# On the Kuramochi boundary of a subsurface of a Riemann surface

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## Introduction

Z. Kuramochi [4] considered a compactification of a subsurface $G$ of a Riemann surface $R$, which is similar to the Kuramochi compactification of $R$. In [4], he introduced the function-theoretic mass for $F_0SH$ functions on $G$ and showed that it is important to investigate the properties of the compactification of $G$. Then, on the subsurface $G$, there are two topologies: one of them is the topology on the compactification of $G$ and the other is the induced topology on $G$ by the Kuramochi compactification of $R$. Z. Kuramochi [3] investigated the relations of these two topologies (Theorem A and B).

In this paper, we shall give some properties of the function-theoretic mass (Proposition 2 and Theorem 1) and that the $F_0H$ function with finite function- theoretic mass is represented by the canonical measure (Theorem 2). In §5, we shall study the relation of the above two topologies (Theorem 4, 5 and 6).

## §1. Notation and terminology

Let $R$ be a hyperbolic Riemann surface. We call a closed or open subset $A$ of $R$ regular if the relative boundary $\partial A$ of $A$ consists of at most a countable number of analytic arcs clustering nowhere in $R$. We fix a closed disk $K_0$ in $R$ and a regular subdomain $G$ of $R$ such that $K_0 \cap G = \phi$. Let $R_0 = R - K_0$. An exhaustion of $R$ will mean an increasing sequence $\{R_n\}$ of relatively compact domains on $R$ such that $\bigcup_{n=1}^{\infty} R_n = R$ and each $\partial R_n$ consists of finite number of closed analytic Jordan curves. We denote by $\{G_n\}$ an exhaustion of $G$.

## §2. $G-F_0SH$ function

We follow [1] for the definition and properties of Dirichlet functions. Let $f$ be a continuous Dirichlet function on $R$ with $f=0$ on $R-G$ and $F$ be a regular closed snbset of $G$. Then there is a uniquely determined Dirichlet function $f^F$ on $R$ which minimizes the Dirichlet norm $\|g\|$ among Dirichlet functions $g$ such that $g=f$ on $F\cup(R-G)$ and which is equal to $f$ on $F\cup(R-G)$ and is harmonic in $G-F$ (Dirichlet principle). If there is a Dirichlet function $f$ on $R$ such that $f=0$ on $R-G$ and $=1$ on $F$, then $f^F$ exists and does not depend on the choice of such $f$. Thus we denote it by $\omega_F=\omega(\partial F, z, G-F)$.

There exists a uniquely determined function $N(z,p)$ $(z, p\in G)$ which has the following conditions (cf. [4]):

a) *$N(z,p)-G(z,p)$ is harmonic in $z\in G$ for each $p\in G$, where $G(z,p)$ is the Green function of $G$.*

b) $N(z,p)=N(p,z)$.

c) $\lim_{z\to\partial G} N(z,p)=0$.

d) *If $K$ be a regular compact set in $G$ which contains $p$ in its interior, then $N(\cdot,p)^K(z)=N(z,p)$. (We set $N(z,p)=0$ on $R-G$).*

e) $\|\min(N(z,p), M)\|^2=2\pi M$ *for any* $M>0$.

We call $N(z,p)$ the *N-function* of $G$. We denote by $L(z,p)$ the $N$-function of $R_0=R-K_0$. As a usual manner in [4], we have the Kuramochi compactification $G^*$ of $G$ and $(z,p)\to N(z,p)$ is extended continuously over $G\times G^*$. In $\Delta^G=G^*-G$, there is only one point $p_0$ such that $N(z,p_0)\equiv 0$. We note that $G^*$ is metrizable. The properties of the Kuramochi compactification $R^*$ of $R$ are found in [1], [2] and [5]. For any non-negative measure $\mu$ on $G^*$ (resp. $R_0^*=R^*-K_0$), we define *N-potential* (resp. *L-potential*) by $N_\mu(z)=\int N(z,p)d\mu(p)$ (resp. $L_\mu(z)=\int L(z,p)d\mu(p)$).

Let $V(z)$ be a non-negative continuous function in $R$ with $V>0$ on $G$ and $V=0$ on $R-G$ such that $V_M(z)=\min(V(z), M)$ is a Dirichlet function for any $M>0$. For any regular compact set $K$ in $G$, we define $V_K(z)=V_K^G(z)$ by increasing limit of $(V_M)^K(z)$ as $M\to\infty$. If $V_K(z)\leq V(z)$ for any regular compact set $K$, then $V(z)$ is called a $G-\mathrm{F_0SH}$ *function*. Any $G-\mathrm{F_0SH}$ function is superharmonic in $G$. If, in addition, $G-\mathrm{F_0SH}$ function is harmonic in $G$, it is called a $G-\mathrm{F_0H}$ *function*. Let $N_\mu$ be a continuous $N$-potential which $\min(N_\mu(z), M)$ is Dirichlet function for any $M>0$. Then $N_\mu$ is a $G-\mathrm{F_0SH}$ function (cf. [5]).

