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A semi-group formula for the Riesz potentials

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Abstract. The purpose of this article is to establish a semi-group formula for the Riesz potentials of L^p-functions. As preparations, we study the Lizorkin space $\Phi(\mathbf{R}^n)$ and investigate integral estimates of the Riesz potentials of functions in the spaces $L^{p:r,s}(\mathbf{R}^n)$.

Key words: Riesz potentials, Lizorkin space, semi-group formula.

1. Introduction

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space. Throughout this paper let $0 < \alpha < \infty$ and $1 < p < \infty$. For real numbers r and s we define the spaces $L^{p:r,s}(\mathbf{R}^n)$ as follows:

$$L^{p:r,s}(\mathbf{R}^n) = \left\{ f : \|f\|_{p:r,s} = \left(\int_{\mathbf{R}^n} |f(x)|^p |x|^{rp} (1+|\log|x||)^{sp} dx \right)^{1/p} < \infty \right\}.$$

We simply write $L^{p:0,0}(\mathbf{R}^n) = L^p(\mathbf{R}^n)$ and $\|f\|_{p:0,0} = \|f\|_p$. Let $G_\alpha(x)$ be the Bessel kernel of order α defined by

$$G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2/\delta} e^{-\delta/(4\pi)} \delta^{(\alpha-n)/2} \frac{d\delta}{\delta}.$$

Since the Bessel kernel $G_\alpha(x)$ is integrable ([St, Proposition 2 in Chap. V]), for $f \in L^p(\mathbf{R}^n)$ the Bessel potential of order α of f

$$G_\alpha f(x) = \int G_\alpha(x-y) f(y) dy$$

belongs to $L^p(\mathbf{R}^n)$. For the Bessel potentials, it is known that the following semi-group formula holds ([St, 3.3 in Chap. V]):

$$G_{\alpha+\beta} f = G_\alpha(G_\beta f), \qquad f \in L^p(\mathbf{R}^n).$$

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The purpose of this article is to establish a semi-group formula for the Riesz potentials of L^p-functions. Let $\mathbf{N}$ be the set of nonnegative integers and $2\mathbf{N}$ stands for the set of nonnegative even numbers. The Riesz kernel $\kappa_\alpha(x)$ of order α is given by

$$\kappa_\alpha(x) = \frac{1}{\gamma_{\alpha,n}} \begin{cases} |x|^{\alpha-n}, & \alpha - n \notin 2\mathbf{N} \\ (\delta_{\alpha,n} - \log|x|)|x|^{\alpha-n}, & \alpha - n \in 2\mathbf{N} \end{cases}$$

with

$$\gamma_{\alpha,n} = \begin{cases} \pi^{n/2}2^\alpha\Gamma(\alpha/2)/\Gamma((n-\alpha)/2), & \alpha - n \notin 2\mathbf{N} \\ (-1)^{(\alpha-n)/2}2^{\alpha-1}\pi^{n/2}\Gamma(\alpha/2)((\alpha-n)/2)!, & \alpha - n \in 2\mathbf{N} \end{cases}$$

and

$$\delta_{\alpha,n} = \frac{\Gamma'(\alpha/2)}{2\Gamma(\alpha/2)} + \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{(\alpha-n)/2} - \mathcal{C}\right) - \log\pi$$

where $\mathcal{C}$ is Euler's constant. For a function f we define the Riesz potential $U_\alpha f$ of order α of f as follows:

$$U_\alpha f(x) = \int \kappa_\alpha(x-y)f(y)dy$$

if it exists. If $\alpha - (n/p) < 0$, then for $f \in L^p(\mathbf{R}^n)$, $U_\alpha f$ exists and satisfies the following inequality ([SW, Theorem B*]):

$$\|U_\alpha f\|_{p,-\alpha,0} \le C\|f\|_p.$$

However, if $\alpha - (n/p) \ge 0$, then for an L^p-function f, $U_\alpha f$ does not necessarily exist. To consider the Riesz potentials of L^p-functions we introduce the Riesz kernels of type (α, k). For an integer k we set

$$\kappa_{\alpha,k}(x,y) = \kappa_\alpha(x-y) - \sum_{|\gamma|\le k} \frac{x^\gamma}{\gamma!} D^\gamma \kappa_\alpha(-y)$$

where we regard the second term of the right-hand side as zero if $k \le -1$, and $\gamma = (\gamma_1, \dots, \gamma_n)$ is a multi-index, $x^\gamma = x_1^{\gamma_1} \dots x_n^{\gamma_n}$ $(x = (x_1, \dots, x_n))$,

$D^\gamma = D_1^{\gamma_1} \dots D_n^{\gamma_n}$ ($D_j = \partial/\partial x_j$), $\gamma! = \gamma_1! \dots \gamma_n!$ and $|\gamma| = \gamma_1 + \cdots + \gamma_n$. We also denote

$$p_{\alpha,k}(x,y) = -\sum_{|\gamma|\le k} \frac{x^\gamma}{\gamma!} D^\gamma \kappa_\alpha(-y).$$

For a function f we define the Riesz potential $U_{\alpha,k}f$ and the Riesz polynomial $P_{\alpha,k}f$ of type (α,k) of f as follows:

$$U_{\alpha,k}f(x) = \int \kappa_{\alpha,k}(x,y)f(y)dy, \qquad P_{\alpha,k}f(x) = \int p_{\alpha,k}(x,y)f(y)dy$$

if they exist. The Riesz polynomial $P_{\alpha,k}f$ is a polynomial of degree k if it exists.

Our plan is as follows. In Section 2 we introduce and study the Lizorkin space $\Phi(\mathbf{R}^n)$. The Lizorkin space $\Phi(\mathbf{R}^n)$ has been studied by several authors (cf. [Sa], [SKM]). It is known that $\Phi(\mathbf{R}^n)$ is invariant with respect to the Riesz potential oparator and a semi-group formula for the Riesz potentials of functions in $\Phi(\mathbf{R}^n)$ holds. We establish the fact that certain subspaces of $\Phi(\mathbf{R}^n)$ are dense in $L^p(\mathbf{R}^n)$ (Proposition 2.9). In Section 3 we give integral estimates for the Riesz potentials of type (α,k) of functions in the spaces $L^{p:r,s}(\mathbf{R}^n)$ (Theorem 3.2 and Corollary 3.8). In particular, it turns out that for a function $f \in L^{p:r,s}(\mathbf{R}^n)$, $U_{\alpha,k}f$ exists if $r > -n/p'$ and $\alpha + r - (n/p) \notin \mathbf{N}$ where k is the integral part of $\alpha + r - (n/p)$. In Section 4 we prove a semi-group formula for the Riesz potentials of L^p-functions (Theorem 4.4). Throughout this paper we use the symbol C for a generic positive constant whose value may be different at each occurrence.

2. The Lizorkin space $\Phi(\mathbf{R}^n)$

We denote the Schwartz space on $\mathbf{R}^n$ by $\mathcal{S}(\mathbf{R}^n)$. That is, $\mathcal{S}(\mathbf{R}^n)$ is the space of all C^∞-functions φ in $\mathbf{R}^n$ such that

$$q_{\gamma,\delta}(\varphi) = \sup_{x\in\mathbf{R}^n} |x^\gamma D^\delta \varphi(x)| < \infty$$

for all multi-indices γ and δ. The space $\mathcal{S}(\mathbf{R}^n)$ is a Fréchet space with a countable family of semi-norms $\{q_{\gamma,\delta}\}$. For a function $f \in \mathcal{S}(\mathbf{R}^n)$ the Riesz potential $U_\alpha f(x)$ exists for any $x \in \mathbf{R}^n$. Moreover in case $k < \alpha$, $U_{\alpha,k}f(x)$

and $P_{\alpha,k}f(x)$ exist for any $x \in \mathbf{R}^n$ and

$$U_{\alpha,k}f(x) = U_\alpha f(x) + P_{\alpha,k}f(x). \tag{2.1}$$

The Lizorkin space $\Phi(\mathbf{R}^n)$ is defined by

$$\Phi(\mathbf{R}^n) = \left\{\varphi \in \mathcal{S}(\mathbf{R}^n) : \int \varphi(x)x^\gamma dx = 0 \quad \text{for all } \gamma\right\}$$

([SKM, Section 25 in Chap. 5]). Further, we introduce the space $\Psi(\mathbf{R}^n)$ as follows:

$$\Psi(\mathbf{R}^n) = \{\psi \in \mathcal{S}(\mathbf{R}^n) : D^\gamma\psi(0) = 0 \quad \text{for all } \gamma\}.$$

The Fourier transform $\mathcal{F}f$ and the inverse Fourier transforms $\overline{\mathcal{F}}f$ of an integrable function f are defined by

$$\mathcal{F}f(x) = \int e^{-ix\cdot y}f(y)dy, \quad \overline{\mathcal{F}}f(x) = \int e^{ix\cdot y}f(y)dy = \mathcal{F}f(-x)$$

where $x \cdot y = x_1y_1 + \cdots + x_ny_n$. By the Fourier inversion formula, for $\varphi \in \mathcal{S}(\mathbf{R}^n)$ we have the equality

$$\overline{\mathcal{F}}\mathcal{F}\varphi = \mathcal{F}\overline{\mathcal{F}}\varphi = (2\pi)^n\varphi. \tag{2.2}$$

