Linearly compact modules and cogenerators II

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Introduction

Let R be a ring with 1-element. A left R-module M is called linearly compact if every finitely solvable system of congruences $x \equiv m_\alpha$ (mod M_α), $\alpha \in I$, where $m_\alpha \in M_\alpha$ and M_α's are submodules of M, is solvable. Linearly compact modules play an essential role in Morita duality without chain conditions. ([9], [12], [14]).

Let ${}_RP$ be a finitely generated projective left R-module and ${}_RQ$ be a cofinitely generated injective[1)] left R-module. We say that the $\{P, Q\}$ is an RZ-pair if every simple homomorphic image of ${}_RP$ is isomorphic to a submodule of ${}_RQ$, and every simple submodule of ${}_RQ$ is a homomorphic image of ${}_RP$. It should be noted that $\{P, P\}$ is an RZ-pair if and only if ${}_RP$ is a finitely generated projective and cofinitely generated injective RZ-module (cf. [11]). Let S and T be the endomorphism rings of ${}_RP$ and ${}_RQ$, respectively. The main purpose of this paper is to show the following

THEOREM (THEOREM 2). *Suppose that the pair $\{P, Q\}$ is an RZ-pair and both ${}_RP$ and ${}_RQ$ are linearly compact. Then both ${}_S Hom_R(P, Q)$ and $Hom_R(P, Q)_T$ are injective cogenerators, and, S and T are naturally isomorphic to the endomorphism rings of $Hom_R(P, Q)_T$ and ${}_S Hom_R(P, Q)$, respectively.*

A ring R is called a *left Morita ring* following [14], if ${}_RR$ and the injective envelope of every simple left R-module are linearly compact. A *right Morita ring* is defined similary. Then we have the following

COROLLARY A (COROLLARY 2 to THEOREM 2). *Let R be a left Morita ring. Then the endomorphism ring of every finitely generated projective left R-module is a left Morita ring, and, the endomorphism ring of every cofinitely generated injective left R-module is a right Morita ring.*

The first half of the corollary was announced by R. W. Miller and D. R. Turnidge [7].

Further, we have the following corollaries.

COROLLARY B (COROLLARY 3 to THEOREM 2). *Let R be a ring such*

1) A module Q is cofinitely generated if and only if the socle of Q is finitely generated and an essential submodule of Q (cf. [12, 13]).

that $_RR$ is linearly compact. Let $_RQ$ be a linearly compact, cofinitely generated injective left R-module, and T be the endomorphism ring of $_RQ$. Then the right T-module Q_T is a linearly compact cogenerator, and the left R'-module $_{R'}Q$ is cofinitely generated and injective, where R' is the double centralizer of $_RQ$, $R'=End(Q)$.

COROLLARY C (COROLLARY 4 to THEOREM 2). *Let R be a ring such that $_RR$ is linearly compact. If there is a (faithful) linearly compact, cofinitely generated injective and flat left R-module, then there exists a (faithful) finitely generated projective and injective right R-module.*

These corollaries afford some useful generalizations in the study of QF-3 rings (cf. [3, 8]).

§1 A pairing of modules

Throught the present paper, $_RP$ and $_RQ$ denote left R-modules, and, S and T denote always the endomorphism rings of $_RP$ and $_RQ$, respectively. $_SHom_R(P, Q)_T$ is then a S–T-bimodule in the following way:

$$(sgt)(p)=g(ps)t,\quad g\in Hom_R(P, Q),\quad s\in S,\quad t\in T,\quad p\in P.$$

Let $_SP^*_R=Hom_R(P, R)$ be the R-dual of $_RP$ there is a mapping $(\ ,\ )$ of $P^*\times Q$ into $Hom_R(P, Q)$ which is defined by

$$(f, q)(p)=f(p)q,\quad f\in P^*,\quad q\in Q,\quad p\in P.$$

It is easy to see that the following identities hold:

$$(f_1+f_2, q)=(f_1, q)+(f_2, q),\qquad (f, q_1+q_2)=(f, q_1)+(f, q_2)$$
$$(sf, q)=s(f, q),\qquad (f, qt)=(f, q)t$$
$$(fr, q)=(f, rq)$$

$f, f_1, f_2\in P^*$, $q, q_1, q_2\in Q$, $s\in S$, $t\in T$, $r\in R$.