Let $V$ be a $G-\mathrm{F_0H}$ function. For any regular closed subset $F$ of $G$, we define $V_F$ by an increasing limit of $V_{K_n}$, where $K_n=F\cap(R_n\cup\partial R_n)$. Let $F\subset F'$. Then $V_F\leq V_{F'}$ and $(V_F)_{F'}=V_F$. For any closed subset $A$ of $\Delta^G$, we set $A(m)=\left\{z\in G \mid d(z,A)\leq\dfrac{1}{m}\right\}$, where $d$ is a metric on $G^*$. Then there exist a decreasing sequence of closed neighbourhoods of $A$ in $G^*$ such that each of their intersection $\{A^{(m)}\}$ with $G$ is a regular closed set in $G$ and $A(\mathrm{m})\subset A^{(m)}\subset A(m-1)$ for each $m$. We define $V_A$ by decreasing limit of

$V_{A(m)}$. Let $V(z)$ be a Dirichlet finite $G-F_0H$ function. If $\bar{F}\not\ni p_0$ ($\bar{F}$ is the closure of $F$ on $G^*$) and $A\not\ni p_0$, then $\|V_{K_n}-V_F\|\to 0$ as $n\to\infty$, $V_F(z)=V^F(z)$ and $\|V_{A(m)}-V_A\|\to 0$ and $n\to\infty$ respectively. (cf. [5])

LEMMA 1. ([2], *cf. Fuglede's lemma*). *Let $f$ be a non-negative Dirichlet function on $R$ such that $f=0$ on $R-G$, and $f_n$ be the harmonic function in $G\cap R_n-F$, which is equal to zero $\partial G\cap(R_n\cup\partial R_n)$ and to $f$ on $\partial F\cap(R_n\cup\partial R_n)$ and whose normal derivative vanishes everywhere on the rest of the boundary. If $\inf_{z\in F} f(z)>0$, then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that*

$$\lim_{k\to\infty}\int_{\partial F_M\cap R_{n_k}}\frac{\partial}{\partial\nu}f_{n_k}ds=\int_{\partial F_M}\frac{\partial}{\partial\nu}f^F ds$$

*for almost all $M$, where $F_M=\{z\in G|f^F(z)\geq M\}$ $(0<M<\inf_{z\in F} f(z))$.*

COROLLARY 1. ([2], [6]). *Let $F$ be a regular closed set of $G$. If $\omega_F$ exist, then $\int_{\partial F_\alpha}\frac{\partial}{\partial\nu}\omega_F ds=\|\omega_F\|^2$ for almost all $\alpha$, where $F_\alpha=\{z\in G|\omega_F(z)\geq\alpha\}$* $(0<\alpha<1)$.

COROLLARY 2. ([4]). *Let $V$ be a $G-F_0$SH function and $F_M=\{z\in G|V(z)\geq M\}$. If $M_1>M_2$, then $M_1\|\omega_{F_{M_1}}\|^2\leq M_2\|\omega_{F_{M_2}}\|^2$. If $V=V_{F_{M_1}}$, then $M_1\|\omega_{F_{M_1}}\|^2=M\|\omega_{F_M}\|^2$ for any $M$: $0<M<M_1$.*

PROOF. Let $M_1>M_2$. Set $F'=\{z\in G|V_{F_{M_1}}(z)\geq M_2\}$. Then $\omega_{F_{M_1}}=\frac{M_2}{M_1}\omega_{F'}$ on $G-F'$ and $\omega_{F'}\leq\omega_{F_{M_2}}$. By COROLLARY 1, there is a $t_0$: $0<t_0<\frac{M_2}{M_1}$ such that

$$\begin{aligned}M_1\|\omega_{F_{M_1}}\|^2&=M_1\int_{\{\omega_{F_{M_1}}=t_0\}}\frac{\partial}{\partial\nu}\omega_{F_{M_1}}ds=M_2\int_{\{\omega_{F_{M_1}}=t_0\}}\frac{\partial}{\partial\nu}\omega_{F'}ds\\&=M_2\int_{\left\{\omega_{F'}=\frac{M_1}{M_2}t_0\right\}}\frac{\partial}{\partial\nu}\omega_{F'}ds=M_2\|\omega_{F'}\|^2\geq M_2\|\omega_{F_{M_2}}\|^2\end{aligned}$$

If $V=V_{F_{M_1}}$ then $\omega_{F'}=\omega_{F_{M_2}}$. Hence we have $M_1\|\omega_{F_{M_1}}\|^2=M_2\|\omega_{F_{M_2}}\|^2$ for any $M_2$: $0<M_2<M_1$.

## § 3. Function-theoretic mass

Let $V(z)$ be $G-F_0$SH function. Set $F_M=\{z\in G|V(z)\geq M\}$ for any $M>0$. By COROLLARY 2 of LEMMA 1, $M\|\omega_{F_M}\|^2$ increases as $M\to 0$. We define

$\mathfrak{M}(V)=\mathfrak{M}^G(V)=\lim_{M\to 0}\frac{M}{2\pi}\|\omega_{F_M}\|^2$, and call it the function-theoretic mass of $V(z)$ ([4]). If a $G-F_0$SH function $V$ satifies $\mathfrak{M}(V)<\infty$, then we say that $V$ is of potential type.

PROPOSITION 1. (i) *Let $V_i$ $(i=1,2)$ be $G-F_0SH$ functions. If $V_1\leq V_2$ on $G$, then $\mathfrak{M}(V_1)\leq\mathfrak{M}(V_2)$.*

(ii) *Let $V$ be a* $G-F_0H$ *function and $F$ be a regular closed set in $G$ such that $\bar{F}\not\ni p_0$. Then $\inf_{z\in F}V(z)>0$ and $\mathfrak{M}(V_F)=\frac{M}{2\pi}\|\omega_{F_M}\|^2_{G-F_M}<\infty$ for any $0<M<\inf_{z\in F}V(z)$.*

PROOF. (i) Obvious from the definition.