Noting that

$$D^\gamma(\mathcal{F}\varphi)(0) = \int \varphi(y)(-iy)^\gamma dy \tag{2.3}$$

and

$$\int \mathcal{F}\psi(y)(iy)^\gamma dy = (2\pi)^n D^\gamma\psi(0) \tag{2.4}$$

for $\varphi, \psi \in \mathcal{S}(\mathbf{R}^n)$, we see that

$$\Phi(\mathbf{R}^n) = \mathcal{F}(\Psi(\mathbf{R}^n)), \quad \Psi(\mathbf{R}^n) = \mathcal{F}(\Phi(\mathbf{R}^n)). \tag{2.5}$$

The symbol $\mathcal{S}'(\mathbf{R}^n)$ (the space of tempered distributions) stands for the topological dual space of $\mathcal{S}(\mathbf{R}^n)$. We use the notation $\langle u, \varphi\rangle$ for the canonical bilinear form on $\mathcal{S}'(\mathbf{R}^n) \times \mathcal{S}(\mathbf{R}^n)$. For $u \in \mathcal{S}'(\mathbf{R}^n)$ we define the Fourier transform $\mathcal{F}u$ (resp. the inverse Fourier transform $\overline{\mathcal{F}}u$) to be the element of $\mathcal{S}'(\mathbf{R}^n)$ whose value at $\varphi \in \mathcal{S}(\mathbf{R}^n)$ is $\langle \mathcal{F}u, \varphi\rangle = \langle u, \mathcal{F}\varphi\rangle$ (resp. $\langle \overline{\mathcal{F}}u, \varphi\rangle = \langle u, \overline{\mathcal{F}}\varphi\rangle$. The Fourier transform of the Riesz kernel $\kappa_\alpha \in \mathcal{S}'(\mathbf{R}^n)$ is given by

$$\mathcal{F}\kappa_\alpha(x) = (2\pi)^\alpha \mathrm{Pf.}|x|^{-\alpha} \tag{2.6}$$

where Pf. stands for the pseudo function ([Sc, Section 4 in Chap VII]). We note that for $\psi \in \Psi(\mathbf{R}^n)$

$$\langle \mathrm{Pf.}|x|^{-\alpha}, \psi\rangle = \int |x|^{-\alpha}\psi(x)dx. \tag{2.7}$$

The Lizorkin space $\Phi(\mathbf{R}^n)$ has the following properties.

Proposition 2.1 ([SKM, Theorem 25.1], [Sa, Theorem 2.16]) *For $\varphi \in \Phi(\mathbf{R}^n)$, $U_\alpha\varphi$ belongs to $\Phi(\mathbf{R}^n)$ and*

$$U_{\alpha+\beta}\varphi = U_\alpha(U_\beta\varphi).$$

Proposition 2.2 ([Sa, Theorem 2.7]) *The Lizorkin space $\Phi(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$.*

We establish that not only the space $\Phi(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$, but also certain subspaces of $\Phi(\mathbf{R}^n)$ are dense in $L^p(\mathbf{R}^n)$. For $\alpha > 0$ and a nonnegative integer k with $k < \alpha$, the space $\Phi_{\alpha,k}(\mathbf{R}^n)$ is defined by

$$\Phi_{\alpha,k}(\mathbf{R}^n) = \left\{\varphi \in \Phi(\mathbf{R}^n) : \int \varphi(x) D^\gamma \kappa_\alpha(x)dx = 0 \quad \text{for } |\gamma| \le k\right\}.$$

In the remainder of this section we prove that if $\alpha - (n/p) \notin \mathbf{N}$, then the space $\Phi_{\alpha,[\alpha-(n/p)]}(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$ where $[\alpha - (n/p)]$ is the integral part of $\alpha - (n/p)$.

To prove the above fact we prepare five lemmas and one remark.

The first lemma is proved by the similar way to [Ku, Lemma 2.2].

Lemma 2.3 *For a nonnegative integer k there exists a function $\theta(t) \in \Phi(R^1)$ such that*

$$D^i\theta(0) = \begin{cases} 1, & i = 0 \\ 0, & i = 1, \dots, k \end{cases} \tag{2.8}$$

where $D^i\theta$ is the derivative of order i of θ.

Lemma 2.4 *For a nonnegative integer k there exists a function $\zeta(x) \in \Phi(\mathbf{R}^n)$ such that*

$$D^\delta\zeta(0) = \begin{cases} 1, & \delta = 0 \\ 0, & 0 < |\delta| \le k. \end{cases} \tag{2.9}$$

Proof. By Lemma 2.3 there exists $\theta(t) \in \Phi(\mathbf{R}^1)$ which satisfies (2.8). We put $\zeta(x) = \theta(x_1)\dots\theta(x_n)$. It is clear that $\zeta \in \Phi(\mathbf{R}^n)$. Moreover we have

$$\zeta(0) = \theta(0)\dots\theta(0) = 1$$

and for $0 < |\delta| \le k$

$$D^\delta\zeta(0) = D^{\delta_1}\theta(0)\dots D^{\delta_n}\theta(0) = 0$$

because there exists i such that $\delta_i \neq 0$. Thus we obtain the lemma. □

Lemma 2.5 *For a nonnegative integer k there exist functions $\{\zeta_\gamma\}_{|\gamma|\le k} \subset \Phi(\mathbf{R}^n)$ such that*

$$D^\delta\zeta_\gamma(0) = \begin{cases} 1, & \delta = \gamma \\ 0, & \delta \neq \gamma \end{cases} \tag{2.10}$$

for $|\delta|, |\gamma| \le k$.

Proof. By Lemma 2.4 there exists a function $\zeta \in \Phi(\mathbf{R}^n)$ which satisfies (2.9). For $|\gamma| \le k$ we put

$$\zeta_\gamma(x) = \omega_\gamma(x)\zeta(x)$$

where $\omega_\gamma(x) = x^\gamma/\gamma!$. It is clear that $\zeta_\gamma \in \Phi(\mathbf{R}^n)$ for $|\gamma| \le k$. We prove (2.10). By Leipniz's formula we have

$$D^\delta\zeta_\gamma(x) = D^\delta(\omega_\gamma(x)\zeta(x)) = \sum_{\eta\le\delta} \binom{\delta}{\eta} D^\eta\omega_\gamma(x)D^{\delta-\eta}\zeta(x)$$

where

$$\binom{\delta}{\eta} = \binom{\delta_1}{\eta_1} \cdots \binom{\delta_n}{\eta_n} \quad \text{and} \quad \binom{\delta_i}{\eta_i} = \frac{\delta_i!}{\eta_i!(\delta_i - \eta_i)!}.$$

Since

$$D^\eta \omega_\gamma(x) = \begin{cases} \omega_{\gamma-\eta}(x), & \eta \le \gamma, \\ 0, & \text{otherwise}, \end{cases}$$

we see that

$$D^\delta \zeta_\gamma(0) = \sum_{\eta \le \min(\delta,\gamma)} \binom{\delta}{\eta} \omega_{\gamma-\eta}(0) D^{\delta-\eta}\zeta(0)$$

where $\min(\delta, \gamma) = (\min(\delta_1, \gamma_1), \ldots, \min(\delta_n, \gamma_n))$. In case of $\delta = \gamma$, by (2.9) and the fact that

$$\omega_\gamma(0) = \begin{cases} 1, & \gamma = 0 \\ 0, & \gamma \ne 0, \end{cases} \tag{2.11}$$

we have

$$D^\delta \zeta_\gamma(0) = D^\gamma \zeta_\gamma(0) = \sum_{\eta \le \gamma} \binom{\gamma}{\eta} \omega_{\gamma-\eta}(0) D^{\gamma-\eta}\zeta(0) = \binom{\gamma}{\gamma} \omega_0(0)\zeta(0) = 1.$$

Next, let $\delta \ne \gamma$. There are two cases. Firstly we consider the case that there exists i such that $\delta_i < \gamma_i$. For $\eta \le \min(\delta, \gamma)$ we obtain that $\eta_i \le \delta_i < \gamma_i$, and hence $\gamma - \eta > 0$. Therefore by (2.11) $\omega_{\gamma-\eta}(0) = 0$ for $\eta \le \min(\delta, \gamma)$, and hence $D^\delta \zeta_\gamma(0) = 0$. Secondly we consider the case that there exists i such that $\delta_i > \gamma_i$. For $\eta \le \min(\delta, \gamma)$ we obtain that $\eta_i \le \gamma_i < \delta_i$, and hence $\delta - \eta > 0$. Therefore by (2.9) $D^{\delta-\eta}\zeta(0) = 0$ for $\eta \le \min(\delta, \gamma)$, and hence $D^\delta \zeta_\gamma(0) = 0$. Consequently, we see that $D^\delta \zeta_\gamma(0) = 0$ for $\delta \ne \gamma$. Thus we obtain (2.10) and complete the proof of the lemma. □

Lemma 2.6 *For $\alpha > 0$ and a nonnegative integer k with $k < \alpha$, there exist functions $\{\mu_\gamma\}_{|\gamma| \le k} \subset \Phi(\mathbf{R}^n)$ such that*

$$\int \mu_\gamma(x) D^\delta \kappa_\alpha(x) dx = \begin{cases} 1, & \gamma = \delta \\ 0, & \gamma \ne \delta \end{cases}$$

for $|\gamma|, |\delta| \leq k$.

