A characterization of $PSU(3, 3^2)$ as a permutation group of rank 4

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1. Introduction

It is known that the simple unitary group $PSU(3, 3^2)$ of order 6048 has a representation as a primitive group of degree 36 with the stabilizer of a point isomorphic to the projective special linear group $PSL(3, 2)$ of order 168. This representation has rank 4 and subdegrees 1, 7, 7, $21=7\cdot 6/2$, and the orbitals of length 7 are paired with each other (for example, see Quirin [6, P. 224]).

The purpose of this note is to prove the following result, which is a supplement of section 2 of [5].

THEOREM. *Let (G, Ω) be a finite primitive permutation group of rank 4 such that the subdegrees are 1, k, k, $k(k-1)/2$ and the orbitals of length k are paired with each other. Then $k=7$ and (G, Ω) is permutation-isomorphic to the simple unitary group $PSU(3, 3^2)$ acting by right multiplication on the cosets of its subgroup $PSL(3, 2)$.*

REMARK. By Proposition 3.6 of [5], if the stabilizer of a point acts doubly transitively on an orbit of length k, the assumption that the orbitals of length k are paired with each other is omitted.

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2. Notation and preliminaries

Our proof is quite elementary and only the familiarity with definitions and basic properties of Higman's intersection numbers ([4]) is assumed. Notation follows [4] and [5], but for convenience we rewrite below. The orbitals of length 1, k, k, $l=k(k-1)/2$ are denoted by Γ_0, $\Gamma_1=\Delta$, $\Gamma_3=\Lambda$, $\Gamma_2=\Gamma$, respectively. Here we may take the orbitals so that $\Gamma_\alpha(a)^g=\Gamma_\alpha(a^g)$ for all $g\in G$ and $a\in\Omega$. The intersection numbers relative to an orbital Γ_α are defined by

$$\mu_{ij}^{(\alpha)}=|\Gamma_\alpha(b)\cap\Gamma_i(a)| \quad \text{for} \quad b\in\Gamma_j(a).$$

The following are fundamental relations among the $\mu_{ij}^{(\alpha)}$ and k, l.

$\mu_{11}^{(1)}=\mu_{13}^{(1)}=\mu_{33}^{(1)}=\mu_{11}^{(3)}=\mu_{31}^{(3)}=\mu_{33}^{(3)}$ (set λ),
$\mu_{12}^{(1)}=\mu_{32}^{(3)}$ (set μ),
$\mu_{21}^{(1)}=\mu_{13}^{(2)}=\mu_{31}^{(2)}=\mu_{23}^{(3)}$ (set ν_1),
$\mu_{22}^{(1)}=\mu_{12}^{(2)}=\mu_{32}^{(2)}=\mu_{22}^{(3)}$ (set μ_1),
$\mu_{23}^{(1)}=\mu_{11}^{(2)}=\mu_{33}^{(2)}=\mu_{21}^{(3)}$ (set λ_1), $\mu_{31}^{(1)}=\mu_{13}^{(3)}$ (set ν_2),
$\mu_{32}^{(1)}=\mu_{12}^{(3)}$ (set μ_2), $\mu_{21}^{(2)}=\mu_{23}^{(2)}$ (set λ') and set $\mu_{22}^{(2)}=\mu'$;
$1+2\lambda+\lambda_1=\mu+\mu_1+\mu_2=\lambda+\nu_1+\nu_2=k$,
$\nu_1+\lambda'+\lambda_1=1+2\mu_1+\mu'=l$;
$k\nu_1=l\mu_2$, $k\lambda'=l\mu_1$, $k\lambda_1=l\mu$.

Intersection matrices $M_\alpha=(\mu_{ij}^{(\alpha)})$ corresponding to $\Gamma_\alpha(\alpha=1, 2, 3)$ are

$$M_1=\begin{pmatrix}0&0&0&1\\k&\lambda&\mu&\lambda\\0&\nu_1&\mu_1&\lambda_1\\0&\nu_2&\mu_2&\lambda\end{pmatrix},\quad M_2=\begin{pmatrix}0&0&1&0\\0&\lambda_1&\mu_1&\nu_1\\l&\lambda'&\mu'&\lambda'\\0&\nu_1&\mu_1&\lambda_1\end{pmatrix},\quad M_3=\begin{pmatrix}0&1&0&0\\0&\lambda&\mu_2&\nu_2\\0&\lambda_1&\mu_1&\nu_1\\k&\lambda&\mu&\lambda\end{pmatrix}.$$