(ii) If $p_0\notin F$, then there is a $q\in G$ and $\delta>0$ such that $F\subset\{z\in G|N(z,q)>\delta\}$. Then $\omega_F$ exists. Let $K$ be a regular compact set in $G$ which contains $q$ in its interior, and set $\min_{z\in\partial K}V_F(z)=\alpha>0$, $\max_{z\in\partial K}N(z,q)=\beta<+\infty$. By $\frac{\alpha}{\beta}N(z,q)\leq V_F(z)$ on $\partial K$, we have $\frac{\alpha}{\beta}N(z,q)=\frac{\alpha}{\beta}N(\cdot,q)_K(z)\leq(V_F)_K(z)\leq V_F(z)$ on $G-K$. Hence $V_F(z)\geq\frac{\alpha}{\beta}\min(\delta,\alpha)>0$ on $F$. Set $\inf_{z\in F}V(z)=M_0>0$. Then $V_F(z)=(V_F)_{F_{M_0}}(z)$. By COROLLARY of LEMMA 1, we have $M\|\omega_{F_M}\|^2=M_0\|\omega_{FM_0}\|^2$ for any $M$: $0<M\leq M_0$. Hence we obtain $\mathfrak{M}(V_F)=\lim_{M\to 0}M\|\omega_{F_M}\|^2=\frac{M}{2\pi}\|\omega_{F_M}\|^2$ for any $M$ $(0<M\leq M_0)$.

COROLLARY. *Let $K$ be a regular compact set in $G$. Then $\mathfrak{M}(V_K)=\frac{M}{2\pi}\|\omega_{F_M}\|^2<\infty$ for any $M$: $0<M\leq\min_{z\in\partial K}V(z)$.*

LEMMA 2. *Let $V$ be a* $G-F_0$H *function and $K$ be a regular compact set in $G$. Then $\mathfrak{M}(V_K)=\frac{1}{2\pi}\int_{\partial K}\frac{\partial}{\partial\nu}V_K ds$.*

PROOF. We may assume $K\subset G\cap R_1$. Let $V_n$ be the harmonic function in $G\cap R_n-K$, which is equal to zero on $\partial G\cap(R_n\cup\partial R_n)$, to $V$ on $\partial K$ and whose normal derivative vanishes everywhere of the rest of the boundary. Fix $M_1$: $0<M_1<\min_{z\in\partial K}V(z)$. By the COROLLARY of PROPOSITION 1, COROLLARY 1 of LEMMA 1 and LEMMA 1, there is a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ and some $M_0$: $0<M_0<M_1$ such that

$$\mathfrak{M}(V_K)=\frac{M_1}{2\pi}\|\omega_{F_{M_1}}\|^2=\frac{1}{2\pi}\int_{\partial F_{M_0}}\frac{\partial}{\partial\nu}V_K ds=\lim_{k\to\infty}\frac{1}{2\pi}\int_{\partial F_{M_0}\cap R_{n_k}}\frac{\partial}{\partial\nu}V_{n_k}ds.$$

By Green's formula, $\int_{\partial F_{M_0}\cap R_{n_k}} \frac{\partial}{\partial\nu} V_{n_k} ds = \int_{\partial K} \frac{\partial}{\partial\nu} V_{n_k} ds$.

Then we have $\mathfrak{M}(V_K)=\lim_{k\to\infty} \frac{1}{2\pi}\int_{\partial K} \frac{\partial}{\partial\nu} V_{n_k} ds = \frac{1}{2\pi}\int_{\partial K} \frac{\partial}{\partial\nu} V_K ds$.

LEMMA 3. *Let $V$ be a $G-F_0H$ function. Then* $\lim_{n\to\infty}\mathfrak{M}(V_{\bar{G}_n})=\mathfrak{M}(V)$.

PROOF. Since $V_{\bar{G}_n}\leq V$, we have $\lim_{n\to\infty}\mathfrak{M}(V_{\bar{G}_n})\leq\mathfrak{M}(V)$ by (i) of PROPOSITION 1. Then we have to prove inverse inequality. For any given $T$: $0<T<\mathfrak{M}(V)$, there exists $M>0$ such that $T<\frac{1}{2\pi M}\|V_{F_M}\|^2_{G-F_M}$, where $F_M =\{z\in G \mid V(z)\geq M\}$. Set $K_n=F_M\cap\bar{G}_n$ and $F_M^{(n)}=\{z\in G \mid V_{K_n}(z)\geq M\}$. Then $\inf_{z\in K_n} V(z)\geq M$ and $K_n\subset F_M^{(n)}\subset F_M$. By COROLLARY of PROPOSITION 1, we have $\mathfrak{M}(V_{K_n})=\frac{1}{2\pi M}\|V_{K_n}\|^2_{G-F_M^{(n)}}\geq\frac{1}{2\pi M}\|V_{K_n}\|^2_{G-F_M}$. Since $V_{K_n}$ converges to $V_{F_M}$ locally uniformly on $G-F_M$, we have $\lim_{n\to\infty}\|V_{K_n}\|^2_{G-F_M}\geq\|V_{F_M}\|^2_{G-F_M}> 2\pi MT$. Then $\lim_{n\to\infty}\mathfrak{M}(V_{\bar{G}_n})\geq\lim_{n\to\infty}\mathfrak{M}(V_{K_n})>T$. Hence $\lim_{n\to\infty}\mathfrak{M}(V_{\bar{G}_n})\geq\mathfrak{M}(V)$.