Proof. By the previous lemma there exist functions $\{\zeta_\gamma\}_{|\gamma|\leq k} \subset \Phi(\mathbf{R}^n)$ which satisfy (2.10). We put

$$\mu_\gamma(x) = \frac{(-1)^{|\gamma|}}{(2\pi)^{\alpha+n}} \overline{\mathcal{F}}(|\xi|^\alpha \mathcal{F}\zeta_\gamma(\xi))(x)$$

for $|\gamma| \leq k$. Since $\zeta_\gamma \in \Phi(\mathbf{R}^n)$, by (2.5) we see that $\mathcal{F}\zeta_\gamma(\xi) \in \Psi(\mathbf{R}^n)$, $|\xi|^\alpha \mathcal{F}\zeta_\gamma(\xi) \in \Psi(\mathbf{R}^n)$ and $\mu_\gamma(x) \in \Phi(\mathbf{R}^n)$. Since $\kappa_\alpha \in \mathcal{S}'(\mathbf{R}^n)$ and $\mu_\gamma \in \Phi(\mathbf{R}^n) \subset \mathcal{S}(\mathbf{R}^n)$, we can consider

$$I = \frac{1}{(2\pi)^n} \big\langle \overline{\mathcal{F}}(D^\delta \kappa_\alpha), \mathcal{F}\mu_\gamma \big\rangle.$$

By (2.2) we have

$$I = \frac{1}{(2\pi)^n} \big\langle D^\delta \kappa_\alpha, \overline{\mathcal{F}}\mathcal{F}\mu_\gamma \big\rangle = \big\langle D^\delta \kappa_\alpha, \mu_\gamma \big\rangle.$$

Moreover, since $D^\delta \kappa_\alpha(x)$ is locally integrable and $D^\delta \kappa_\alpha(x) \mu_\gamma(x)$ is integrable for $|\delta| < \alpha$, we have

$$I = \int D^\delta \kappa_\alpha(x) \mu_\gamma(x) dx \tag{2.12}$$

for $|\delta|, |\gamma| \leq k(< \alpha)$. On the other hand, since $\overline{\mathcal{F}}\kappa_\alpha(\xi) = (2\pi)^\alpha \mathrm{Pf.}|\xi|^{-\alpha}$ in $\mathcal{S}'(\mathbf{R}^n)$ by (2.6), we have

$$\begin{aligned}
I &= \frac{(2\pi)^\alpha (-i)^{|\delta|}}{(2\pi)^n} \big\langle \xi^\delta \mathrm{Pf.}|\xi|^{-\alpha}, \mathcal{F}\mu_\gamma \big\rangle \\
&= \frac{(2\pi)^\alpha (-i)^{|\delta|}}{(2\pi)^n} \left\langle \xi^\delta \mathrm{Pf.}|\xi|^{-\alpha}, \frac{(-1)^{|\gamma|}}{(2\pi)^{\alpha+n}} \mathcal{F}\overline{\mathcal{F}}(|\xi|^\alpha \mathcal{F}\zeta_\gamma(\xi)) \right\rangle \\
&= \frac{(-i)^{|\delta|}}{(2\pi)^n} \big\langle \xi^\delta \mathrm{Pf.}|\xi|^{-\alpha}, (-1)^{|\gamma|} |\xi|^\alpha \mathcal{F}\zeta_\gamma(\xi) \big\rangle \\
&= \frac{(-i)^{|\delta|}}{(2\pi)^n} \big\langle \mathrm{Pf.}|\xi|^{-\alpha}, (-1)^{|\gamma|} \xi^\delta |\xi|^\alpha \mathcal{F}\zeta_\gamma(\xi) \big\rangle.
\end{aligned}$$

Since $(-1)^{|\gamma|}\xi^\delta|\xi|^\alpha\mathcal{F}\zeta_\gamma(\xi) \in \Psi(\mathbf{R}^n)$, by (2.7) and (2.10) we obtain that

$$\begin{aligned} I &= \frac{(-i)^{|\delta|}}{(2\pi)^n}\int |\xi|^{-\alpha}(-1)^{|\gamma|}\xi^\delta|\xi|^\alpha\mathcal{F}\zeta_\gamma(\xi)d\xi \\ &= \frac{(-i)^{|\delta|}(-1)^{|\gamma|}}{(2\pi)^n i^{|\delta|}}\int (i\xi)^\delta\mathcal{F}\zeta_\gamma(\xi)d\xi \\ &= \frac{(-1)^{|\delta|+|\gamma|}}{(2\pi)^n}D^\delta(\overline{\mathcal{F}}\mathcal{F}\zeta_\gamma)(0) = (-1)^{|\delta|+|\gamma|}D^\delta\zeta_\gamma(0) \\ &= \begin{cases} 1, & \delta = \gamma \\ 0, & \delta \neq \gamma \end{cases} \end{aligned} \tag{2.13}$$

for $|\delta|, |\gamma| \leq k$. By (2.12) and (2.13) we get

$$\int \mu_\gamma(x)D^\delta\kappa_\alpha(x)dx = \begin{cases} 1, & \delta = \gamma \\ 0, & \delta \neq \gamma \end{cases}$$

for $|\delta|, |\gamma| \leq k$. Thus we obtain the lemma. □

Here, we remark the following fact.

Remark 2.7 Let $H(x) = |x|^{2\ell}\log|x|$ where ℓ is a nonnegative integer. Then

$$D^\delta H(x) = \begin{cases} P(x)\log|x| + Q(x), & |\delta| \leq 2\ell \\ Q(x), & |\delta| \geq 2\ell + 1 \end{cases}$$

where $P(x)$ is a homogeneous polynomial of degree $2\ell - |\delta|$ and $Q(x)$ is a homogeneous function of degree $2\ell - |\delta|$.

Lemma 2.8 *Let $\alpha - (n/p) > 0$ and $\alpha - (n/p) \notin \mathbf{N}$. Then there exist functions $\{\mu_{\gamma.m}\}_{|\gamma|\leq[\alpha-(n/p)],m=1,2,\ldots} \subset \Phi(\mathbf{R}^n)$ such that*

(i) *for $|\gamma|, |\delta| \leq [\alpha - (n/p)]$*

$$\int \mu_{\gamma,m}(x)D^\delta\kappa_\alpha(x)dx = \begin{cases} 1, & \gamma = \delta \\ 0, & \gamma \neq \delta \end{cases}$$

and

(ii) *for* $|\gamma| \le [\alpha - (n/p)]$

$$\|\mu_{\gamma,m}\|_p \to 0 \quad (m \to \infty).$$

Proof. Since $[\alpha - (n/p)] < \alpha$, by Lemma 2.6 there exist functions $\{\mu_\gamma\}_{|\gamma| \le [\alpha-(n/p)]} \subset \Phi(\mathbf{R}^n)$ such that

$$\int \mu_\gamma(x) D^\delta \kappa_\alpha(x) dx = \begin{cases} 1, & \gamma = \delta \\ 0, & \gamma \ne \delta \end{cases} \tag{2.14}$$

for $|\gamma|, |\delta| \le [\alpha - (n/p)]$. We put

$$\mu_{\gamma,m}(x) = \frac{1}{m^{\alpha-|\gamma|}} \mu_\gamma\left(\frac{x}{m}\right)$$

for $|\gamma| \le [\alpha - (n/p)]$ and $m = 1, 2, \ldots$. It is clear that $\mu_{\gamma,m} \in \Phi(\mathbf{R}^n)$. First we consider the case that $\alpha - n$ is not a nonnegative even number. In this case $D^\delta \kappa_\alpha(x)$ is a homogeneous function of degree $\alpha - n - |\delta|$. Hence for $|\gamma|, |\delta| \le [\alpha - (n/p)]$, by the change of variables we have

$$\begin{aligned} &\int \mu_{\gamma,m}(x) D^\delta \kappa_\alpha(x) dx \\ &\quad = \frac{1}{m^{\alpha-|\gamma|}} \int \mu_\gamma\left(\frac{x}{m}\right) D^\delta \kappa_\alpha(x) dx = m^{|\gamma|-\alpha+n} \int \mu_\gamma(y) D^\delta \kappa_\alpha(my) dy \\ &\quad = m^{|\gamma|-|\delta|} \int \mu_\gamma(y) D^\delta \kappa_\alpha(y) dy = \begin{cases} 1. & \gamma = \delta \\ 0, & \gamma \ne \delta \end{cases} \end{aligned}$$

on account of (2.14). Next we consider the case that $\alpha - n$ is a nonnegative even number. In this case, since $[\alpha - (n/p)] \le \alpha - n$, by Remark 2.7 for $|\delta| \le [\alpha - (n/p)]$

