$

(2) *If $|x|/2 \le |y| \le 2|x|$, then*

$$\left| I_\alpha(x-y) - \sum_{\{\mu:|\mu|\le k-1\}} \frac{x^\mu}{\mu!}(D^\mu I_\alpha)(-y) \right| \le C|x-y|^{\alpha-N};$$

(3) *If $1 \le |y| \le |x|/2$, then*

$$\left| I_\alpha(x-y) - \sum_{\{\mu:|\mu|\le k-1\}} \frac{x^\mu}{\mu!}(D^\mu I_\alpha)(-y) \right| \le C|x|^{k-1}|y|^{\alpha-N-(k-1)}.$$

Lemma 4.2 *Let k be the integer defined by (4.1). Then there exists a constant $C > 0$ such that*

$$\left| I_{\alpha,k}(f\chi_{\mathbf{R}^N\setminus B(0,2R)})(x) \right| \le CR^{\alpha-(N-\nu)/p(\infty)}$$

for all $x \in B(0,R)$ with $R \ge 1$ and $f \ge 0$ with $\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} \le 1$.

Proof. Let f be a nonnegative function on $\mathbf{R}^N$ such that $\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} \le 1$. Let $R \ge 1$ and $x \in B(0,R)$. First note from Lemma 4.1 (1) that

$$\left| I_{\alpha,k}(f\chi_{\mathbf{R}^N\setminus B(0,2R)})(x) \right| \le CR^k \int_{\mathbf{R}^N\setminus B(0,2R)} |y|^{\alpha-N-k} f(y)\,dy.$$

Hence, we have by Lemma 3.3

$$
\begin{aligned}
&\left|I_{\alpha,k}(f\chi_{\mathbf{R}^N\setminus B(0,2R)})(x)\right| \\
&\quad\le CR^k\sum_{j=1}^{\infty}\int_{B(0,2^{j+1}R)\setminus B(0,2^jR)}|y|^{\alpha-N-k}f(y)\,dy \\
&\quad\le CR^k\sum_{j=1}^{\infty}(2^jR)^{\alpha-k}\frac{1}{|B(0,2^{j+1}R)|}\int_{B(0,2^{j+1}R)\setminus B(0,2^jR)}f(y)\,dy \\
&\quad\le CR^k\sum_{j=1}^{\infty}(2^jR)^{\alpha-k-(N-\nu)/p(\infty)} \\
&\quad= CR^{\alpha-(N-\nu)/p(\infty)}\sum_{j=1}^{\infty}2^{j\{\alpha-k-(N-\nu)/p(\infty)\}} \\
&\quad\le CR^{\alpha-(N-\nu)/p(\infty)},
\end{aligned}
$$

as required. □

Lemma 4.3 *Let $k\ge 1$ be an integer. Then there exists a constant $C>0$ such that*

(1) *in case $k-1<\alpha-(N-\nu)/p(\infty)<k$,*

$$|x|^{k-1}\int_{B(0,|x|/2)\setminus B(0,1)}|y|^{\alpha-N-(k-1)}f(y)\,dy\le CR^{\alpha-(N-\nu)/p(\infty)};$$

(2) *in case $k-1=\alpha-(N-\nu)/p(\infty)$,*

$$|x|^{k-1}\int_{B(0,|x|/2)\setminus B(0,1)}|y|^{\alpha-N-(k-1)}f(y)\,dy\le CR^{\alpha-(N-\nu)/p(\infty)}\log R$$

for all $x\in B(0,R)$ with $R\ge 2$ and $f\ge 0$ with $\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)}\le 1$.

Proof. Let f be a nonnegative function on $\mathbf{R}^N$ such that $\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)}\le 1$. Let $R\ge 2, k\ge 1$ and $x\in B(0,R)$. We may assume that $|x|\ge 2$. We take an integer $j_0\ge 1$ such that $2^{-j_0-1}|x|<1\le 2^{-j_0}|x|$.