Here we consider the following conditions:

(A) $(f, q)=0$ for all $f\in P^*$ implies $q=0$.[2)]

(B) $(f, q)=0$ for all $q\in Q$ implies $f=0$.

If these two conditions are fullfiled, then we say that the pair $\{P, Q\}$ is a regular pair. It is easy to see that if $_RP$ is a generator, then $\{P, Q\}$ satisfies the condition (A) for every left R-module Q.

LEMMA 1. *Let $\{P, Q\}$ satisfy the condition (A) and $_RM$ be a left R-module. Then, for any submodule X of $_SHom_R(P, M)$ and for any S-*

2) Condition (A) is equivalent to say that Q is P-distinguished in the terminology of [6].

homomorphism δ *of* X *into* ${}_S Hom_R(P, Q)$, *the following mapping,*

$$X(P) \ni \sum_{finite} f_i(p_i) \longrightarrow \sum \delta(f_i)(p_i) \in Q, \quad f_i \in X, \quad p_i \in P,$$

is an well defined R-homomorphism.

PROOF. By assumption, it suffices to show that if $\sum f_i(p_i)=0$, then $(f, \sum \delta(f_i)(p_i))=0$ for all $f \in P^*$. Let s_i be the element of S which is defined by $ps_i = f(p)p_i$, $p \in P$. Then we have $\sum s_i f_i = 0$, because $(\sum s_i f_i)(p) = \sum f_i(ps)$ $= \sum f(p) f_i(p_i) = 0$ for all $p \in P$. Now, $(f, \sum \delta(f_i)(p_i))(p) = \sum f(p)(\delta(f_i)(p_i)) =$ $\sum \delta(f_i)(f(p)p_i) = \sum \delta(f_i)(ps_i) = \sum \delta(s_i f_i)(p) = \delta(\sum s_i f_i)(p) = 0$ for all $p \in P$. Thus $(f, \sum \delta(f_i)(p_i)) = 0$. Since f is arbitrary element of P^*, we have obtained our assertion.

COROLLARY. *Let* ${}_R P$ *be a generator. Then, for any left R-modules* ${}_R A$ *and* ${}_R B$, $Hom_S(Hom_R(P, A), Hom_R(P, B))$ *is naturally isomorphic to* $Hom_R(A, B)$.

PROOF. Let δ be an element of $Hom_S(Hom_R(P, A), Hom_R(P, B))$. Then, by Lemma 1, the mapping

$$g:\ A = Hom_R(P, A)(P) \in \sum_{finite} f_i(p_i) \to \sum \delta(f_i)(p_i) \in B,\ f_i \in Hom_R(P, A),\ p_i \in P,$$

is an well defined R-homomorphism. This implies that $\delta(f) = gf$ for all $f \in Hom_R(P, A)$. On the other hand, if g is an element of $Hom_R(A, B)$ and $gf=0$ for all $f \in Hom_R(P, A)$, then $g=0$, because ${}_R P$ is a generator. Thus $Hom_S(Hom_R(P, A), Hom_R(P, B))$ is naturally isomorphic to $Hom_R(A, B)$.

LEMMA 2. *Let* $\{P, Q\}$ *satisfy the condition* (A), *and* ${}_R M$ *be a left R-module. Then the following conditions are equivalent.*

(1) ${}_R Q$ *is M-injective.*[3]

(2) ${}_S Hom_R(P, Q)$ *is* ${}_S Hom_R(P, M)$*-injective and* $Hom_S(Hom_R(P, M),$ $Hom_R(P, Q))$ *is naturally isomorphic to* $Hom_R(M, Q)$.