$$D^\delta \kappa_\alpha(x) = P(x) \log |x| + Q(x)$$

where $P(x)$ is a homogeneous polynomial of degree $\alpha - n - |\delta|$ and $Q(x)$ is a homogeneous function of degree $\alpha - n - |\delta|$. Hence for $|\gamma|, |\delta| \le [\alpha - (n/p)]$, we have

$$
\begin{aligned}
&\int \mu_{\gamma,m}(x) D^\delta \kappa_\alpha(x)dx \\
&= \frac{1}{m^{\alpha-|\gamma|}} \int \mu_\gamma\left(\frac{x}{m}\right) D^\delta \kappa_\alpha(x)dx \\
&= m^{|\gamma|-\alpha+n} \int \mu_\gamma(y) D^\delta \kappa_\alpha(my)dy \\
&= m^{|\gamma|-\alpha+n} \int \mu_\gamma(y)(P(my)\log(m|y|) + Q(my))dy \\
&= m^{|\gamma|-\alpha+n} \int (\mu_\gamma(y)(m^{\alpha-n-|\delta|}P(y)(\log m + \log|y|) + m^{\alpha-n-|\delta|}Q(y))dy \\
&= m^{|\gamma|-|\delta|}\left(\int \mu_\gamma(y)(P(y)\log|y| + Q(y))dy + \log m \int \mu_\gamma(y)P(y)dy\right) \\
&= m^{|\gamma|-|\delta|} \int \mu_\gamma(y) D^\delta \kappa_\alpha(y)dy
\end{aligned}
$$

because $P(y)$ is a polynomial and $\mu_\gamma \in \Phi(\mathbf{R}^n)$. Therefore, by (2.14)

$$
\int \mu_{\gamma,m}(x) D^\delta \kappa_\alpha(x)dx = \begin{cases} 1, & \gamma = \delta \\ 0, & \gamma \neq \delta \end{cases}
$$

for $|\gamma|, |\delta| \leq [\alpha - (n/p)]$. Thus we obtain (i). Further, by the change of variables we have

$$
\begin{aligned}
\|\mu_{\gamma,m}\|_p = \left(\int |\mu_{\gamma,m}(x)|^p dx\right)^{1/p} &= \left(\int \frac{1}{m^{(\alpha-|\gamma|)p}}\left|\mu_\gamma\left(\frac{x}{m}\right)\right|^p dx\right)^{1/p} \\
&= m^{(n/p)-\alpha+|\gamma|}\left(\int |\mu_\gamma(y)|^p dy\right)^{1/p}.
\end{aligned}
$$

Since $\alpha-(n/p) \notin N$, the condition $|\gamma| \leq [\alpha-(n/p)]$ implies $(n/p)-\alpha+|\gamma| < 0$, and hence $\|\mu_{\gamma,m}\|_p \to 0$ $(m \to \infty)$ for $|\gamma| \leq [\alpha - (n/p)]$. This shows (ii). Thus we complete the proof of the lemma. □

Now we prove the denseness of $\Phi_{\alpha,[\alpha-(n/p)]}(\mathbf{R}^n)$ in $L^p(\mathbf{R}^n)$.

Proposition 2.9 *Let $\alpha - (n/p) \notin \mathbf{N}$. Then the space $\Phi_{\alpha,[\alpha-(n/p)]}(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$.*

Proof. Since $\Phi(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$ by Proposition 2.2, it is sufficient to show that $\Phi_{\alpha,[\alpha-(n/p)]}(\mathbf{R}^n)$ is dense in $\Phi(\mathbf{R}^n)$ with respect to $L^p(\mathbf{R}^n)$-norm. In case $\alpha - (n/p) < 0$, $\Phi_{\alpha,[\alpha-(n/p)]}(\mathbf{R}^n) = \Phi(\mathbf{R}^n)$, and hence the assertion is obvious. Let $\alpha - (n/p) > 0$. Then there exist functions $\{\mu_{\gamma,m}\}_{|\gamma|\le[\alpha-(n/p)],m=1,2,\ldots} \subset \Phi(\mathbf{R}^n)$ which satisfy (i) and (ii) in Lemma 2.8. For $\varphi \in \Phi(\mathbf{R}^n)$ we put

$$0\varphi_m(x) = \varphi(x) - \sum_{|\delta|\le[\alpha-(n/p)]} \left(\int \varphi(y) D^\delta \kappa_\alpha(y) dy \right) \mu_{\delta,m}(x).$$

It is clear that $\varphi_m \in \Phi(\mathbf{R}^n)$. Moreover for $|\gamma| \le [\alpha-(n/p)]$, by (i) in Lemma 2.8 we have

$$\begin{aligned}
&\int \varphi_m(x) D^\gamma \kappa_\alpha(x) dx \\
&\quad = \int \varphi(x) D^\gamma \kappa_\alpha(x) dx \\
&\qquad - \sum_{|\delta|\le[\alpha-(n/p)]} \left(\int \varphi(y) D^\delta \kappa_\alpha(y) dy \right) \left(\int \mu_{\delta,m}(x) D^\gamma \kappa_\alpha(x) dx \right) \\
&\quad = \int \varphi(x) D^\gamma \kappa_\alpha(x) dx - \int \varphi(y) D^\gamma \kappa_\alpha(y) dx = 0.
\end{aligned}$$

Hence $\varphi_m \in \Phi_{\alpha,[\alpha-(n/p)]}(\mathbf{R}^n)$. Further, by (ii) in Lemma 2.8 we obtain

$$\|\varphi_m - \varphi\|_p \le \sum_{|\delta|\le[\alpha-(n/p)]} \left| \int \varphi(y) D^\delta \kappa_\alpha(y) dy \right| \|\mu_{\delta.m}\|_p \to 0 \quad (m \to \infty).$$

Namely, φ_m converges to φ with respect to $L^p(\mathbf{R}^n)$-norm as $m \to \infty$. Thus $\Phi_{\alpha.[\alpha-(n/p)]}(\mathbf{R}^n)$ is dense in $\Phi(\mathbf{R}^n)$ with respect to $L^p(\mathbf{R}^n)$-norm. This completes the proof of Proposition 2.9. □

3. Riesz potentials on the spaces $L^{p:r,s}(\mathbf{R}^n)$

As defined in section 1, for $p > 1$ and $r, s \in \mathbf{R}$ the spaces $L^{p:r,s}(\mathbf{R}^n)$ are given by

$$L^{p:r,s}(\mathbf{R}^n) = \left\{ f : \|f\|_{p:r,s} = \left(\int_{\mathbf{R}^n} |f(x)||x|^{rp}(1+|\log|x||)^{sp}dx \right)^{1/p} < \infty \right\}.$$

In this section we investigate integral estimates for Riesz potentials of functions in $L^{p:r,s}(R^n)$. To do so we introduce a kernel $K_{\alpha,\ell}(x) = K_\alpha(x)(1+\log|x|)^\ell$ ($\alpha > 0$, $\ell \in \mathbf{N}$) where $K_\alpha(x)$ is a homogeneous function of degree $\alpha - n$ which is infinitely differentiable in $\mathbf{R}^n - \{0\}$. For multi-index γ we see that

$$D^\gamma K_{\alpha,\ell}(x) = \sum_{j=0}^{\min(|\gamma|,\ell)} H_{\gamma,j}(x)(1+\log|x|)^{\ell-j}$$

where $H_{\gamma.j}(x)$ is a homogeneous function of degree $\alpha - n - |\gamma|$. Hence

$$|D^\gamma K_{\alpha,\ell}(x)| \le C|x|^{\alpha-n-|\gamma|}(1+|\log|x||)^\ell. \tag{3.1}$$

Further, for an integer k we set

$$K_{\alpha,\ell:k}(x,y) = K_{\alpha,\ell}(x-y) - \sum_{|\gamma|\le k} \frac{x^\gamma}{\gamma!} D^\gamma K_{\alpha,\ell}(-y)$$

where we regard the second term of the right-hand side as zero if $k \le -1$. For $x \in \mathbf{R}^n$ we put $\ell_x = \{tx : 0 \le t \le 1\}$ and denote by $d(y,\ell_x)$ the distance between y and ℓ_x.