PROPOSITION 2. *Let $V_n$ $(n=1,2,\cdots)$ and $V$ be $G-F_0H$ functions.*

(i) $\mathfrak{M}(aV_1+V_2)=a\mathfrak{M}(V_1)+\mathfrak{M}(V_2)$ *for any positive constant $a>0$.*

(ii) *If $V_n$ converges to $V$ locally uniformly, then* $\underline{\lim}_{n\to\infty}\mathfrak{M}(V_n)\geq\mathfrak{M}(V)$.

(iii) *If $\{V_n\}$ be increasing sequence and $\lim_{n\to\infty}V_n(z)=V(z)$, then* $\lim_{n\to\infty}\mathfrak{M}(V_n) =\mathfrak{M}(V)$.

(iv) *If $p\in G$, then $\mathfrak{M}(N(\cdot,p))=1$. If $q\in\Delta^G$, then $\mathfrak{M}(N(\cdot,q))\leq 1$.*

(v) *$\mathfrak{M}(N(\cdot,q))$ is lower semi-continuous on $G^*$.*

PROOF. (i) By LEMMA 2, $\mathfrak{M}((aV_1+V_2)_{\bar{G}_n})=a\mathfrak{M}((V_1)_{\bar{G}_n})+\mathfrak{M}(V_2)$. By LEMMMA 3, as $n\to\infty$, we have $\mathfrak{M}(aV_1+V_2)=a\mathfrak{M}(V_1)+\mathfrak{M}(V_2)$.

(ii) Since $V_n$ converges to $V$ uniformly on $\partial G_k$, we have $\lim_{n\to\infty}\mathfrak{M}((V_n)_{\bar{G}_k}) =\mathfrak{M}(V_{\bar{G}_k})$ by LEMMA 2. Then $\underline{\lim}_{n\to\infty}\mathfrak{M}(V_n)\geq\mathfrak{M}(V_{\bar{G}_k})$ for any $k$. As $k\to\infty$, $\underline{\lim}_{n\to\infty}\mathfrak{M}(V_n)\geq\mathfrak{M}(V)$ by LEMMA 3.

(iii) Since $V_n$ are harmonic in $G$, $V_n$ converges to $V$ locally uniformly. Then, by above (ii) and (i) of PROPOSITION 1, $\lim_{n\to\infty}\mathfrak{M}(V_n)=\mathfrak{M}(V)$.

(iv) Use properties (e) and (d) of $N$-function and above (ii).

(v) Obvious from (iv).

REMARK 1. (i) *Let $V$ be a $G-F_0H$ function and $K$ be a regular compact set in $G$. Then there is a unique measure $\mu$ supported by $K$ such that $V_K=N_\mu$* ([1], [2] and [5]). By LEMMA 2 we have $\mathfrak{M}(V_K)=\mu(K)$.

(ii) *Let* $G=R_0$. *Then any* $R_0-F_0H$ *function* $V(z)$ *has an L-potential representation* (*i.e. there is a measure* $\mu$ *on* $\Delta(=\Delta^{R_0})$ *such that* $V=L_\mu$) *and furthermore* $V$ *satisfies* $\mathfrak{M}^{R_0}(V)=\frac{1}{2\pi}\int_{\partial K_0}\frac{\partial}{\partial\nu}Vds=\mu(\Delta)<\infty$.

PROPOSITION 3 ([4]). *If V is of potential type, then V is N-potential.*

PROOF. Let $\mu_n$ be the associated measure on $\partial G_n$ with $V_{\bar{G}_n}=N_{\mu_n}$. By REMARK 1, $\mu_n(\partial G_n)=\mathfrak{M}(V_{\bar{G}_n})\leq\mathfrak{M}(V)<\infty$ for any $n$. Then there is a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ and some measure $\mu$ on $\Delta$ such that $\mu_{n_k}\to\mu$ (vague) as $k\to\infty$. Then $V(z)=N_\mu(z)$ and $\mu(\Delta^G)=\lim_{k\to\infty}\mu_{n_k}(\partial G_{n_k})=\lim_{n\to\infty}\mathfrak{M}(V_{\bar{G}_n})=\mathfrak{M}(V)$ by LEMMA 3.

COROLLARY 1. *Let V be a* $G-F_0H$ *function and F be a regular closed subset of G. If* $\inf_{z\in F}V(z)>0$, *then there is a measure* $\mu$ *on* $\bar{F}$ *such that* $V_F=N_\mu$ and $\mathfrak{M}(V_F)=\mu(\bar{F})$.

PROOF. By Proposition 1, $\mathfrak{M}(V_F)<\infty$. Then use Theorem 14 in [5] and (iii) of PROPOSITION 2.

COROLLARY 2. *Let V be a* $G-F_0H$ *function and A be a closed set in* $\Delta^G$ *with* $p_0\notin A$. *Then there is a measure* $\mu$ *on A such that* $V_A=N_\mu$ *and* $\mu(A)=\lim_{n\to\infty}\mathfrak{M}(V_{A(n)})$.