PROOF. (1)$\Rightarrow$(2). Let X be a submodule of ${}_S Hom_R(P, M)$ and δ be a S-homomorphism of X into ${}_S Hom_R(P, Q)$. Then, since ${}_R Q$ is M-injective, by Lemma 1, there exists an element $g \in Hom_R(M, Q)$ such that $\delta(f) = gf$ for all $f \in X$. It follows that ${}_S Hom_R(P, Q)$ is ${}_S Hom_R(P, M)$-injective, and, by taking ${}_S Hom_R(P, M)$ as X, $Hom_S(Hom_R(P, M), Hom_R(P, Q))$ is homomorphic to $Hom_R(M, Q)$. It remains to show that if g is an element of $Hom_R(M, Q)$ such that $gf=0$ for all $f \in Hom_R(P, M)$, then $g=0$. Let m be an element of M. Then we have $(h, g(m))(p) = h(p)g(m) = g(h(p)m) = 0$ for all $h \in P^*$,

3) Q is called M-injective if every homomorphism of a submodule of M into Q is extended to that of M into Q (cf. [1]).

and for all $p \in P$. This implies, by assumption, that $g(m)=0$. Since m is an arbitrary element of M, we have $g=0$. This proves our assertion.
(2)$\Rightarrow$(1). Let M' be a submodule of M and φ be an R-homomorphism of M' into Q. Then, by assumptions, there exists an element $g \in Hom_R(M, Q)$ such that $\varphi \cdot f = g \cdot f$ for all $f \in Hom_R(P, M')$. Then we have $\varphi(m') = g(m')$ for all $m' \in M'$, since $(h, \varphi(m') - g(m'))(p) = (\varphi - g)(h(p)m') = 0$ for all $h \in P^*$, and for all $p \in P$. Thus Q is M-injective.

COROLLARY 1. Let $\{P, Q\}$ satisfy the condition (A). If ${}_RQ$ is injective, then ${}_S Hom_R(P, Q)$ is injective, and the endomorphis ring of ${}_RQ$ is naturally isomorphic to that of ${}_S Hom_R(P, Q)$.

PROOF. Setting $M=P$ in Lemma 2, we see that ${}_S Hom_R(P, Q)$ is injective, and, setting $M=Q$ we obtain the latter half of our assertions.

COROLLARY 2. (Pahl).[4] *Let ${}_RP$ be a generator and S be the endomorphism ring of ${}_RP$. Then the following conditions are equivalent.*

(1) *${}_RP$ is quasi-injective.*
(2) *${}_SS$ is injective.*
(3) *${}_RP$ is injective.*

PROOF. Since $\{P, P\}$ satisfies the condition (A), the equivalence (1)$\Leftrightarrow$(2) follows direct from Lemma 2 by setting $M=Q=P$, while the equivalence (1)$\Leftrightarrow$(3) is well known.[5]

§ 2 *RZ*-pairs

Let ${}_RP$ be a finitely generated left R-module and ${}_RQ$ be a cofinitely generated injective left R-module. We say that the pair $\{P, Q\}$ is an *RZ*-pair if every homomorphic image of P is isomorphic to a submodule of Q, and every simple submodule of Q is a homomorphic image of P.

LEMMA 3. *If the pair $\{P, Q\}$ is an RZ-pair, then it is a regular pair.*

PROOF. Let $(P^*, q)=0$, $q \in Q$. Suppose $q \neq 0$, and let Rq_0 be a simple submodule of Rq. Then, since Rq_0 is a homomorphic image of ${}_RP$ and ${}_RP$ is projective, there exists an element $f \in P^*$ such that $f(P)q_0 = Rq_0$. But this means that $(f, q_0) \neq 0$, and we have a contradiction. Next, let $(f, Q)=0$, $f \in P^*$. Suppose $f \neq 0$, and, let Rq_0 be a simple homomorphic image of $f(P)$. Then, since ${}_RQ$ is injective, there exists an element $q_1 \in Q$ such that $Rq_0 = f(P)q_1$. But this means that $(f, q_1) \neq 0$, and we have a contradiction.

Following lemma is easy to show and we omitt here the proof for it.

4) Cf. [5], Theorem 3.
5) Cf. [1].