Lemma 3.1 *Let k be a nonnegative integer. Then for $d(y,\ell_x) > |x|/2$*

$$|K_{\alpha,\ell:k}(x,y)| \le C|x|^{k+1}|y|^{\alpha-n-k-1}(1+|\log|y||)^\ell.$$

Proof. Let $x = 0$. Then $d(y,\ell_x) > |x|/2$ means $y \ne 0$. For $y \ne 0$ we see that

$$K_{\alpha,\ell:k}(0,y) = K_{\alpha,\ell}(-y) - K_{\alpha,\ell}(-y) = 0,$$

and the right-hand side of the required inequality is zero. Hence the lemma holds. Let $x \ne 0$. We note that $K_{\alpha,\ell}(z-y)$ is a C^∞-function as a function of z in $\mathbf{R}^n - \{y\}$. Therefore, for $d(y,\ell_x) > |x|/2$, $K_{\alpha,\ell}(z-y)$ is a C^∞-function as a function of z in the open set $U_x = \{z : d(z,\ell_x) < |x|/2\}$. Noting that

$\ell_x \subset U_x$ for $z \in U_x$ and U_x contains 0, we apply the integral remainder formula for Taylor' theorem to $K_{\alpha,\ell}(z-y)$ in U_x. Then we get

$$K_{\alpha,\ell}(z-y) = \sum_{|\gamma|\le k} \frac{z^\gamma}{\gamma!} D^\gamma K_{\alpha,\ell}(-y) + (k+1) \times \sum_{|\gamma|=k+1} \int_0^{|z|} \frac{(|z|-t)^k}{\gamma!} (z')^\gamma D^\gamma K_{\alpha,\ell}(tz'-y)dt$$

for $z \in U_x$ where $z' = z/|z|$ $(z \neq 0)$ and $0' = 0$. In particular, since x belongs to U_x, we have

$$K_{\alpha,\ell:k}(x,y) = (k+1) \sum_{|\gamma|=k+1} \int_0^{|x|} \frac{(|x|-t)^k}{\gamma!} (x')^\gamma D^\gamma K_{\alpha,\ell}(tx'-y)dt.$$

We also note that $d(y,\ell_x) > |x|/2$ implies that $|y|/3 < |tx'-y| < 3|y|$ for $0 \le t \le |x|$. Therefore by (3.1), for $d(y,\ell_x) > |x|/2$

$$\begin{aligned} &|K_{\alpha,\ell:k}(x,y)| \\ &\le (k+1) \sum_{|\gamma|=k+1} \int_0^{|x|} \frac{(|x|-t)^k}{\gamma!} |D^\gamma K_{\alpha,\ell}(tx'-y)|dt \\ &\le C(k+1) \sum_{|\gamma|=k+1} \int_0^{|x|} \frac{(|x|-t)^k}{\gamma!} |tx'-y|^{\alpha-n-|\gamma|}(1+|\log|tx'-y||)^\ell dt \\ &\le C \sum_{|\gamma|=k+1} |y|^{\alpha-n-|\gamma|}(1+|\log|y||)^\ell \int_0^{|x|} (|x|-t)^k dt \\ &= C|x|^{k+1}|y|^{\alpha-n-k-1}(1+|\log|y||)^\ell. \end{aligned}$$

Thus we obtain the lemma. □

For a function f we set

$$K_{\alpha,\ell:k}f(x) = \int K_{\alpha,\ell:k}(x,y)f(y)dy.$$

The main purpose of this section is to prove the following integral estimate.

Let $(1/p) + (1/p') = 1$.

Theorem 3.2 *Let $\alpha > 0$, $p > 1$, $r > -n/p'$, $\ell \in \mathbf{N}$, $s \in \mathbf{R}$ and $\alpha + r - (n/p) \notin \mathbf{N}$. Then for $k = [\alpha + r - (n/p)]$*

$$\|K_{\alpha,\ell:k} f\|_{p:-r-\alpha,s-\ell} \leq C\|f\|_{p:-r,s}.$$

For $k, \ell \in \mathbf{N}$ and $r, s \in \mathbf{R}$ we set

$$K^{r,s}_{\alpha,\ell:k}(x,y) = |x|^{-\alpha-r}(1+|\log|x||)^{s-\ell} K_{\alpha,\ell:k}(x,y)|y|^r(1+|\log|y||)^{-s}$$

and

$$K^{r,s}_{\alpha,\ell:k} f(x) = \int K^{r,s}_{\alpha,\ell:k}(x,y)f(y)dy.$$

Obviously, in order to prove Theorem 3.2 it is sufficient to show the following proposition.

Proposition 3.3 *Let $\alpha > 0$, $p > 1$, $r > -n/p'$, $\ell \in \mathbf{N}$, $s \in \mathbf{R}$ and $\alpha + r - (n/p) \notin \mathbf{N}$. Then for $k = [\alpha + r - (n/p)]$*

$$\|K^{r,s}_{\alpha,\ell:k} f\|_p \leq C\|f\|_p.$$

To show Proposition 3.3 we prepare four lemmas. The first lemma is a special case of the inequality by G. O. Okikiolu [Ok, Theorem 2.1].

Lemma 3.4 *Let $K(x,y)$ be a nonnegative measurable function on $\mathbf{R}^n \times \mathbf{R}^n$. Suppose that there are a measurable function $\varphi(x) > 0$ on $\mathbf{R}^n$ and constants $M_1 > 0$, $M_2 > 0$ such that*

$$\int \varphi(y)^{p'} K(x.y)dy \leq M_1^{p'} \varphi(x)^{p'} \tag{3.2}$$

$$\int \varphi(x)^{p} K(x.y)dx \leq M_2^{p} \varphi(y)^{p}. \tag{3.3}$$

If the operator K is defined by

$$Kf(x) = \int K(x,y)f(y)dy,$$

then

$$\|Kf\|_p \leq M_1 M_2 \|f\|_p.$$

Lemma 3.5 *Let $\alpha > 0$, $\ell \in \mathbf{N}$, $r > -n/p'$ and $s \in \mathbf{R}$. Then*

$$\Big(\int \Big| \int_{|x-y|\leq 3|x|/2} |x|^{-\alpha-r}(1+|\log|x||)^{s-\ell}|x-y|^{\alpha-n}(1+|\log|x-y||)^{\ell}|y|^{r} \times (1+|\log|y||)^{-s} f(y)dy \Big|^p dx \Big)^{1/p} \leq C\|f\|_p.$$

Proof. Let

$$K(x,y) = \begin{cases} |x|^{-\alpha-r}(1+|\log|x||)^{s-\ell}|x-y|^{\alpha-n} \\ \quad \times (1+|\log|x-y||)^{\ell}|y|^r(1+|\log|y||)^{-s}, & |x-y| \leq 3|x|/2 \\ 0, & |x-y| > 3|x|/2. \end{cases}$$

The condition $r > -n/p'$ implies that $-((r+n)/p') < r/p$, and hence we can take a number a such that $-((r+n)/p') < a < r/p$. For the above $K(x,y)$ and $\varphi(x) = |x|^a(1+|\log|x||)^b$ with $b > \max(s/p', (\ell-s)/p)$ we prove (3.2) and (3.3). First we have

$$\begin{aligned} I(x) &= \int \varphi(y)^{p'} K(x,y)dy \\ &= \int_{|x-y|\leq 3|x|/2} |y|^{ap'}(1+|\log|y||)^{bp'}|x|^{-\alpha-r}(1+|\log|x||)^{s-\ell}|x-y|^{\alpha-n} \\ &\quad \times (1+|\log|x-y||)^{\ell}|y|^r(1+|\log|y||)^{-s}dy \\ &= \int_{|x'-(y/|x|)|\leq 3/2} |y|^{ap'+r}(1+|\log|y||)^{bp'-s}|x|^{-\alpha-r}(1+|\log|x||)^{s-\ell} \\ &\quad |x|^{\alpha-n}\left|x'-\frac{y}{|x|}\right|^{\alpha-n}\left(1+\left|\log\left(|x|\left|x'-\frac{y}{|x|}\right|\right)\right|\right)^{\ell} dy. \end{aligned}$$

By putting $z = y/|x|$, we obtain

$$I(x) = \int_{|x'-z|\le 3/2} |x|^{ap'+r}|z|^{ap'+r}(1+|\log(|x||z|)|)^{bp'-s}|x|^{-r-n}$$
$$\times (1+|\log|x||)^{s-\ell}|x'-z|^{\alpha-n}(1+|\log(|x||x'-z|)|)^{\ell}|x|^n dz.$$

Noting that $1+|\log(uv)| \le (1+|\log u|)(1+|\log v|)$ for $u, v > 0$ and $bp'-s > 0$, we get

$$I(x) = |x|^{ap'}(1+|\log|x||)^{bp'} \int_{|x'-z|\le 3/2} |z|^{ap'+r}(1+|\log|z||)^{bp'-s}$$
$$\times |x'-z|^{\alpha-n}(1+|\log|x'-z||)^{\ell} dz.$$

Since $\alpha > 0$ and $a > -((r+n)/p')$, the integral

$$\int_{|x'-z|\le 3/2} |z|^{ap'+r}(1+|\log|z||)^{bp'-s}|x'-z|^{\alpha-n}(1+|\log|x'-z||)^{\ell} dz$$

exists and is a constant. Hence

$$I(x) \le C|x|^{ap'}(1+|\log|x||)^{bp'} = C\varphi(x)^{p'}.$$

Next we have

$$\begin{aligned} J(y) &= \int \varphi(x)^p K(x,y)dx \\ &= \int_{|x|\ge 2|x-y|/3} |x|^{ap}(1+|\log|x||)^{bp}|x|^{-\alpha-r}(1+|\log|x||)^{s-\ell}|x-y|^{\alpha-n} \\ &\quad \times (1+|\log|x-y||)^{\ell}|y|^r(1+|\log|y||)^{-s}dx \\ &= \int_{|x/|y||\ge 2|(x/|y|)-y'|/3} |x|^{ap-\alpha-r}(1+|\log|x||)^{bp+s-\ell}|y|^{\alpha-n}\left|\frac{x}{|y|}-y'\right|^{\alpha-n} \\ &\quad \times \left(1+\left|\log\left(|y|\left|\frac{x}{|y|}-y'\right|\right)\right|\right)^{\ell}|y|^r(1+|\log|y||)^{-s}dx. \end{aligned}$$

