Operational calculus for tensor products of linear operators in Banach spaces*,†

By Takashi ICHINOSE

Introduction

Given two matrices A and B and a polynomial $P(\xi, \eta)=\sum_{j,k} c_{jk}\xi^{j}\cdot\eta^{k}$ (c_{jk} complex numbers) in ξ and η, a polynomial operator $P(A\otimes I, I\otimes B)=\sum_{j,k} c_{jk}A^{j}\otimes B^{k}$ is defined, where I is the identity matrix. Stéphanos proved (see [13]) that the set of the eigenvalues of $P(A\otimes I, I\otimes B)$ coincides with the set of the complex numbers $\sum_{j,k} c_{jk}\alpha^{j}\beta^{k}$ with α eigenvalue of A and β eigenvalue of B.

The aim of the present paper is to extend this result and to prove the spectral mapping theorem for tensor products of densely defined closed linear operators in complex Banach spaces. We develop an operational calculus defined for holomorphic functions of several variables in a neighbourhood of the product of the extended spectra of those operators, which may be considered roughly as the tensor product of the operational calculi defined for holomorphic functions of one variable developed by I. Gelfand, Dunford and Taylor ([15], [3], [20], [8]). Brown and Pearcy [2] have proved for bounded operators A and B on a Hilbert space and $P(\xi, \eta)=\xi\cdot\eta$ that the spectrum of the tensor product $A\otimes B$ is the set of the products of the spectra of A and B. Ichinose [9] has extended it for some unbounded operators A and B in Banach spaces. The result of Stéphanos for several bounded operators in Banach spaces has also been obtained by Schechter [19]. Our results include those of Brown and Pearcy, Schechter and partly Ichinose. The spectral mapping theorem enables us, in particular, to find the resolvent of a polynomial operator defined for tensor products of closed operators, from the knowledge of the spectra of those operators.

Section 1 deals with the basic notions about linear operators, their maximal extensions and their spectra, and tensor products of spaces and operators.

---

* This research was supported in part by the DAAD at University of Frankfurt.

† This paper was primarily received by the editors of the Scripta Mathematica September 1970, accepted by them and transferred to this Hokkaido Mathematical Journal, since that journal does not seem to be able to publish it immediately.

In Section 2, we develop an operational calculus for tensor products of Banach algebras with unit elements. Generalized tensor products are defined by the use of the Cauchy integral for functions holomorphic in a neighbourhood of the product of the spectra of Banach algebra elements. We prove the spectral mapping theorem first for tensor products of operator algebras and then for tensor products of general Banach algebras, by considering a left regular representation of a Banach algebra to some operator algebra.

In Section 3, we extend the previous results and present an operational calculus for tensor products of closed operators in a similar way to that of Taylor [20]. For polynomials of several variables, we define polynomial operators with closable operators and prove their closability and the spectral mapping theorem for polynomials continuous at the product of the extended spectra. In particular, our results give a precise knowledge of the spectrum of the tensor product $A \otimes B$ of two densely defined closable operators in Banach spaces.

In Section 4, we give an application to the initial value problem of a partial differential equation.

The author would like to express his hearty thanks to Professor Gottfried Köthe for his constant encouragement and to Professor G. Trautmann, Dr. N. Adasch and Mr. G. Biebinger for many valuable discussions.

1. Basic notions

1.1. *Linear operators*

We shall recall the definitions and properties of the spectrum and its parts of a linear operator in a normed linear space (see e. g. [3], [11], [9]).

Let X be a complex normed linear space. For a linear operator T: $D[T] \subset X \to X$ with domain $D[T]$ and range $R[T]$ both in X, we define the (resp. *extended*) *resolvent set* $\rho(T)$ (resp. $\rho_e(T)$) and the (resp. *extended*) *spectrum* $\sigma(T)$ (resp. $\sigma_e(T)$) of T which are subsets of the complex plane $\boldsymbol{C}$ (resp. the extended complex plane $\boldsymbol{C}^* = \boldsymbol{C} \cup \{\infty\}$). $\rho(T)$ is the set of all $\lambda \in \boldsymbol{C}$ for which $T - \lambda I$ has a densely defined bounded inverse in X, where I is the identity operator in X. $\sigma(T)$ is the complement in $\boldsymbol{C}$ of $\rho(T)$, and closed in $\boldsymbol{C}$. $\rho_e(T)$ is defined as $\rho(T) \cup \{\infty\}$ if T is densely defined and bounded in X, and otherwise, we set $\rho_e(T) = \rho(T)$. $\sigma_e(T)$ is the complement in $\boldsymbol{C}^*$ of $\rho_e(T)$, and a non-empty compact subset of $\boldsymbol{C}^*$ considered as the Riemann sphere.