First we show the case $k-1<\alpha-(N-\nu)/p(\infty)<k$. Then we have by Lemma 3.3

$$
\begin{aligned}
&|x|^{k-1} \int_{B(0,|x|/2)\setminus B(0,1)} |y|^{\alpha-N-(k-1)} f(y)\, dy \\
&\le |x|^{k-1} \sum_{j=1}^{j_0} \int_{B(0,2^{-j}|x|)\setminus B(0,2^{-j-1}|x|)} |y|^{\alpha-N-(k-1)} f(y)\, dy \\
&\le C|x|^{k-1} \sum_{j=1}^{\infty} (2^{-j}|x|)^{\alpha-(k-1)} \frac{1}{|B(0,2^{-j}|x|)|} \int_{B(0,2^{-j}|x|)\setminus B(0,2^{-j-1}|x|)} f(y)\, dy \\
&\le CR^{k-1} \sum_{j=1}^{j_0} (2^{-j}R)^{\alpha-(k-1)-(N-\nu)/p(\infty)} \\
&\le CR^{\alpha-(N-\nu)/p(\infty)}.
\end{aligned}
$$

Next we deal with the case $k-1 = \alpha - (N-\nu)/p(\infty)$. Since $j_0 \le \log|x|/\log 2 < j_0 + 1$, we see from Lemma 3.3 that

$$
\begin{aligned}
&|x|^{k-1} \int_{B(0,|x|/2)\setminus B(0,1)} |y|^{\alpha-N-(k-1)} f(y)\, dy \\
&\le CR^{k-1} \sum_{j=1}^{j_0} (2^{-j}R)^{\alpha-(k-1)-(N-\nu)/p(\infty)} \\
&\le CR^{\alpha-(N-\nu)/p(\infty)} j_0 \\
&\le CR^{\alpha-(N-\nu)/p(\infty)} \log R,
\end{aligned}
$$

as required. □

Set

$$1/p^\sharp(x) = 1/p(x) - \alpha/N.$$

Lemma 4.4 ([18, Theorem 4.1]) *Suppose $1/p^+ - \alpha/N > 0$. Then there exists a constant $c_1 > 0$ such that*

$$\|I_\alpha f\|_{L^{p^\sharp(\cdot)}(\mathbf{R}^N)} \le c_1 \|f\|_{L^{p(\cdot)}(\mathbf{R}^N)}$$

for all $f \in L^{p(\cdot)}(\mathbf{R}^N)$ with compact support.

Now we show the Sobolev type inequality for generalized Riesz potentials in the central Morrey spaces of variable exponents, as an extension of Fu, Lin and Lu [9] in the constant exponent case.

Theorem 4.5 (cf. [9, Proposition 1.1]) *Suppose* $1/p^+ - \alpha/N > 0$ *and* $k-1 < \alpha - (N-\nu)/p(\infty) < k$. *Then there exists a constant* $C > 0$ *such that*

$$\sup_{R \geq 1} R^{-\nu/p(\infty)} \|I_{\alpha,k} f\|_{L^{p^\sharp(\cdot)}(B(0,R))} \leq C$$

for all $f \geq 0$ *with* $\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} \leq 1$.

Proof. Let f be a nonnegative function on $\mathbf{R}^N$ such that $\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} \leq 1$. For $R \geq 1$, set

$$f = f\chi_{B(0,2R)} + f\chi_{\mathbf{R}^N \setminus B(0,2R)} = f_1 + f_2.$$

First we find by Lemmas 3.2 and 4.2

$$\begin{aligned} \|I_{\alpha,k} f_2\|_{L^{p^\sharp(\cdot)}(B(0,R))} &\leq C R^{\alpha-(N-\nu)/p(\infty)} \|1\|_{L^{p^\sharp(\cdot)}(B(0,R))} \\ &\leq C R^{\alpha-(N-\nu)/p(\infty)} R^{N/p^\sharp(\infty)} \\ &= C R^{\nu/p(\infty)}. \end{aligned}$$