By putting $w = x/|y|$, we get

$$J(y) = \int_{|w|\geq 2|w-y'|/3} |y|^{ap-\alpha-r}|w|^{ap-\alpha-r}(1+|\log(|w||y|)|)^{bp+s-\ell}|y|^{\alpha-n+r}$$
$$\times |w-y'|^{\alpha-n}(1+|\log(|y||w-y'|)|)^{\ell}(1+|\log|y||)^{-s}|y|^n dw.$$

Noting that $bp+s-\ell>0$, we have

$$J(y) \leq |y|^{ap}(1+|\log|y||)^{bp}\int_{|w|\geq 2|w-y'|/3} |w|^{ap-r-\alpha}(1+|\log|w||)^{bp+s-\ell}$$
$$\times |w-y'|^{\alpha-n}(1+|\log|w-y'||)^{\ell}dw.$$

Since $\alpha>0$ and $a<r/p$, the integral

$$\int_{|w|\geq 2|w-y'|/3} |w|^{ap-r-\alpha}(1+|\log|w||)^{bp+s-\ell}|w-y'|^{\alpha-n}(1+|\log|w-y'||)^{\ell}dw$$

exists and is a constant. Hence

$$J(y) \leq C|y|^{ap}(1+|\log|y||)^{bp} = C\varphi(y)^p.$$

Thus we obtain (3.2) and (3.3). This proves the lemma by Lemma 3.4. □

Lemma 3.6 *Let $t-(n/p)>0$ and $u\in\mathbf{R}$. Then*

$$\left(\int\left|\int_{|y|\leq 2|x|} |x|^{-t}(1+|\log|x||)^{-u}|y|^{t-n}(1+|\log|y||)^{u}f(y)dy\right|^p dx\right)^{1/p}$$
$$\leq C\|f\|_p.$$

Proof. Let

$$K(x,y) = \begin{cases} |x|^{-t}(1+|\log|x||)^{-u}|y|^{t-n}(1+|\log|y||)^{u}, & |y|\leq 2|x| \\ 0, & |y|>2|x|. \end{cases}$$

For the above $K(x,y)$ and $\varphi(x) = |x|^{-n/(pp')}(1+|\log|x||)^b$ with $b > \max(-u/p', u/p)$ we prove (3.2) and (3.3). First, by $bp'+u>0$ we have

$$
\begin{aligned}
I(x) &= \int \varphi(y)^{p'} K(x,y)dy \\
&= \int_{|y|\le 2|x|} |y|^{-n/p}(1+|\log|y||)^{bp'}|x|^{-t}(1+|\log|x||)^{-u}|y|^{t-n} \\
&\quad \times (1+|\log|y||)^{u}dy \\
&= |x|^{-t}(1+|\log|x||)^{-u}\int_{|y/|x||\le 2} |x|^{t-(n/p)-n}\left|\frac{y}{|x|}\right|^{t-(n/p)-n} \\
&\quad \times \left(1+\left|\log\left(|x|\left|\frac{y}{|x|}\right|\right)\right|\right)^{bp'+u}dy \\
&\le |x|^{-(n/p)-n}(1+|\log|x||)^{bp'}\int_{|y/|x||\le 2}\left|\frac{y}{|x|}\right|^{t-(n/p)-n} \\
&\quad \times \left(1+\left|\log\left|\frac{y}{|x|}\right|\right|\right)^{bp'+u}dy.
\end{aligned}
$$

By putting $z = y/|x|$ we get

$$
\begin{aligned}
I(x) &\le |x|^{-n/p}(1+|\log|x||)^{bp'}\int_{|z|\le 2}|z|^{t-(n/p)-n}(1+|\log|z||)^{bp'+u}dz \\
&= C|x|^{-n/p}(1+(|\log|x||)^{bp'} = C\varphi(x)^{p'}
\end{aligned}
$$

because of $t-(n/p)>0$. Next, by $bp-u>0$ we have

$$
\begin{aligned}
J(y) &= \int \varphi(x)^{p} K(x,y)dx \\
&= \int_{|x|\ge |y|/2} |x|^{-n/p'}(1+|\log|x||)^{bp}|x|^{-t}(1+|\log|x||)^{-u}|y|^{t-n} \\
&\quad \times (1+|\log|y||)^{u}dx \\
&= |y|^{t-n}(1+|\log|y|)^{u}\int_{|x/|y||\ge 1/2}|y|^{-t-(n/p')}\left|\frac{x}{|y|}\right|^{-t-(n/p')} \\
&\quad \times \left(1+\left|\log\left(|y|\left|\frac{x}{|y|}\right|\right)\right|\right)^{bp-u}dx
\end{aligned}
$$

$$\le |y|^{-(n/p')-n}(1+|\log|y||)^{bp}\int_{|x/|y||\ge 1/2}\left|\frac{x}{|y|}\right|^{-t-(n/p')} \times \left(1+\left|\log\left|\frac{x}{|y|}\right|\right|\right)^{bp-u}dx.$$

By putting $w = x/|y|$ we obtain

$$J(y) \le |y|^{-n/p'}(1+|\log|y||)^{bp}\int_{|w|\ge 1/2}|w|^{-t-(n/p')}(1+|\log|w||)^{bp-u}dw$$
$$= C|y|^{-n/p'}(1+|\log|y||)^{bp} = C\varphi(y)^p$$

because $t-(n/p)>0$ implies $-t-(n/p')<-n$. Thus we obtain (3.2) and (3.3). Therefore the lemma is proved by Lemma 3.4. □

Lemma 3.7 *Let $t-(n/p)<0$ and $u\in\mathbf{R}$. Then*

$$\left(\int\left|\int_{|y|\ge|x|/2}|x|^{-t}(1+|\log|x||)^{-u}|y|^{t-n}(1+|\log|y||)^{u}f(y)dy\right|^p dx\right)^{1/p} \le C\|f\|_p.$$

Proof. We denote the left-hand side by I. Since the Jacobian of the change of variables $y = z/|z|^2$ is $1/|z|^{2n}$, by the change of variables and putting $g(z) = |z|^{-2n/p}f(z/|z|^2)$ we have

$$I = \left(\int\left|\int_{|z|\le 2/|x|}|x|^{-t}(1+|\log|x||)^{-u}|z|^{n-t}\left(1+\left|\log\frac{1}{|z|}\right|\right)^{u} \times |z|^{2n/p}|z|^{-2n/p}f\left(\frac{z}{|z|^2}\right)\frac{1}{|z|^{2n}}dz\right|^p dx\right)^{1/p}$$
$$= \left(\int\left|\int_{|z|\le 2/|x|}|x|^{-t}(1+|\log|x||)^{-u}|z|^{-n-t+(2n/p)} \times (1+|\log|z||)^{u}g(z)dz\right|^p dx\right)^{1/p}.$$

Again by using the change of variables $x = w/|w|^2$ we get

$$I = \left(\int \left| \int_{|z| \le 2|w|} |w|^t \left(1 + |\log \frac{1}{|w|}| \right)^{-u} |z|^{-n-t+(2n/p)} \right. \right.$$
$$\left. \left. \times (1 + |\log |z||)^u g(z) dz \right|^p \frac{dw}{|w|^{2n}} \right)^{1/p}$$
$$= \left(\int \left| \int_{|z| \le 2|w|} |w|^{t-(2n/p)} (1 + |\log |w||)^{-u} |z|^{-t+(2n/p)-n} \right. \right.$$
$$\left. \left. \times (1 + |\log |z||)^u g(z) dz \right|^p dw \right)^{1/p}.$$

By putting $v = -t + (2n/p)$, we see that

$$I = \left(\int \left| \int_{|z| \le 2|w|} |w|^{-v} (1+|\log |w||)^{-u} |z|^{v-n} (1+|\log |z||)^u g(z) dz \right|^p dw \right)^{1/p}.$$

Since $t - (n/p) < 0$ implies $v - (n/p) > 0$, Lemma 3.6 gives $I \le C\|g\|_p$. Noting that $\|g\|_p = \|f\|_p$, we obtain the lemma. $\square$

Now we are in a position to prove Proposition 3.3.