The *approximate point spectrum* $\pi(T)$ of T is the set of all $\lambda\in\boldsymbol{C}$ for which there exists a sequence $\{x_\nu\}_{\nu=1}^\infty$ in $D[T]$ with $\|x_\nu\|=1$, $\nu=1,2,\cdots$, such that $\|(T-\lambda I)x_\nu\|\to 0$ as $\nu\to\infty$. The *point spectrum* $P_\sigma(T)$ of T is a subset of $\pi(T)$, and clearly $\pi(T)\subset\sigma(T)$. The boundary of $\sigma(T)$ is shown to be a subset of the boundary of $\pi(T)$, so that the set $\gamma(T)=\sigma(T)\backslash\pi(T)$ is open in $\boldsymbol{C}$.

The *compression spectrum* $\Gamma(T)$ of T is the set of all $\lambda\in\boldsymbol{C}$ for which the range of $T-\lambda I$ is not dense in X. Clearly $\Gamma(T)\subset\sigma(T)$.

We refer now to the spectra of linear extensions of T.

LEMMA 1.1. *Let X and T be as above. If X is a subspace of another complex normed linear space Y and $T_1: D[T_1]\subset Y\to Y$ is a linear operator with $T_1x=Tx$ in $D[T]\subset D[T_1]$, then we have*

$$\pi(T)=\pi(T|Y)\subset\pi(T_1),$$

where $\pi(T)$ and $\pi(T|Y)$ denote the approximate point spectra of T considered respectively as an operator in X with domain $D[T]$, as one in Y with the same domain $D[T]$.

We denote the graph of T by $G(T)$ and its closure in $X\times X$ by $\overline{G(T)}$. An extension $\tilde{T}$ of T is called *maximal* if $\tilde{T}$ satisfies $\overline{G(\tilde{T})}=\overline{G(T)}$ and if $\tilde{T}$ has no proper extension $\check{T}$ such that $\overline{G(\check{T})}=\overline{G(T)}$. Every linear opeartor T has in virtue of Zorn's lemma a maximal extension $\tilde{T}$ in X. The domain $D[\tilde{T}]$ of $\tilde{T}$ is the projection H_T of $\overline{G(T)}\subset X\times X$ into the first X. All maximal extensions of T have the same domain $D[\tilde{T}]$. T is maximal iff $D[T]=H_T$. The closure, the smallest closed extension, of a closable operator T, in particular, the continuous extension of a bounded operator T to the closure of $D[T]$, is a unique maximal extension of T, so that we shall employ the same notation $\tilde{T}$ for the closure of a closable operator T (cf. [11]).

It is shown ([9]) that under extensions of a linear operator T the spectra and their parts may change, but under maximal extensions the spectrum, approximate point spectrum and compression spectrum remain the same as those of T, i.e. $\sigma(T)=\sigma(\tilde{T})$, $\pi(T)=\pi(\tilde{T})$, $\Gamma(T)=\Gamma(\tilde{T})$.

If T is a densely defined linear operator in X, so that the adjoint T' of T is well-defined in the dual space X' of X, then we have

PROPOSITION 1.2.

1) $\rho(T)=\rho(T')$, $\sigma(T)=\sigma(T')$.
2) $\gamma(T)=\sigma(T)\backslash\pi(T)\subset\Gamma(T)=P_\sigma(T')\subset\pi(T')$.
3) $\gamma(T')=\sigma(T')\backslash\pi(T')\subset\pi(T)$.

1.2. *Tensor products*

We formulate some basic results on tensor products, following Grothendieck [5], [6], Schatten [18] and Ruston [17]. For the facts on topological linear spaces used here, see Köthe [10].

For $j=1, 2, \cdots, n$, let X_j be a complex Banach space, X'_j be its (topological) dual space and $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ be the algebraic tensor product of $X_1, X_2, \cdots, X_n$.

A norm α or $\|\cdot\|_\alpha$ on $X_1 \otimes \cdots \otimes X_n$ is called a *crossnorm* if

$$\alpha(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = \|x_1 \otimes x_2 \otimes \cdots \otimes x_n\|_\alpha = \|x_1\| \cdot \|x_2\| \cdots \|x_n\|$$

for $(x_1, x_2, \cdots, x_n) \in \prod_{j=1}^{n} X_j$. A norm α or $\|\cdot\|_\alpha$ on $X_1 \otimes \cdots \otimes X_n$ is said to be *reasonable* if α is a crossnorm on $X_1 \otimes \cdots \otimes X_n$ and the dual norm α' induced by the dual space of $X_1 \otimes \underset{\alpha}{\cdots} \otimes X_n$ equipped with α is also a crossnorm on $X'_1 \otimes \cdots \otimes X'_n$.