LEMMA 4. *Let $_RP$ be a projective left R-module. Then, for a simple left R-module $_RM$, $_SHom_R(P, M)$ is either* 0 *or simple.*

THEOREM 1.[6] *Let $\{P, Q\}$ be an RZ-pair. Then $_SHom_R(P, Q)$ is a cofinitely generated injective cogenerator, and the endomorphism ring of $_RQ$ is naturally isomorphic to that of $_SHom_R(P, Q)$.*

PROOF. Let Q_0 be the socle of Q. Then, by Lemma 4, $_SHom_R(P, Q_0)$ is a sum of simple submodules. We show that $_SHom_R(P, Q_0)$ is an essential submodule of $_SHom_R(P, Q)$. Let f be a non-zero element of $_SHom_R(P, Q)$ and Q' be a simple submodule of $f(P)$. Then $Hom_R(P, Q')$ is a simple submodule of $Hom_R(P, Q_0)$ which is contained in $Hom_R(P, f(P))=Sf$. Thus $_SHom_R(P, Q)$ is cofinitely generated. Next, we show that $_SHom_R(P, Q)$ contains an isomorphic image of every simple left S-module. For this purpose, let $\mathfrak{l}$ be a maximal left ideal of S. Then, since $_RP$ is finitely generated and projective, we have $P\mathfrak{l}\neq P$. Let f be a non-zero R-homomorphism of $P/P\mathfrak{l}$ into Q. Then $\tilde{f}=f\cdot\nu$ is an element of $_SHom_R(P, Q)$ such that $S\tilde{f}\cong S/\mathfrak{l}$, where ν is the natural homomorphism of $_RP$ onto $P/P\mathfrak{l}$. Thus $_SHom_R(P, Q)$ contains an isomorphic image of $S/\mathfrak{l}$. Since $_SHom_R(P, Q)$ is injective by Corollary 1 to Lemma 2, it is an injective cogenerator. The last assertion of the theorem follows direct from Lemma 2.

COROLLARY. *If $\{P, P\}$ is an RZ-pair, or equivalently, if $_RP$ is a finitely generated projective and cofinitely injective RZ-module, then the endomorphism ring of $_RP$ is a left injectiv cogenerator.*

§3 Endomorphism rings of linearly compact modules

We begin this section with the following

PROPOSITION 1. *Let $_RP$ be a finitely generated projective left R-module. If $_RQ$ is linearly compact, then $_SHom_R(P, Q)$ is linearly compact.*

PROOF. This is proved in [14]. But for the sake of completeness, we give here the proof for it. Let $g\equiv g_\alpha$ (mod $\mathfrak{U}_\alpha$), $\alpha\in I$, where $g_\alpha\in Hom_R(P, Q)$ and $\mathfrak{U}_\alpha$'s are submodules of $_SHom_R(P, Q)$, be a finitely solvable system of congruences. Then, as is easily seen, for each element p of P, the system of congruences $x\equiv g_\alpha(p)$ (mod $\mathfrak{U}_\alpha(P)$) is finitely solvable, and, because $_RQ$ is linearly compact, it is solvable. Let $K=\prod_{\alpha\in I}Q/\mathfrak{U}_\alpha(P)$ be the direct product of $Q/\mathfrak{U}_\alpha(P)$'s and K_0 be the submodule $\{\prod(q \bmod \mathfrak{U}_\alpha(P))|q\in Q\}$ of K. Let h be the homomorphism of $_RP$ into K wich is defined by,

$$h(p)=\prod\big(g_\alpha(p) \bmod \mathfrak{U}_\alpha(P)\big),\quad p\in P.$$

6) Cf. [6], Theorem 4.

Because ${}_RP$ is projective, there is a homomorphism k of ${}_RP$ in ${}_RQ$ such that $k(p)\equiv g_\alpha(p)$ (mod $\mathfrak{U}_\alpha(P)$) for all $p\in P$. Our proof is therefore complete if we prove the following

LEMMA 5. *Let ${}_RP$ be a finitely generated projective left R-module and ${}_RQ$ be a left R-module. Let $\mathfrak{U}$ be a submodule of ${}_S Hom_R(P, Q)$, and, k be a homomorphism of ${}_RP$ in ${}_RQ$ such that $k(P)\subseteq\mathfrak{U}(P)$. Then we have $k\in\mathfrak{U}$.*

PROOF. Let $P=\sum_{i=1}^{n}Rp_i$ and $k(p_i)=u_1^{(i)}(p_1^{(i)})+\cdots+u_{l_i}^{(i)}(p_{l_i}^{(i)})$, $u_j^{(i)}\in\mathfrak{U}$, $p_j^{(i)}\in P$, $i=1, 2, \cdots, n$. Let $t=l_1+\cdots+l_n$ and ν be the homomorphism of $P^{(t)}$, the direct sum of t-copies of ${}_RP$, in Q which is defined by,

$$P^{(t)}(x_1, \cdots, x_t)\longrightarrow u_1^{(1)}(x_1)+\cdots+u_{l_n}^{(n)}(x_t)\in Q\,.$$