Proof of Proposition 3.3. We put $I = \|K^{r,s}_{\alpha,\ell:k} f\|_p$. In case of $\alpha + r - (n/p) > 0$ we have

$$I = \left(\int \left| \int_{d(y,\ell_x) \le |x|/2} |x|^{-\alpha-r} (1 + |\log |x||)^{s-\ell} K_{\alpha,\ell:k}(x,y) |y|^r \right. \right.$$
$$\times (1 + |\log |y||)^{-s} f(y) dy$$
$$+ \int_{d(y,\ell_x) > |x|/2} |x|^{-\alpha-r} (1 + |\log |x||)^{s-\ell} K_{\alpha,\ell:k}(x,y) |y|^r$$
$$\left. \left. \times (1 + |\log |y||)^{-s} f(y) dy \right|^p dx \right)^{1/p}$$
$$\le \left(\int \left(\int_{d(y,\ell_x) \le |x|/2} |x|^{-\alpha-r} (1 + |\log |x||)^{s-\ell} \right. \right.$$
$$\times \left| K_{\alpha,\ell}(x-y) - \sum_{|\gamma| \le k} \frac{x^\gamma}{\gamma!} D^\gamma K_{\alpha,\ell}(-y) \right|$$

$$\times |y|^r(1+|\log|y||)^{-s}|f(y)|dy\bigg)^p dx\bigg)^{1/p}$$

$$+\bigg(\int\bigg(\int_{d(y,\ell_x)>|x|/2}|x|^{-\alpha-r}(1+|\log|x||)^{s-\ell}|K_{\alpha,\ell:k}(x,y)||y|^r$$

$$\times(1+|\log|y||)^{-s}|f(y)|dy\bigg)^p dx\bigg)^{1/p}$$

$$= I_1 + I_2.$$

Since $d(y,\ell_x) \le |x|/2$ implies that $|x-y| \le 3|x|/2$ and $|y| \le 3|x|/2$, by (3.1) we see that

$$I_1 \le \bigg(\int\bigg(\int_{|x-y|\le 3|x|/2}|x|^{-\alpha-r}(1+|\log|x||)^{s-\ell}|K_{\alpha,\ell}(x-y)||y|^r$$

$$\times(1+|\log|y||)^{-s}|f(y)|dy\bigg)^p dx\bigg)^{1/p}$$

$$+\sum_{|\gamma|\le k}\frac{1}{\gamma!}\bigg(\int\bigg(\int_{|y|\le 3|x|/2}|x|^{-\alpha-r+|\gamma|}(1+|\log|x||)^{s-\ell}|D^\gamma K_{\alpha,\ell}(-y)|$$

$$\times|y|^r(1+|\log|y||)^{-s}|f(y)|dy\bigg)^p dx\bigg)^{1/p}$$

$$\le C\bigg(\int\bigg(\int_{|x-y|\le 3|x|/2}|x|^{-\alpha-r}(1+|\log|x||)^{s-\ell}|x-y|^{\alpha-n}$$

$$\times(1+|\log|x-y||)^{\ell}|y|^r(1+|\log|y||)^{-s}|f(y)|dy\bigg)^p dx\bigg)^{1/p}$$

$$+C\sum_{|\gamma|\le k}\bigg(\int\bigg(\int_{|y|\le 3|x|/2}|x|^{-(\alpha+r-|\gamma|)}(1+|\log|x||)^{s-\ell}|y|^{\alpha+r-|\gamma|-n}$$

$$\times(1+|\log|y||)^{\ell-s}|f(y)|dy\bigg)^p dx\bigg)^{1/p}$$

$$= I_{11} + I_{12}.$$

Since $\alpha > 0$, $r > -n/p'$, $\ell \in \mathbf{N}$ and $s \in \mathbf{R}$, Lemma 3.5 gives that $I_{11} \le$

$C\|f\|_p$. The conditions $\alpha+r-(n/p)>0$ and $\alpha+r-(n/p)\notin \mathbf{N}$ imply that $\alpha+r-|\gamma|-(n/p)>0$ for $|\gamma|\le k=[\alpha+r-(n/p)]$. Hence, by Lemma 3.6 we get

$$I_{12}\le C\sum_{|\gamma|\le k}\|f\|_p=C\|f\|_p.$$

By Lemma 3.1 and the fact that $d(y,\ell_x)>|x|/2$ implies $|y|>|x|/2$, we see that

$$I_2\le\Big(\int\Big(\int_{|y|>|x|/2}|x|^{-(\alpha+r-k-1)}(1+|\log|x||)^{s-\ell}|y|^{\alpha+r-k-1-n}\\ \times(1+|\log|y||)^{\ell-s}|f(y)|dy\Big)^p dx\Big)^{1/p}.$$

Since $k=[\alpha+r-(n/p)]$, we have $\alpha+r-k-1-(n/p)<0$, and hence Lemma 3.7 gives $I_2\le C\|f\|_p$. Consequently $I\le C\|f\|_p$. Next we consider the case $\alpha+r-(n/p)<0$. In this case, since $K_{\alpha,\ell:k}(x,y)=K_{\alpha,\ell}(x-y)$, by (3.1) we have

$$\begin{aligned}I&=\Big(\int\Big|\int|x|^{-\alpha-r}(1+|\log|x||)^{s-\ell}K_{\alpha,\ell}(x-y)|y|^r\\&\qquad\times(1+|\log|y||)^{-s}f(y)dy\Big|^p dx\Big)^{1/p}\\&\le C\Big(\int\Big(\int_{d(y,\ell_x)\le|x|/2}|x|^{-\alpha-r}(1+|\log|x||)^{s-\ell}|x-y|^{\alpha-n}\\&\qquad\times(1+|\log|x-y||)^{\ell}|y|^r(1+|\log|y||)^{-s}|f(y)|dy\Big)^p dx\Big)^{1/p}\\&\quad+C\Big(\int\Big(\int_{d(y,\ell_x)>|x|/2}|x|^{-\alpha-r}(1+|\log|x||)^{s-\ell}|x-y|^{\alpha-n}\\&\qquad\times(1+|\log|x-y||)^{\ell}|y|^r(1+|\log|y||)^{-s}|f(y)|dy\Big)^p dx\Big)^{1/p}\\&=J_1+J_2.\end{aligned}$$

Since $d(y,\ell_x) \le |x|/2$ implies $|x-y| \le 3|x|/2$, the conditions $\alpha > 0$, $\ell \in \mathbf{N}$, $r > -n/p'$ and $s \in \mathbf{R}$ allows us to apply Lemma 3.5 to J_1. Then we get $J_1 \le C\|f\|_p$. Since $d(y,\ell_x) > |x|/2$ implies $|y| > |x|/2$ and $|y|/3 < |x-y| < 3|y|$, the condition $\alpha + r - (n/p) < 0$ and Lemma 3.7 gives

$$\begin{aligned} J_2 &\le \left(\int \left(\int_{|y|>|x|/2} |x|^{-\alpha-r}(1+|\log|x||)^{s-\ell}|y|^{\alpha+r-n} \right.\right. \\ &\qquad \left.\left. \times (1+|\log|y||)^{\ell-s}|f(y)|dy \right)^p dx \right)^{1/p} \\ &\le C\|f\|_p. \end{aligned}$$

Therefore $I \le C\|f\|_p$. Thus we complete the proof of Proposition 3.3. □

By applying Theorem 3.2 to the Riesz potentials we obtain the following corollary.

Corollary 3.8 *Let $r > -n/p'$, $s \in \mathbf{R}$ and $\alpha + r - (n/p) \notin \mathbf{N}$. Then for $k = [\alpha + r - (n/p)]$*

$$\begin{cases} \|U_{\alpha,k}f\|_{p,-\alpha-r,s} \le C\|f\|_{p,-r,s}, & \alpha - n \notin 2\mathbf{N} \\ \|U_{\alpha,k}f\|_{p,-\alpha-r,s-1} \le C\|f\|_{p,-r,s}, & \alpha - n \in 2\mathbf{N}. \end{cases}$$

4. A semi-group formula for Riesz potentials of L^p-functions

In Section 2 we stated that for $\varphi \in \Phi(\mathbf{R}^n)$, $U_\beta\varphi \in \Phi(\mathbf{R}^n)$ and hence $U_\alpha(U_\beta\varphi) \in \Phi(\mathbf{R}^n)$. Moreover, we refered to the fact that the equality $U_{\alpha+\beta}\varphi = U_\alpha(U_\beta\varphi)$ holds for $\varphi \in \Phi(\mathbf{R}^n)$. Let $f \in L^p(\mathbf{R}^n)$. We consider the case $\beta - (n/p) \notin \mathbf{N}$ and $\alpha + \beta - (n/p) \notin \mathbf{N}$. According to Theorem 3.2 $U_{\beta,[\beta-(n/p)]}f$ belongs to $L^{p,-\beta,-1}(\mathbf{R}^n)$. Therefore again by Theorem 3.2 $U_{\alpha,[\alpha+\beta-(n/p)]}(U_{\beta,[\beta-(n/p)]}f)$ belongs to $L^{p,-\alpha-\beta,-2}(\mathbf{R}^n)$. On the other hand, it follows also from Theorem 3.2 that $U_{\alpha+\beta,[\alpha+\beta-(n/p)]}f$ belongs to $L^{p,-\alpha-\beta,-1}(\mathbf{R}^n)$. The purpose of this section is to prove that the both are equal (a semi-group formula).

We begin with some remarks.

Remark 4.1 We denote by $L^1_{loc}(\mathbf{R}^n)$ the space of all locally integrable functions in $\mathbf{R}^n$. If $r > -n/p'$, then $L^{p,-r,s}(\mathbf{R}^n) \subset L^1_{loc}(\mathbf{R}^n)$.

Remark 4.2 Let $f \in L^{p,-r,s}(\mathbf{R}^n) \cap \mathcal{S}(\mathbf{R}^n)$. Then the Riesz polynomial $P_{\alpha,k}f$ of type (α, k) of f exists for $k < \alpha + r - (n/p)$.