PROOF. By $A\not\ni p_0$, we may assume $A^{(1)}\not\ni p_0$. Then $\mathfrak{M}(V_{A(1)})<\infty$ by PROPOSITION 1. Since $V_A\leq V_{A(1)}$, we obtain $\mathfrak{M}(V_A)<\infty$. Then, by same method of the proof of PROPOSITION 3, we see $\mu(A)=\lim_{n\to\infty}\mathfrak{M}(V_{\bar{G}_n})$.

THEOREM 1. *Let* $N_\mu$ *a* $G-F_0H$ *function. Then*

$$\mathfrak{M}(N_\mu)=\int_{\Delta^G}\big(N(\cdot,p)\big)d\mu(p).$$

PROOF. Let $\mu_{p,n}$ be the associated measure on $\partial G_n$ with $N(\cdot,p)_{\bar{G}_n}(z)=N_{\mu_{p,n}}(z)$ for any $p\in\Delta^G$ and $G_n$. Then there is a measure $\nu_n$ on $\partial G_n$ such that $\nu_n=\int\mu_{p,n}d\mu(p)$ (i.e. $\int fd\nu_n=\int\int fd\mu_{p,n}d\mu(p)$ for any $f\in C(\Delta)$) (see p 297 in [5]). Then $(N_\mu)_{\bar{G}_n}(z)=N_{\nu_n}(z)$. By REMARK 1, we have $\mathfrak{M}((N_\mu)_{\bar{G}_n})=\nu_n(\partial G_n)=\int\int 1\,d\mu_{p,n}d\mu(p)=\int\mathfrak{M}(N(\cdot,p)_{\bar{G}_n})d\mu(p)$. As $n\to\infty$, we obtain $\mathfrak{M}(N_\mu)=\int\mathfrak{M}(N(\cdot,p)d\mu(p)$ by LEMMA 3.

COROLLARY 1. *For any regular closed subset F of G,*

$$\mathfrak{M}\left((N_\mu)_F\right)=\int\mathfrak{M}\left(N(\cdot,p)_F\right)d\mu(p)$$

PROOF. By Theorem 15, in [5], $(N_\mu)_F(z)=\int N(\cdot,p)_F(z)d\mu(p)$. Set $F_n=F\cap\bar{G}_n$. Let $\nu_{p,n}$ be the associated measure with $N(\cdot,p)_{F_n}=N_{\nu_{p,n}}$. By a way similar to the proof of THEOREM 1, we have $\mathfrak{M}((N_\mu)_{F_n})=\int\int 1 d\nu_{p,n}d\mu(p)=\int\mathfrak{M}(N(\cdot,p)_{F_n})d\mu(p)$. Then, as $n\to\infty$, $\mathfrak{M}((N_\mu)_F)=\int\mathfrak{M}(N(\cdot,p)_F)d\mu(p)$ by LEMMA 3.

COROLLARY 2. $\mathfrak{M}(N_\mu)\leq\mu(\Delta)$.

PROOF. By (iv) of PROPOSITION 2, $\mathfrak{M}(N_\mu)=\int_\Delta\mathfrak{M}(N(\cdot,p))d\mu(p)\leq\mu(\Delta)$.

LEMMA 4. *Let $V$ be a $G-F_0H$ function and $A$ be a closed set in $\Delta^G$ with $A\not\ni p_0$. If $(V_A)_A=V_A$, then* $\lim_{n\to\infty}\mathfrak{M}(V_{A(n)})=\mathfrak{M}(V_A)$.

PROOF. By $(V_A)_A=V_A$, we see $(V_A)_{A(n)}=V_A$ for any $n$. We may assume $A_1\not\ni p$. Then $\inf_{z\in A^{(1)}}V_A(z)>0$ by PROPOSITION 1. Fix $M$: $0<M<\inf_{z\in A^{(1)}}V_A(z)$ and set $F_M=\{z\in G\mid V_A(z)\geq M\}$, $F_M^{(n)}=\{z\in G\mid V_{A(n)}(z)\geq M\}$ and $V_n(z)=\min(V_{(n)}(z),M)$. Then, by PROPOSITION 1, $\mathfrak{M}(V_{A(n)})=\dfrac{1}{2\pi M}\|V_{A(n)}\|^2_{G-F_M^{(n)}}$ and $\mathfrak{M}(V_A)=\mathfrak{M}((V_A)_{A(1)})=\dfrac{1}{2\pi M}\|V_A\|^2_{G-F_M}$. Let $n>m$. Since $V_m-V_{A(n)}=0$ on $\partial F_M^{(n)}$, $(V_{A(n)},V_m-V_{A(n)})=0$ by Dirichlet principle. (For Dirichlet function $f$, $g$, we denote by $(f,g)$ the mixed Dirichlet integral). Then

$$\begin{aligned}\|V_n-V_m\|^2_{G-F_M}&=\|V_{A(n)}-V_m\|^2_{G-F_M^{(n)}}\\&=\|V_{A(m)}\|^2_{G-F_M^{(m)}}-\|V_{A(n)}\|^2_{G-F_M^{(n)}}.\end{aligned}$$

Since $\{\|V_{A(n)}\|^2_{G-F_M^{(n)}}\}_{n=1}^\infty$ are decreasing, we have

$$\lim_{n\to\infty}\|V_n-V_A\|^2_{G-F_M^{(n)}}=0.$$

Hence

$$\lim_{n\to\infty}\|V_{A(n)}\|^2_{G-F_M^{(n)}}=\|V_A\|^2_{G-F_M},$$

and

$$\lim_{n\to\infty}\mathfrak{M}(V_{A(n)})=\mathfrak{M}(V_A).$$

COROLLARY. *For the above $V$ and $A$, there is a measure $\mu$ on $A$ such that $V_A(z)=N_\mu(z)$ and $\mathfrak{M}(V_A)=\mu(A)$.*