It is clear that $k(P)\subseteq\nu(P^{(t)})$. Since ${}_RP$ is projective, there exist $s_1, \cdots, s_t\in S$ such that $u_1^{(1)}(ps_1)+\cdots+u_{l_n}^{(n)}(ps_t)=k(p)$ for all $p\in P$. It follows that $k=s_1u_1^{(1)}+\cdots+s_tu_{l_n}^{(n)}\in\mathfrak{U}$, as asserted.

COROLLARY. *Let ${}_RP$ be a linearly compact projective left R-module. Then ${}_SS$ is linearly compact.*

PROOF.[7] It suffices to show that ${}_RP$ is finitely generated. Since ${}_RP$ is complemented, $Ra(P)$, the radical of ${}_RP$, is small and $P/Ra(P)$ is semi-simple artinian. It follows that ${}_RP$ is finitely generated.

PROPOSITION 2. *Let ${}_RP$ be a generator and ${}_RQ$ be a left R-module. If the left S-module ${}_SHom_R(P, Q)$ is linearly compact, then ${}_RQ$ is linearly compact.*

PROOF. Let $x\equiv q_\alpha$ (mod $\mathfrak{U}_\alpha$), $\alpha\in I$, where $q_\alpha\in Q$ and $\mathfrak{U}_\alpha$'s are submodules of ${}_RQ$, be a finitely solvable system of congruences. Since ${}_RP$ is a generator, there exist $f_1, \cdots, f_n\in Hom_R(P, R)$ and $p_1, \cdots, p_n\in P$ such that $\sum_{i=1}^{n}f_i(p_i)=1$. For each i and for each α, let $g_\alpha^{(i)}$ be a homomorphism of ${}_RP$ in ${}_RQ$ which is defined by,

$$g_\alpha^{(i)}:\quad P\ni p\longrightarrow f_i(p)q_\alpha\in Q\,.$$

Then, as is easily seen, for each i, the system of congruences $g=g_\alpha^{(i)}$ (mod $Hom_R(P, \mathfrak{U}_\alpha)$), $\alpha\in I$, is finitely solvable. Since ${}_SHom_R(P, Q)$ is linearly compact, the system has a solution $g^{(i)}$. Let $q_0=\sum_{i=1}^{n}g^{(i)}(p_i)$. Then we have, for each $\alpha\in I$, $q_0-q_\alpha=\sum_{i=1}^{n}g^{(i)}(p_i)-(\sum_{i=1}^{n}f_i(p_i))q_\alpha=\sum_{i=1}^{n}g^{(i)}(p_i)-g_\alpha^{(i)}(p_i)=\sum_{i=1}^{n}(g^{(i)}-g_\alpha^{(i)})(p_i)\in\mathfrak{U}_\alpha$. Thus our assertion is proved.

7) Cf. [4, 15].

From Propositions 1 and 2 we have direct the following

PROPOSITION 3. *Let ${}_RP$ be a progenerator. Then ${}_S Hom_R(P, Q)$ is linearly compact if and only if ${}_RQ$ is linearly compact.*

We have also the following

PROPOSITION 4. *Let ${}_RQ$ be an injective left R-module with essential socle and ${}_RP$ be a linearly compact left R-module. Then the right T-module $Hom_R(P, Q)_T$ is linearly compact.*

PROOF. For each submodule $\mathfrak{U}$ of $Hom_R(P, Q)_T$, we have $Ann_{Hom_R(P,Q)}(Ann_P(\mathfrak{U}))=\mathfrak{U}$.[8] Let $g\equiv g_\alpha \pmod{\mathfrak{U}_\alpha}$, $\alpha\in I$, be a finitely solvable system of congruences, where $g_\alpha \in Hom_R(P, Q)$ and $\mathfrak{U}_\alpha$'s are submodules of $Hom_R(P, Q)_T$. Then, as is easily seen, the mapping