Remark 4.3 Let $r - (n/p) > 0$, $r - (n/p) \notin \mathbf{N}$, $s \in \mathbf{R}$ and $P(x)$ is a polynomial of degree $[r - (n/p]$. If $P(x) \in L^{p,-r,s}(|x| \leq 1)$, then $P = 0$ where

$$L^{p,-r,s}(|x| \leq 1) = \left\{ f : \int_{|x|\leq 1} |f(x)|^p |x|^{-rp}(1 + |\log |x||)^{sp} dx < \infty \right\}.$$

Now we prove our main theorem.

Theorem 4.4 *Let $\beta - (n/p) \notin \mathbf{N}$, $\alpha + \beta - (n/p) \notin \mathbf{N}$ and $f \in L^p(\mathbf{R}^n)$. Then*

$$U_{\alpha+\beta,[\alpha+\beta-(n/p)]} f = U_{\alpha,[\alpha+\beta-(n/p)]}(U_{\beta,[\beta-(n/p)]} f).$$

Proof. Let $f \in L^p(\mathbf{R}^n)$. Since $\Phi_{\beta,[\beta-(n/p)]}(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$ by Proposition 2.9, there exists a sequence $\{\varphi_m\} \subset \Phi_{\beta,[\beta-(n/p)]}(\mathbf{R}^n)$ such that φ_m converges to f in $L^p(\mathbf{R}^n)$ as $m \to \infty$. Since $\varphi_m \in \Phi(\mathbf{R}^n)$, Propoition 2.1 gives

$$U_{\alpha+\beta}(\varphi_m) = U_\alpha(U_\beta \varphi_m). \tag{4.1}$$

Moreover, since $\varphi_m \in \Phi(\mathbf{R}^n) \subset \mathcal{S}(\mathbf{R}^n)$ and $[\alpha + \beta - (n/p)] < \alpha + \beta$, by (2.1) we have

$$U_{\alpha+\beta,[\alpha+\beta-(n/p)]}\varphi_m = U_{\alpha+\beta}\varphi_m + P_{\alpha+\beta,[\alpha+\beta-(n/p)]}\varphi_m. \tag{4.2}$$

On the other hand, the fact $\varphi_m \in \Phi_{\beta,[\beta-(n/p)]}(\mathbf{R}^n)$ gives $P_{\beta,[\beta-(n/p)]}\varphi_m = 0$, and hence $U_{\beta,[\beta-(n/p)]}\varphi_m = U_\beta \varphi_m$. By using Proposition 2.1 and Theorem 3.2 we see that $U_\beta \varphi_m \in \Phi(\mathbf{R}^n) \cap L^{p,-\beta,-1}(\mathbf{R}^n)$. The fact $U_\beta \varphi_m \in \Phi(\mathbf{R}^n)$ implies the existence of $U_\alpha(U_\beta \varphi_m)$, and the fact $U_\beta \varphi_m \in \Phi(\mathbf{R}^n) \cap L^{p,-\beta,-1}(\mathbf{R}^n)$ gives the existence of $P_{\alpha,[\alpha+\beta-(n/p)]}(U_\beta \varphi_m)$ by $[\alpha+\beta-(n/p)] < \alpha+\beta-(n/p)$ and Remark 4.2. Hence

$$\begin{aligned} U_{\alpha,[\alpha+\beta-(n/p)]}(U_{\beta,[\beta-(n/p)]}\varphi_m) &= U_{\alpha,[\alpha+\beta-(n/p)]}(U_\beta \varphi_m) \\ &= U_\alpha(U_\beta \varphi_m) + P_{\alpha,[\alpha+\beta-(n/p)]}(U_\beta \varphi_m). \end{aligned} \tag{4.3}$$

By (4.1), (4.2) and (4.3) we obtain

$$\begin{aligned} &U_{\alpha+\beta,[\alpha+\beta-(n/p)]}\varphi_m - U_{\alpha,[\alpha+\beta-(n/p)]}(U_{\beta,[\beta-(n/p)]}\varphi_m) \\ &\quad = P_{\alpha,[\alpha+\beta-(n/p)]}(U_\beta\varphi_m) - P_{\alpha+\beta,[\alpha+\beta-(n/p)]}\varphi_m. \end{aligned} \tag{4.4}$$

Since $\beta-(n/p) \notin \mathbf{N}$ and $\alpha+\beta-(n/p) \notin \mathbf{N}$, Theorem 3.2 implies that the left-hand side of (4.4) belongs to $L^{p,-\alpha-\beta,-2}(\mathbf{R}^n)$. Theorefore the right-hand side of (4.4) also belongs to $L^{p,-\alpha-\beta,-2}(\mathbf{R}^n)$, and is a polynomial of degree $[\alpha+\beta-(n/p)]$. This shows that the right-hand side of (4.4) is zero by Remark 4.3. Thus we obtain

$$U_{\alpha+\beta,[\alpha+\beta-(n/p)]}\varphi_m = U_{\alpha,[\alpha+\beta-(n/p)]}(U_{\beta,[\beta-(n/p)]}\varphi_m). \tag{4.5}$$

Next we consider the limit process as $m \to \infty$ in (4.5). Since φ_m converges to f in $L^p(\mathbf{R}^n)$ as $m \to \infty$ and $\alpha+\beta-(n/p) \notin \mathbf{N}$, by Theorem 3.2 $U_{\alpha+\beta,[\alpha+\beta-(n/p)]}\varphi_m$ converges to $U_{\alpha+\beta,[\alpha+\beta-(n/p)]}f$ in $L^{p,-\alpha-\beta,-1}(\mathbf{R}^n)$, and hence in $L^1_{loc}(\mathbf{R}^n)$ as $m \to \infty$ by $\alpha+\beta > 0 > -n/p'$ and Remark 4.1. On the other hand, $U_{\beta,[\beta-(n/p)]}\varphi_m$ converges to $U_{\beta,[\beta-(n/p)]}f$ in $L^{p,-\beta,-1}(\mathbf{R}^n)$ as $m \to \infty$ on account of $\beta-(n/p) \notin \mathbf{N}$ and Theorem 3.2. Hence by using Theorem 3.2 again, we see that $U_{\alpha,[\alpha+\beta-(n/p)]}(U_{\beta,[\beta-(n/p)]}\varphi_m)$ converges to $U_{\alpha,[\alpha+\beta-(n/p)]}(U_{\beta,[\beta-(n/p)]}f)$ in $L^{p,-\alpha-\beta,-2}(\mathbf{R}^n)$, and hence in $L^1_{loc}(\mathbf{R}^n)$ as $m \to \infty$ because of $\alpha+\beta-(n/p) \notin \mathbf{N}$. This fact and (4.5) implies that

$$U_{\alpha+\beta,[\alpha+\beta-(n/p)]}f = U_{\alpha,[\alpha+\beta-(n/p)]}(U_{\beta,[\beta-(n/p)]}f).$$

We complete the proof in Theorem 4.4. □

Finaly, we give an improvement of the integral estimates in corollary 3.8 by using the semi-group formula in Theorem 4.4.

Corollary 4.5 *Let $\alpha-(n/p) \notin \mathbf{N}$ and $f \in L^p(\mathbf{R}^n)$. Then for $k = [\alpha-(n/p)]$*

$$\|U_{\alpha,k}f\|_{p,-\alpha,0} \le C\|f\|_p.$$

Proof. In case of $\alpha-n \notin 2\mathbf{N}$, this is nothing but Corollary 3.8. Let $\alpha-n \in 2\mathbf{N}$. We can take positive numbers β and ζ such that $\alpha=\beta+\zeta$, $\beta-n \notin 2\mathbf{N}$, $\zeta-n \notin 2\mathbf{N}$ and $\zeta-(n/p) \notin \mathbf{N}$. Since $\alpha-(n/p) = \beta+\zeta-(n/p) \notin \mathbf{N}$ and

$\zeta - (n/p) \notin \mathbf{N}$, by using the semi-group formula in Theorem 4.4 we see that

$$U_{\alpha,k}f = U_{\beta+\zeta,[\beta+\zeta-(n/p)]}f = U_{\beta,[\beta+\zeta-(n/p)]}(U_{\zeta,[\zeta-(n/p)]}f).$$

Moreover, by $\beta + \zeta - (n/p) \notin \mathbf{N}$ and $\beta - n \notin 2\mathbf{N}$ Theorem 3.2 implies that

$$\begin{aligned} \|U_{\alpha,k}f\|_{p,-\alpha,0} &= \left\|U_{\beta,[\beta+\zeta-(n/p)]}(U_{\zeta,[\zeta-(n/p)]}f)\right\|_{p,-\beta-\zeta,0} \\ &\le C\|U_{\zeta,[\zeta-(n/p)]}f\|_{p,-\zeta,0}. \end{aligned} \tag{4.6}$$

Further, since $\zeta - (n/p) \notin \mathbf{N}$, $\zeta - n \notin 2\mathbf{N}$, by Theorem 3.2 again we have

$$\|U_{\zeta,[\zeta-(n/p)]}f\|_{p,-\zeta,0} \le C\|f\|_{p,0,0} = \|f\|_p. \tag{4.7}$$

By combining (4.6) and (4.7) we obtain the required estimate. □

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