By (4. 10) of Higman [4], any two intersection matrices commute with each other. In particular, (3, 4)-entries of $M_1M_2=M_2M_1$ gives an additional relation

$$\nu_1^2+\mu_1\lambda'+\lambda_1^2=l+\mu'\lambda_1+2\lambda\lambda'. \qquad (*)$$

3. Proof of Theorem

To begin with, as in [5], we shall determine the value of k. Since $k\lambda_1=l\mu$ and $\lambda_1\leqq k-1$, we have $\mu\leqq 2$. Similarly, $k\nu_1=l\mu_2$ and $\nu_1\leqq k$ imply $\mu_2\leqq 2$ (in case $\mu_2=3$, we have $k=2$ or 3 and these are easily excluded). Since $\mu+\mu_1+\mu_2=k$ and $k\lambda'=l\mu_1$, if $\mu=0$ and $\mu_2=0$, then $\lambda'=l$, which contradicts the primitivity of G by Lemma 1. 3 of [5]. Hence we have the possibilities listed in the table of the next page.

In Cases (1), (3), (4), (6) and (8), by the equality (*) we have a contradiction. In Case (2), $\lambda+\nu_1+\nu_2=k$ yields $(k-1)/2+(k-1)\leqq k$, that is, $k\leqq 3$, which is impossible. Similarly, in Case (5), it follows that $(k-1)/4+(k-1)\leqq k$, that is, $k\leqq 5$. Since $\lambda=(k-1)/4$ must be an integer, $k=5$. Thus, in Case (5) we have $k=5$. In Case (7), by the equality (*), $k=7$ follows necessarily.

Next, we examine the both Cases (5) and (7).

Case (5): Firstly we show that $\Gamma_2(a)$ is identified with the set of all unordered pairs of $\Gamma_1(a)$. In fact, since $\mu_{32}^{(1)}=\mu_2=2$, for every point x in

Case	μ	$\lambda_1=l\mu/k$	$\lambda=(k-1-\lambda_1)/2$	μ_2	$\nu_1=l\mu_2/k$	$\mu_1=k-(\mu+\mu_2)$	$\lambda'=l\mu_1/k$	$\mu'=l-1-2\mu_1$
(1)	0	0	$(k-1)/2$	1	$(k-1)/2$	$k-1$	$(k-1)^2/2$	$(k-1)(k-4)/2-1$
(2)				2	$k-1$	$k-2$	$(k-1)(k-2)/2$	$(k-2)(k-3)/2$
(3)	1	$(k-1)/2$	$(k-1)/4$	0	0	$k-1$	$(k-1)^2/2$	$(k-1)(k-4)/2-1$
(4)				1	$(k-1)/2$	$k-2$	$(k-1)(k-2)/2$	$(k-2)(k-3)/2$
(5)				2	$k-1$	$k-3$	$(k-1)(k-3)/2$	$(k^2-5k+10)/2$
(6)	2	$k-1$	0	0	0	$k-2$	$(k-1)(k-2)/2$	$(k-2)(k-3)/2$
(7)				1	$(k-1)/2$	$k-3$	$(k-1)(k-3)/2$	$(k^2-5k+10)/2$
(8)				2	$k-1$	$k-4$	$(k-1)(k-4)/2$	$(k^2-5k+14)/2$

$\Gamma_2(a)$ we may set $\Gamma_1(a)\cap\Gamma_3(x)=\{x_1, x_2\}$. Also, since $\mu_{11}^{(1)}=\mu_{13}^{(1)}=\lambda=1$ and $\mu_{12}^{(1)}=\mu=1$, it follows that $\Gamma_1(x_1)\cap\Gamma_1(x_2)=\{x\}$. Hence we see easily that the mapping $x\mapsto\{x_1, x_2\}$ is a bijection from $\Gamma_2(a)$ onto the set of all unordered pairs of $\Gamma_1(a)$ and they are identified since $x^g\mapsto\{x_1^g, x_2^g\}$ for all $g\in G_a$. Next, let x_1 be an element of $\Gamma_1(a)$. Since $|\Gamma_1(x_1)\cap\Gamma_1(a)|=\mu_{11}^{(1)}=\lambda=1$ and $|\Gamma_2(x_1)\cap\Gamma_1(a)|=\mu_{11}^{(2)}=\lambda_1=2$, we may take elements x_2, x_3 of $\Gamma_1(a)$ such that $x_2\in\Gamma_1(x_1)$ and $x_3\in\Gamma_2(x_1)$. Let x, y be the elements of $\Gamma_2(a)$ corresponding to $\{x_1, x_2\}$, $\{x_1, x_3\}$, respectively, and let g be an element of G_a with $x^g=y$. Then $\{x_1^g, x_2^g\}=\{x_1, x_3\}$, which is a contradiction[1] since $x_2^g\in\Gamma_1(x_1^g)$ and $x_3\in\Gamma_2(x_1)$. Thus Case (5) cannot occur.