Next, we see from Lemmas 4.1 and 4.3 (1) that

$$\begin{aligned} |I_{\alpha,k} f_1(x)| \leq{}& \left|I_{\alpha,k}(f\chi_{B(0,2R)\setminus B(0,2|x|)})(x)\right| + \left|I_{\alpha,k}(f\chi_{B(0,2|x|)\setminus B(0,|x|/2)})(x)\right| \\ &+ \left|I_{\alpha,k}(f\chi_{B(0,|x|/2)\setminus B(0,1)})(x)\right| \\ \leq{}& C\left\{I_\alpha f_1(x) + R^{\alpha-(N-\nu)/p(\infty)}\right\} \end{aligned}$$

for $x \in B(0,R)$ since $|x|^k |y|^{\alpha-N-k} \leq C|x-y|^{\alpha-N}$ for $2|x| < |y|$, so that we have by Lemmas 3.2 and 4.4

$$\begin{aligned} \|I_{\alpha,k} f\|_{L^{p^\sharp(\cdot)}(B(0,R))} &\leq \|I_{\alpha,k} f_1\|_{L^{p^\sharp(\cdot)}(B(0,R))} + \|I_{\alpha,k} f_2\|_{L^{p^\sharp(\cdot)}(B(0,R))} \\ &\leq C\left\{\|f\|_{L^{p(\cdot)}(B(0,2R))} + R^{\nu/p(\infty)}\right\} \\ &\leq C\left\{(2R)^{\nu/p(\infty)} + R^{\nu/p(\infty)}\right\} \leq C R^{\nu/p(\infty)}, \end{aligned}$$

so that

$$\sup_{R \ge 1} R^{-\nu/p(\infty)} \|I_{\alpha,k} f\|_{L^{p^\sharp(\cdot)}(B(0,R))} \le C.$$

Thus we completes the proof. $\square$

Remark 4.6 Suppose $1/p^+ - \alpha/N > 0$ and $k - 1 = \alpha - (N - \nu)/p(\infty)$. Then there exists a constant $C > 0$ such that

$$\sup_{R \ge 2} R^{-\nu/p(\infty)} (\log R)^{-1} \|I_{\alpha,k} f\|_{L^{p^\sharp(\cdot)}(B(0,R))} \le C$$

for all $f \ge 0$ with $\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} \le 1$.

5. Exponential integrability

Our aim in this section is to discuss the exponential integrability.

Theorem 5.1 *Let $p = N/\alpha$ and $k - 1 < \alpha - (N - \nu)/p < k$. Then there exist constants $c_1, c_2 > 0$ such that*

$$\sup_{R \ge 1} R^{-N} \int_{B(0,R)} \exp\left(\{c_1 R^{-\nu/p} |I_{\alpha,k} f(x)|\}^{p'}\right) dx \le c_2$$

for all $f \ge 0$ with $\|f\|_{\mathcal{B}^{p,\nu}(\mathbf{R}^N)} \le 1$.

Proof. Let f be a nonnegative function on $\mathbf{R}^N$ such that $\|f\|_{\mathcal{B}^{p,\nu}(\mathbf{R}^N)} \le 1$ and let $x \in B(0, R)$. For $R \ge 1$, set

$$f = f\chi_{B(0,2R)} + f\chi_{\mathbf{R}^N \setminus B(0,2R)} = f_1 + f_2.$$

For $0 < \delta \le R$, write

$$\begin{aligned} I_\alpha f_1(x) &= \int_{B(x,\delta)} |x - y|^{\alpha - N} f(y)\, dy + \int_{B(0,2R) \setminus B(x,\delta)} |x - y|^{\alpha - N} f(y)\, dy \\ &= U_1(x) + U_2(x). \end{aligned}$$

First we find

$$U_1(x) \le C\delta^\alpha M f_1(x).$$

Next we have by Hölder's inequality

$$U_2(x) \le C(\log(2R/\delta))^{1/p'} \|f_1\|_{L^p(B(0,2R))},$$

so that

$$I_\alpha f_1(x) \le C\big\{\delta^\alpha Mf_1(x) + (\log(2R/\delta))^{1/p'} R^{\nu/p}\big\}.$$