PROOF. By the above LEMMA and COROLLARY 2 of PROPOSITION 3, we have

$$\mu(A)=\lim_{n\to\infty}\mathfrak{M}(V_{A(n)})=\mathfrak{M}(V_A).$$

REMARK 2. *By Theorem* 19 *and* 20 *in* [5], *we see that* $(N(\cdot,p)_{\{p\}})_{\{p\}}=N(\cdot,p)_{\{p\}}$ *for any* $p\in\Delta^G-\{p_0\}$.

## § 4. Classification of the boundary

We set $\alpha(p)=\mathfrak{M}(N(\cdot,p)_{\{p\}})$ for any $p\in\Delta^G-\{p_0\}$. Then we have $\alpha(p)=1$ or 0 (See Theorem 21 in [5] and Theorem 4 in [4]). We set $\Delta_0^G=\{p\in\Delta^G-\{p_0\}|\alpha(p)=0\}$ and $\Delta_1^G=\Delta^G-\{p_0\}-\Delta_0^G$. We call point in $\Delta_1^G$ a minimal point. Let $\mu$ be a measure on $\Delta^G$. If $\mu(\Delta_0^G\cup\{p_0\})=0$, then we call $\mu$ a canonical measure. We note $\mathfrak{M}(N(\cdot,p))=1$ for any $p\in\Delta_1^G$. Because $1\geq\mathfrak{M}(N(\cdot,p))\geq\mathfrak{M}(N(\cdot,p)_{\{p\}})=1$.

PROPOSITION 4. (i) *$\Delta_0^G\cup\{p_0\}$ is an $F_\sigma$-set.*

(ii) *Let $V$ be a $G-F_0H$ function, and $E$ be a closed set of $\Delta_0^G$. Then $V_E=0$.*

The proof of PROPOSITION 4 is similar to the proofs of Theorem 22 and Theorem 23 in [5], where we replace $\frac{1}{2\pi}\int_{\partial K_0}\frac{\partial}{\partial\nu}Vds$ with $\mathfrak{M}(V)$ and use PROPOSITION 2, COROLLARY of THEOREM 1 and COROLLARY of LEMMA 4. By PROPOSITION 4, we have

PROPOSITION 5. (*cf. Theorem* 24 *and* 25 *in* [5]).

(i) *The measure $\mu$ which is defined in* COROLLARY 2 *of* PROPOSITION 3 *is canonical.*

(ii) *For any closed set $A$ in $\Delta^G-\{p_0\}$ and any $G-F_0H$ function $V(z)$, $(V_A)_A(z)=V_A(z)$.*

THEOREM 2. *Let $V(z)$ be a $G-F_0H$ function with $\mathfrak{M}(V)<\infty$. Then there exist a canonical measure $\mu$ such that $V=N_\mu$.*

PROOF. We shall show that the measure $\mu$ in Proposition 3 is canonical. Let $A$ be a closed subset of $\Delta_0^G$. Fix $m$ and an open set $O$ of $G^*$ such that $A\subset O\subset A^{(m)}$. Let $\nu_k$ be the restriction of $\mu_{n_k}$ to $O$. Then $\mathfrak{M}(N_{\nu_k})=\nu_k(G)$. By $(N_{\nu_k})_{A(m)}=N_{\nu_k}$, $N_{\nu_k}\leq V_{A(m)}$.

Then

$$\mu(A)\leq\mu(0)\leq\varliminf_{k\to\infty}\mu_{n_m}(0)=\varliminf_{k\to\infty}\nu_k(0)$$
$$=\varliminf_{k\to\infty}\mathfrak{M}(N_{\nu_k})\leq\mathfrak{M}(V_{A(m)}).$$

Then by LEMMA 4 and PROPOSITION 4, we have $\mu(A)\leq\lim_{m\to\infty}\mathfrak{M}(V_{A(m)})=\mathfrak{M}(V_A)=0$. Hence $\mu(\Delta_0^G)=0$.

COROLLARY. ([4]). *If $\mu$ is canonical, then $\mathfrak{M}(N_\mu)=\mu(\Delta_1^G)$.*

## § 5. Relation between $\Delta^G$ and $\Delta^{R_0}$

In this section, we denote by $\Delta$ the Kuramochi boundary of $R_0$ and by

$\Delta_1$ the set of all minimal points of $\Delta$. Let $F$ be a closed set of $R$. When $L(\cdot,p)_F(z)\neq L(z,p)$, we call that $F$ is thin at $p$ (cf. p 221 in [1]). We set $\Delta_1(G)=\{p\in\Delta_1|R-G$ is thin at p$\}$. We shall study the relation between $G\cup\Delta_1^G$ and $G\cup\Delta_1(G)$.