Case (7): As in Case (5), $\Gamma_2(a)$ is identified with the set of all unordered pairs of $\Gamma_1(a)$. In fact, since $\mu_{12}^{(1)}=\mu=2$, for every $x\in\Gamma_2(a)$ we may set $\Gamma_1(x)\cap\Gamma_1(a)=\{x_1, x_2\}$. Also, since $\mu_{31}^{(3)}=\mu_{33}^{(3)}=\lambda=0$ and $\mu_{32}^{(3)}=\mu=2$, that is $\mu_{3*}^{(3)}\leqq 2$, we have $\Gamma_3(x_1)\cap\Gamma_3(x_2)=\{a, x\}$ and the mapping $x\mapsto\{x_1, x_2\}$ gives a bijection from $\Gamma_2(a)$ onto the set of all unordered pairs of $\Gamma_1(a)$ and they are identified. Next, let g by any element of G_a fixing all the points of $\Gamma_1(a)$. From the above, g fixes $\Gamma_2(a)$ pointwise. Further, by Proposition 3. 1. (*a*) of Quirin [7] g also fixes $\Gamma_3(a)$ pointwise. Thus G_a acts faithfully on $\Gamma_1(a)$. Hence the following hold.

I. If $G_a^{\Gamma_1(a)}$ is not doubly transitive, then $|G_a|=7$, 14 or 21.

II. If $G_a^{\Gamma_1(a)}$ is doubly transitive, then G_a is isomorphic to one of the following groups: (i) the Frobenius group of order 42, (ii) $PSL(3, 2)$, (iii) A_7, (iv) S_7.

1 The author would like to thank Mr. H. Enomoto for pointing out this contradiction and the improvement of his original statement.

In Case I, clearly $|G_a|=7$ or 14 cannot occur since $|\Gamma_2(a)|=21$ must divide $|G_a|$. In case $|G_a|=21$, since $|G|=(1+7+21+7)\cdot 21=36\cdot 21$, G is not simple and let N be a minimal normal subgroup of G. Since 36 is not a power of a prime, N is not solvable and $|N|=36\cdot 7$. N is characteristically simple and $|N|$ contains the prime 7 to the first power only, N must be simple. But this is impossible from the order of N. Thus Case I cannot occur.

Subcase (i) of Case II may be eliminated as follows. By the same reason as above, G is not simple and a minimal normal subgroup N of G must be simple. Thus $|N|=36\cdot 2\cdot 7$ and N is isomorphic to $PSL(2, 8)$. Therefore we see that G is isomorphic to the automorphism group of $PSL(2, 8)=P\Gamma L(2, 8)$. $G=P\Gamma L(2, 8)$ acts naturally on the projective line L over the finite field $GF(8)$ and let $G_{\alpha\beta}$ be the pointwise stabilizer of two points α, β of L. Up to conjugacy, there exists uniquely the subgroup of G with index 36, which is the normalizer of $G_{\alpha\beta}$ for some $\alpha, \beta \in L$. But, we see that G acting by conjugation on $\{G_{\alpha\beta};\ \alpha\neq\beta\in L\}$ has rank 3 and subdegrees 1, $|\{G_{\alpha\beta};|\{\alpha, \beta\}\cap\{\alpha_0, \beta_0\}|=1\}|=14$ and $|\{G_{\alpha\beta};|\{\alpha, \beta\}\cap\{\alpha_0, \beta_0\}|=0\}|=21$ (where α_0, β_0 are the fixed two points of L). Thus subcase (i) cannot occur.

In subcase (ii), if G is not simple, a minimal normal subgroup of G is of order $2^2\cdot 3^2$ and solvable, but $2^2\cdot 3^2$ is not a power of a prime. Thus G is simple and $|G|=2^2\cdot 3^2\cdot|PSL(3, 2)|=|PSU(3, 3^2)|$. Hence, by Brauer [2] G is isomorphic to $PSU(3, 3^2)$.

Subcases (iii) and (iv) cannot occur by Bannai [1] or by the fact that there exist no simple groups of order $2^2\cdot 3^2\cdot|A_7|$ and $2^2\cdot 3^2\cdot|S_7|$ (e.g., Hall [3]). Thus the theorem is proved.

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