Here, letting $\delta/(2R) = \{R^{-\nu/p+\alpha} Mf_1(x)\}^{-1/\alpha}(\log(R^{-\nu/p+\alpha}Mf_1(x)))^{1/(\alpha p')} < 1$, we establish

$$I_\alpha f_1(x) \le C(\log(R^{-\nu/p+\alpha}Mf_1(x)))^{1/p'} R^{\nu/p};$$

if $\{R^{-\nu/p+\alpha}Mf_1(x)\}^{-1/\alpha}(\log(R^{-\nu/p+\alpha}Mf_1(x)))^{1/(\alpha p')} \ge 1$, then, letting $\delta = R$, we have

$$I_\alpha f_1(x) \le CR^{\nu/p}.$$

As in the proof of Theorem 4.5, we see from Lemmas 4.1 and 4.3 (1) that

$$|I_{\alpha,k} f_1(x)| \le C\big\{I_\alpha f_1(x) + R^{\alpha-(N-\nu)/p}\big\} = C\big\{I_\alpha f_1(x) + R^{\nu/p}\big\}$$

for $x \in B(0,R)$, since $\alpha = N/p$. Therefore, we obtain

$$|I_{\alpha,k} f_1(x)| \le C\big\{(\log(e + R^{-\nu/p+\alpha}Mf_1(x)))^{1/p'} R^{\nu/p} + R^{\nu/p}\big\}.$$

On the other hand, we obtain by Lemma 4.2

$$|I_{\alpha,k} f_2(x)| \le CR^{\alpha-(N-\nu)/p} = CR^{\nu/p},$$

since $\alpha = N/p$. Hence, we find

$$\{c_1 R^{-\nu/p}|I_{\alpha,k} f(x)|\}^{p'} \le \log(e + R^{(N-\nu)/p}Mf_1(x)),$$

so that we have by boundedness of maximal operators on $L^p(\mathbf{R}^N)$

$$\int_{B(0,R)} \exp(\{c_1 R^{-\nu/p}|I_{\alpha,k} f(x)|\}^{p'})\, dx \le C \int_{B(0,R)} \big[1 + R^{N-\nu}\{Mf_1(x)\}^p\big]\, dx$$

$$
\begin{aligned}
&\le C\left(R^N + R^{N-\nu}\int_{\mathbf{R}^N} f_1(y)^p\,dy\right)\\
&\le CR^N,
\end{aligned}
$$

as required. □

Remark 5.2 Let $p = N/\alpha$ and $k-1 = \alpha - (N-\nu)/p$. Then there exist constants $c_1, c_2 > 0$ such that

$$\sup_{R\ge 2} R^{-N}\int_{B(0,R)} \exp\left(\{c_1 R^{-\nu/p}(\log R)^{-1}|I_{\alpha,k}f(x)|\}^{p'}\right)dx \le c_2$$

for all $f \ge 0$ with $\|f\|_{\mathcal{B}^{p,\nu}(\mathbf{R}^N)} \le 1$.

Remark 5.3 If $p^- \ge p(\infty)$, then $\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N) \subset \mathcal{B}^{p(\infty),\nu}(\mathbf{R}^N)$, and moreover

$$\|f\|_{\mathcal{B}^{p(\infty),\nu}(\mathbf{R}^N)} \le C\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)}.$$

In fact, for $R \ge 1$ and $a > N/p(\infty)$,

$$
\begin{aligned}
&R^{-\nu}\int_{B(0,R)} |f(x)|^{p(\infty)}\,dx\\
&\quad = R^{-\nu}\int_{\{x\in B(0,R):|f(x)|\ge 1\}} |f(x)|^{p(\infty)}\,dx\\
&\qquad + R^{-\nu}\int_{\{x\in B(0,R):(1+|x|)^{-a}<|f(x)|\le 1\}} |f(x)|^{p(\infty)}\,dx\\
&\qquad + R^{-\nu}\int_{\{x\in B(0,R):|f(x)|\le(1+|x|)^{-a}\}} |f(x)|^{p(\infty)}\,dx\\
&\quad \le R^{-\nu}\int_{\{x\in B(0,R):|f(x)|\ge 1\}} |f(x)|^{p(x)}\,dx\\
&\qquad + R^{-\nu}\int_{\{x\in B(0,R):(1+|x|)^{-a}<|f(x)|\le 1\}} |f(x)|^{p(x)}|f(x)|^{p(\infty)-p(x)}\,dx\\
&\qquad + CR^{-\nu}\int_{B(0,R)} (1+|x|)^{-ap(\infty)}\,dx
\end{aligned}
$$

$$\leq C\left\{R^{-\nu}\int_{B(0,R)}|f(x)|^{p(x)}\,dx + R^{-\nu}\right\}$$
$$\leq C$$

when $\|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbf{R}^N)} \leq 1$.