$$\sum Ann_P(\mathfrak{U}_\alpha) \ni \sum_{finite} p_{\alpha_i} \longrightarrow \sum g_{\alpha_i}(p_{\alpha_i}) \in Q\,,$$

where $p_{\alpha_i} \in Ann_P(\mathfrak{U}_{\alpha_i})$, is an well defined R-homomorphism. Since ${}_RQ$ is injective, there exists $g\in Hom_R(P, Q)$ such that $g(p_\alpha)=g_\alpha(p_\alpha)$ for all $p_\alpha\in Ann_P(\mathfrak{U}_\alpha)$. It follows that, for every $\alpha\in I$, $g-g_\alpha\in Ann_{Hom_R(P,Q)}(Ann_P(\mathfrak{U}_\alpha))=\mathfrak{U}_\alpha$. This proves our assertion.

§4 Main results

THEOREM 2. *Suppose that $\{P, Q\}$ is an RZ-pair and both ${}_RP$ and ${}_RQ$ are linearly compact. Then both ${}_S Hom_R(P, Q)$ and $Hom_R(P, Q)_S$ are injective cogenerators, and, S and T are naturally isomorphic to the endomorphism rings of $Hom_R(P, Q)_T$ and ${}_S Hom_R(P, Q)$, respectively, where S is the endomorphism ring of ${}_RP$ and T is that of ${}_RQ$.*

PROOF. By Theorem 1 ${}_S Hom_R(P, Q)$ is a cofinitely generated injective cogenerator and T is isomorphic to the endomorphism ring of ${}_S Hom_R(P, Q)$. Further, by Proposition 1, both ${}_S S$ and ${}_S Hom_R(P, Q)$ are linearly compact, whence ${}_S Hom_R(P, Q)$ is balanced and $Hom_R(P, Q)_T$ is an injective cogenerator.[9]

COROLLARY 1. *Let ${}_RP$ be a linearly compact, cofinitely generated, injective and projective left R-module. If $\{P, P\}$ is an RZ-pair, then the endomorphism ring of ${}_RP$ is a two-sided cogenerator ring.*

A ring R is called *left Morita ring*[10] if both ${}_RR$ and the injective envelope of every simple left R-module are linearly compact. A *right Morita*

8) Cf. [12], Theorem 5. Here, $Ann_X(Y)$ denotes, as usual, the annihilator of Y in X.
9) Cf. [12], Theorem 2, Theorem 7.
10) Cf. [7].

ring is defined similarly. Then we have the following

COROLLARY 2. *Let R be a left Morita ring. Then the endomorphism ring of a finitely generated projective left R-module is a left Morita ring, and, the endomorphism ring of a cofinitely generated injective left R-module is a right Morita ring.*

PROOF. Let ${}_RP$ be a finitely generated projective left R-module. Since R is a semi-perfect ring,[11] there exists a cofinitely generated injective left R-module ${}_RQ$ such that $\{P, Q\}$ is an RZ-pair. Further, since R is a left Morita ring, both ${}_RP$ and ${}_RQ$ are linearly compact, whence, by Theorem 2, ${}_SS$ is linearly compact and ${}_SHom_R(P, Q)$ is a linearly compact injective cogenerator. Thus S is a left Morita ring. The latter half of the corollary is also proved similarly.

COROLLARY 3. *Let R be a ring such that ${}_RR$ is linearly compact. Let ${}_RQ$ be a linearly compact, cofinitely generated injective left R-module and T be the endomorphism ring of ${}_RQ$. Then Q_T is a linearly compact cogenerator, and, ${}_{R'}Q$ is a cofinitely generated injective left R'-module, where R' is the endomorphism ring of Q_T.*

PROOF. By assumption, there is a finitely generated projective left R-module ${}_RP$ such that $\{P, Q\}$ is an RZ-pair. Then, by Theorem 2 $Hom_R(P, Q)_T$ is an injective cogenerator. Since $Hom_R(P, Q)_T$ is isomorphic to a direct product of Q_T, Q_T is a cogenerator. By Proposition 4, Q_T is linearly compact and whence ${}_{R'}Q$ is injective.[12] Further, ${}_{R'}Q$ is cofinitely generated by ([13], Lemma 8).[13]

COROLLARY 4. *Let R be a two-sided cogenerator ring. Let ${}_RP$ be a finitely generated projective left R-module. Then P_S is a linearly compact cogenerator and ${}_{R'}P$ is a faithful finitely generated projective and cofinitely generated injective left R'-module, where S is the endomorphism ring of ${}_RP$ and R' is the double centralizer of ${}_RP$.*

PROOF. Since ${}_RR$, whence ${}_RP$ is linearly compact, cofinitely generated and injective, our assertion follows direct from the above Corollary 3.