THEOREM A (*Kuramochi* [3]). *Let* $q\in\Delta_1^G$ *and* $B(q;R_0)=\{p\in\Delta\,|$ *There is a sequence* $\{z_n\}\subset G$ *such that* $z_n\to p$ *(L-top.) and* $z_n\to q$ *(N-top.)*$\}$. *Then* $B(q;R_0)$ *consists of only one point and* $B(q;R_0)\in\Delta_1(G)$. *We define a mapping* $jG\cup\Delta_1^G$ *into* $G\cup\Delta_1(G)$ *by* $j(z)=z$ *for any* $z\in G$ *and* $j(q)=B(q;R_0)$ *for any* $q\in\Delta_1^G$. *Then* $j$ *is a one to one continuous mapping of* $G\cup\Delta_1^G$ *onto* $G\cup\Delta_1(G)$ *and furthermore* $j$ *satisfies*

$$N(z,q)=L\big(z,j(q)\big)-L\big(\cdot,j(q)\big)_{R-G}(z)$$

*for any* $q\in G\cup\Delta_1^G$.

THEOREM B (*Kuramochi* [3]). *Let* $V(z)$ *be a* $G-F_0H$ *function with* $\mathfrak{M}(V)<\infty$. *Then there exists a* $R_0-F_0H$ *function* $U(z)$ *such that* $V(z)=U(z)-U_{R-G}(z)$.

For continuity of $j^{-1}$, we have THEOREM 3 and 4.

THEOREM 3. *If* $G$ *satisfies*

*Condition* $(I)$: $\quad\bar{G}\cap\overline{R-G}\cap\Delta_0=\phi$,

*then* $j^{-1}$ *is continuous on* $(\bar{G}-\overline{\partial G})\cap\Delta_1$.

PROOF. Let $p\in(\bar{G}-\overline{\partial G})\cap\Delta_1$. Set $B(p,G)=\{q\in\Delta^G\,|$ *There is a sequence* $\{z_n\}\subset G$ *such that* $z_n\to p$ *(L-top.) and* $z_n\to q$ *(N-top.)*$\}$. By LEMMA 3 of [3], $B(p;G)\cap\Delta_1^G$ consists of only one point. Let $z_n\to p$ (*L-top.*), $z_n\to q$ (*N-top.*) and $\mu_n$ be the canonical measure such that $L(\cdot,z_n)_{R-G}(z)=L_{\mu_n}(z)$. Since the support $S_{\mu_n}$ *of* $\mu_n$ contains $\bar{G}\cap\overline{R-G}$ (Theorem 1 in [7]) and $\mu_n\,(\bar{G}\cap\overline{R-G})\leq 1$, there is a subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ and some measure $\mu$ on $\bar{G}\cap\overline{R-G}$ such that $\mu_{n_k}\to\mu$ (vague) as $k\to\infty$. Then $\lim_{n\to\infty}L(\cdot,z_n)_{R-G}(z)$ and $L_\mu(z)$ are superharmonic in $G$. Since $R\cap\bar{G}\cap\overline{R-G}=\partial G$ and $\partial G$ is locally Lebesgue measure 0, we have $\lim_{n\to\infty}L(\cdot,z_n)_{R-G}(z)=L_\mu(z)$ except for of locally Lebesgue measure 0. Hence we have $\lim_{n\to\infty}L(\cdot,z_n)_{R-G}(z)=L_\mu(z)$. By Lemma 1 of [3], there is a constant $c$ $(0\leq c\leq 1)$ such that $L_\mu(z)=cL(z,p)+(1-c)\;L(\cdot,p)_{R-G}(z)$. By assumption, $S_\mu\subset\bar{G}\cap\overline{R-G}\subset R_0\cup\Delta_1$ and $\mu$ is canonical. Then, by the uniqueness of the canonical measure and $\{p\}\cap\overline{R-G}=\phi$, we have $\mu(\{p\})=c=0$ and $\lim_{n\to\infty}L(\cdot,z_n)_{R-G}(z)=L(\cdot,p)_{R-G}(z)$. Hence $N(z,q)=L(z,p)-L(\cdot,p)_{R-G}(z)$ for any $q\in B(p;G)$. Then B$(p;G)$ consists of only one point and $B(p;G)\in\Delta_1^G$. Then we have $j^{-1}(p)=B(p;G)$ for any $p\in(\bar{G}-\overline{\partial G})\cap\Delta_1$ and $j^{-1}$ is continuous on $(\bar{G}-\overline{\partial G})\cap \Delta_1$.

THEOREM 4. *Let* $p\in(\bar{G}-\overline{\partial G})\cap \Delta_1$. *If*

*Condition (II):* $\varlimsup_{z\to p} L(\cdot, z)_{R-G}(z)<\infty$

*is satisfied, then* $j^{-1}$ *is continuous at* $p$.

PROOF. We show that the $\mu$ in THEOREM 3 is canonical. We denote by $\|\mu_n\|$ the energy of $\mu_n$. ($\|\mu_n\|^2=\int L_{\mu_n}d\mu_n$)

$$\|\mu_n\|^2=\int L_{\mu_n}d\mu_n=\int L(\cdot, z_n)_{R-G}(z)d\mu_n(z)$$
$$\leq\int L(z, z_n)d\mu_n(z)=L_{\mu_n}(z_n)=L(\cdot, z_n)_{R-G}(z_n).$$

By assumption, $\{\|\mu_n\|\}$ are bounded. Then, by Satz 17. 4 in [1], we have that $\mu$ is canonical measure. Hence, on the analogy of the proof of THEOREM 3, we complete the proof.