References

[1] Adams D. R. and Hedberg L. I., Function Spaces and Potential Theory, Springer, 1996.

[2] Almeida A. and Drihem D., *Maximal, potential and singular type operators on Herz spaces with variable exponents*. J. Math. Anal. Appl. **394** (2012), 781–795.

[3] Alvarez J., Guzmán-Partida M. and Lakey J., *Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measures*. Collect. Math. **51** (2000), 1–47.

[4] Beurling A., *Construction and analysis of some convolution algebras*. Ann. Inst. Fourier **14** (1964), 1–32.

[5] Bojarski B. and Hajłasz P., *Pointwise inequalities for Sobolev functions and some applications*. Studia Math. **106**(1) (1993), 77–92.

[6] Cruz-Uribe D., Fiorenza A. and Neugebauer C. J., *The maximal function on variable L^p spaces*. Ann. Acad. Sci. Fenn. Math. **28** (2003), 223–238; Ann. Acad. Sci. Fenn. Math. **29** (2004), 247–249.

[7] Diening L., Harjulehto P., Hästö P. and Růžička M., Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Mathematics, **2017**, Springer, Heidelberg, 2011.

[8] Feichtinger H., An elementary approach to Wiener's third Tauberian theorem on Euclidean n-space, Proceedings, Conference at Cortona 1984, Symposia Mathematica 29 (Academic Press, New York, 1987), 267–301.

[9] Fu Z., Lin Y. and Lu S., *λ-central BMO estimates for commutators of singular integral operators with rough kernels*. Acta Math. Sin. (Engl. Ser.) **24** (2008), 373–386.

[10] García-Cuerva J., *Hardy spaces and Beurling algebras*. J. London Math. Soc. **39** (1989), 499–513.

[11] García-Cuerva J. and Herrero M. J. L., *A theory of Hardy spaces assosiated to the Herz spaces*. Proc. London Math. Soc. **69** (1994), 605–628

[12] Herz C., *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms*. J. Math. Mech. **18** (1968), 283–324.

[13] Komori Y., *Notes on singular integrals on some inhomogeneous Herz spaces.*

Taiwanese J. Math. **8** (2004), 547–556.

[14] Lewis J. L., *On very weak solutions of certain elliptic systems.* Comm. Partial Differential Equations **18**(9) (10) (1993), 1515–1537.

[15] Matsuoka K. and Nakai E., Fractional integral operators on $B^{p,\lambda}$ with Morrey-Campanato norms, Function Spaces IX (Krakow, Poland, 2009), 249–264, Banach Center Publications, Vol. 92, Institute of Mathematics, Polish Academy of Sciences, Warszawa, 2011.

[16] Mizuta Y., Potential theory in Euclidean spaces, Gakkōtosho, Tokyo, 1996.

[17] Mizuta Y., *Integral representations, differentiability properties and limits at infinity for Beppo Levi functions.* Potential Analysis **6** (1997), 237–276.

[18] Mizuta Y., Nakai E., Ohno T. and Shimomura T., *Maximal functions, Riesz potentials and Sobolev embeddings on Musielak-Orlicz-Morrey spaces of variable exponent in* $\mathbf{R}^n$ Rev. Mat. Complut. **25** (2012), 413–434.

[19] Shimomura T. and Mizuta Y., *Taylor expansion of Riesz potentials.* Hiroshima Math. J. **25** (1995), 595–621.

[20] Stein E. M., Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, 1970.

[21] Wiener N., *Generalized harmonic analysis.* Acta Math. **55** (1930), 117–258.

[22] Wiener N., *Taubcrian theorems.* Ann. Math. **33** (1932), 1–100.

Yoshihiro Mizuta
Department of Mechanical Systems Engineering
Hiroshima Institute of Technology
2-1-1 Miyake Saeki-ku Hiroshima 731-5193, Japan
E-mail: y.mizuta.5x@it-hiroshima.ac.jp

Takao Ohno
Faculty of Education and Welfare Science
Oita University
Dannoharu Oita-city 870-1192, Japan
E-mail: t-ohno@oita-u.ac.jp

Tetsu Shimomura
Department of Mathematics
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739-8524, Japan
E-mail: tshimo@hiroshima-u.ac.jp