COROLLARY 5. *Let R be a ring such that ${}_RR$ is linearly compact. If there is a (faithful) linearly compact, cofinitely generated injective and flat left R-module, then there exists a (faithful) finitely generated projective and injective right R-module.*

PROOF. Let ${}_RQ$ be a linearly compact, cofinitely generated injective and

11) Cf. [12], Corollary to Theorem 5.
12) Cf. [12], Corollary 1 to Theorem 2.
13) Note that T is a semi-perfect ring.

flat left R-module, and, T be the endomorphism ring of ${}_RQ$. Let ${}_RP$ be a finitely generated projective left R-module such that $\{P, Q\}$ forms an RZ-pair. Then, by Theorem 2, both ${}_SHom_R(P, Q)$ and $Hom_R(P, Q)_T$ are injective cogenerators, and S, T are naturally isomorphic to the endomorphism rings of $Hom_R(P, Q)_T$, ${}_SHom_R(P, Q)$ respectively, where S is the endomorphism rings of ${}_RP$. We show that the right R-module $P^* = Hom_R(P, R)$ is injective. Since Q_T is linearly compact and $\{P, Q\}$ is a regular pair, ${}_SP^*$ is isomorphic to the $Hom_R(P, Q)$-dual of Q_T, that is, ${}_SP^*_R \cong Hom_T({}_RQ_T, Hom_R(P, Q)_T)$.[14] It follows that P^*_R is injective, because ${}_RQ$ is flat and $Hom_R(P, Q)_T$ is injective. It is clear that P^*_R is finitely generated and projective. Further, let ${}_RQ$ be faithful. Then $P^*r = 0$, $r \in R$, implies that $0 = (P^*r, Q) = (P^*, rQ)$. It follows that $rQ = 0$, whence $r = 0$. Thus P^*_R is faithful. This completes our proof.

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References

[1] G. AZUMAYA: *M*-projective and *M*-injective modules (to appear).

[2] K. R. FULLER: On indecomposable injectives over artinian rings, Pacific J. Math. 29 (1969), 115–135.

[3] M. HARADA: *QF*-3 and semiprimary *PP*-rings II, Osaka J. Math. 3 (1966), 21–27.

[4] F. KASCH and E. A. MARES: Eine Kenzeichung semiperfecter Moduln, Nagoya Math. J. 27 (1966), 525–529.

[5] F. KASCH, H.-J. SCHNEIDER and H. J. STOLBERG: On injective modules and cogenerators, Report 69–23, Dept. Math., Carnegie Institut of Technology, Carnegie Mellon Univ., 1969.

[6] T. KATO: *U*-distinguished modules (to appear).

[7] R. W. MILLER and D. R. TURNIDGE: Morita duality for endomorphism rings, Proc. Amer. Math. Soc. 31 (1972) 91–94.

[8] K. MORITA: Duality in *QF*-3 rings, Math. *Z.* 108 (1960), 237–252.

[9] B. MÜLLER: Linear compactness and Morita duality, J. Algebra 16 (1970), 60–66.

[10] T. ONODERA: Uber Kogeneratoren, Arch. Math. 16 (1968), 402–410.

[11] T. ONODERA: Ein Satz über koendlich erzeugte *RZ*-Moduln, Tōhoku Math. J. 23 (1971), 691–695.

[12] T. ONODERA: Linearly compact modules and cogenerators, J. Fac. Sci., Hokkaido Univ., Ser. I. 12. No. 3. 4 (1972), 116–125.

[13] T. ONODERA: Koendlich erzeugte Moduln und Kogeneratoren Hokkaido M. J. 2 (1973) 69–83.

[14] F. L. SANDOMIERSKI: Linearly compact modules and local Morita duality, Ring Theory, Academic Press, 1972.

14) Cf. [12], Theorem 7.

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