Let $V(z)$ be a $G-F_0H$ function and $K$ be a regular compact set in $G$. Then $V_K$ is extended continuously on $G\cup\Delta^G$. C. Constantinescu and A. Cornea [1] defined the value of $V$ on $\Delta^G$ by $V(p)=\sup_K V(p)=\lim_{K\nearrow G} V_K(p)$ for any $p\in\Delta^G$. Then, by Satz 17.2 of [1], we have $V(p)=\varlimsup_{G\ni z\to p} V(z)$ for any $p\in\Delta_1^G$.

THEOREM 5. *Let* V *be a* $G-F_0H$ *function with* $\mathfrak{M}(V)<\infty$ *and* $p\in(\bar{G}-\overline{\partial G})\cap\Delta_1$. *If* $j^{-1}$ *is continuous at* $p$, *then for* $U(z)$ *which satisfies the condition in* THEOREM *B and* $U_{R-G}(p)<\infty$, *we have* $V(j^{-1}(p))=U(p)-U_{R-G}(p)$.

PROOF. By [3], $V_{\bar{G}_n}(z)+U_{R-G}(z)=U_{G_n\cup(R-G)}(z)$. Let $p\in(\bar{G}-\overline{\partial G})\cap\Delta_1$. $j$ and $j^{-1}$ are continuous at $p$ and $j^{-1}(p)$ respectively. Then for any $\frac{1}{n}$-neighbourhood $D\left(j(p),\frac{1}{n}\right)$ of $j(p)$ $\left(\text{resp. } \frac{1}{n}\text{-neighbourhood } D\left(p,\frac{1}{n}\right) \textit{ of } p\right)$, there is a $\frac{1}{m}$-neighbourhood $D\left(p,\frac{1}{m}\right)$ of $p$ $\left(\text{resp. } \frac{1}{m}\text{-neighbourhood}\right.$ $\left.D\left(j^{-1}(p),\frac{1}{m}\right)\right)$ such that $D\left(p,\frac{1}{m}\right)\cap G\subset D\left(j(p),\frac{1}{n}\right)\cap G$ $\left(\text{resp. } D\left(j(p),\frac{1}{m}\right)\cap G\subset D\left(p,\frac{1}{n}\right)\cap G\right)$. Then

$$\varlimsup_{G\ni z\to j^{-1}(p)}\left(V^G_{\bar{G}n}(z)+U_{R-G}(z)\right)=\varlimsup_{G\ni z\to p} U_{\bar{G}_n\cup(R-G)}(z).$$

By

$$\lim_{\bar{G}\ni z\to j^{-1}(p)} V^{G}_{\bar{G}_n}(z) = \lim_{\bar{G}\ni z\to j^{-1}(p)} V^{G}_{\bar{G}_n}(z) = V^{G}_{\bar{G}_n}\left(j^{-1}(p)\right)$$

and

$$\lim_{\bar{G}\ni z\to j^{-1}(p)} V_{R-G}(z) = \lim_{\bar{G}\ni z\to p} V_{R-G}(z) = V_{R-G}(p)\,,$$

we have

$$V^{G}_{\bar{G}_n}\left(j^{-1}(p)\right) + U_{R-G}(p) = U_{\bar{G}_n\cup(R-G)}(p)\,.$$

as $n\to\infty$, we obtain $V(j^{-1}(p)) = U(p) - U_{R-G}(p)$.

COROLLARY. *Let $p\in(\bar{G}-\partial\bar{G})\cap\Delta_1$. If Condition (I) or (II) is satisfied, we have $V(j^{-1}(p)) = U(p) - U_{R-G}(p)$.*

The function $V$ in Theorem B is not necessary uniquely determined. But $V$ is uniquely determined in the sense of the following theorem.

THEOREM 6. *Let $V(z)$ be a $G-F_0H$ function and $\mu_i$ $(i=1,2)$ be canonical measures. If $V(z) = L_{\mu_i}(z) - (L_{\mu_i})_{R-G}(z)$ $(i=1,2)$, then $\mu_1|\Delta_1(G) = \mu_2|\Delta_1(G)$. ($\mu_i|\Delta_1(G)$ means the restriction of $\mu$ to $\Delta_1(G)$ of $\mu$). And furthermore $V(z) = L_\nu(z) = (L_\nu)_{R-G}(z)$ on $G$ where $\nu = \mu_i|G$.*

PROOF. Let $\mu_{i,R-G}$ be a canonical measure of $(L_{\mu_i})_{R-G}(z)$. By Folgesatz 16. 4 of [1], $\mu_{i,R-G}$ is a measure on $\{p\in R_0\cup\Delta_1 | L(\cdot,p)_{R-G}(z)\equiv L(z,p)\}$. Hence $\mu_{i,R-G}(G\cup\Delta_1(G)) = 0$. Since $L_{\mu_1}(z) + L_{\mu_2,R-G}(z) = L_{\mu_2}(z) + L_{\mu_1,R-G}(z)$, we have $\mu_1 + \mu_{2,R-G} = \mu_2 + \mu_{1,R-G}$ by the uniqueness of a canonical measure. Then we see $\mu_1|\Delta_1(G) = \mu_2|\Delta_1(G)$. Let $\mu_1|\Delta_1(G) = \nu$ and $\mu_1 - \nu = \nu'$. Then $L_{\mu_1}(z) - (L_{\mu_1})_{R-G}(z) = (L_\nu + L_{\nu'})(z) - (L_\nu + L_{\nu'})_{R-G}(z) = L_\nu(z) - (L_\nu)_{R-G}(z)$.

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