We define the norms ε, π as follows:

$$\varepsilon(u) = \|u\|_\varepsilon = \sup\left\{|\langle u, x'_1 \otimes \cdots \otimes x'_n \rangle| \; ; \; x'_j \in X'_j, \|x'_j\| \leq 1, 1 \leq j \leq n\right\},$$

$$\pi(u) = \|u\|_\pi = \sup\left\{|\langle u, v \rangle| \; ; \; v \in B(X_1, \cdots, X_n), \|v\| \leq 1\right\},$$

for $u \in X_1 \otimes \cdots \otimes X_n$. $B(X_1, \cdots, X_n)$ is the space of all continuous multilinear forms on $\prod_{j=1}^{n} X_j$ which is a Banach space with the usual norm. The norm π is also equivalent to

$$\pi(u) = \inf\left\{\sum_k \|x_1^{(k)}\| \cdot \|x_2^{(k)}\| \cdot \cdots \cdot \|x_n^{(k)}\| \; ; \; u = \sum_k x_1^{(k)} \otimes \cdots \otimes x_n^{(k)}\right\}.$$

Every crossnorm $\alpha \geq \varepsilon$ on $X_1 \otimes \cdots \otimes X_n$ is reasonable. The norm π is the greatest reasonable norm and the norm ε the smallest reasonable norm on $X_1 \otimes \cdots \otimes X_n$. Therefore every norm α with $\varepsilon \leq \alpha \leq \pi$ is reasonable.

The dual norm α' of a reasonable norm α is also a reasonable norm on $X'_1 \otimes \cdots \otimes X'_n$. Given a crossnorm or a reasonable norm α, $X_1 \widehat{\otimes \underset{\alpha}{\cdots} \otimes} X_n$ denotes the completion of $X_1 \otimes \cdots \otimes X_n$ with respect to the norm α and $X'_1 \widehat{\otimes \underset{\alpha'}{\cdots} \otimes} X_n$ the completion of $X'_1 \otimes \cdots \otimes X'_n$ with respect to the dual norm α'(when α is reasonable) which is identified with a closed subspace of the Banach space $(X_1 \widehat{\otimes \underset{\alpha}{\cdots} \otimes} X_n)' = (X_1 \otimes \underset{\alpha}{\cdots} \otimes X_n)'$.

We define tensor products of linear operators as follows. Let A_j: $D[A_j] \subset X_j \rightarrow X_j$, $j=1, 2, \cdots, n$, be linear operators. The mapping $(x_1, x_2, \cdots, x_n) \rightarrow Ax_1 \otimes \cdots \otimes Ax_n$ is multilinear of $D[A_1] \times \cdots \times D[A_n]$ into $X_1 \otimes \cdots \otimes X_n$; the corresponding linear mapping of $D[A_1] \otimes \cdots \otimes D[A_n]$ into $X_1 \otimes \cdots \otimes X_n$ is denoted by $A_1 \otimes \cdots \otimes A_n$ and called the tensor product of $A_1, \cdots, A_n$.

$A_1\otimes\cdots\otimes A_n$ is a linear operator in $X_1\otimes\cdots\otimes X_n$ with domain $D[A_1\otimes\cdots\otimes A_n]=D[A_1]\otimes\cdots\otimes D[A_n]$, and will often be considered as an operator in the completed tensor product $X_1\widehat{\otimes_\alpha\cdots\otimes} X_n$ for a reasonable norm α with the same demain. We can consider a maximal extension of $A_1\otimes\cdots\otimes A_n$ in $X_1\widehat{\otimes_\alpha\cdots\otimes} X_n$, which is denoted by $A_1\widetilde{\otimes_\alpha\cdots\otimes} A_n$.

For $j=1, 2, \cdots, n$, $L(X_j)$ denotes the Banach algebra of all continuous linear operators of X_j into itself. A crossnorm or a reasonable norm α on $X_1\otimes\cdots\otimes X_n$ is said to be *uniform*, if for any $(A_1, \cdots, A_n)\in\prod_{j=1}^n L(X_j)$ we have

$$\begin{aligned}\sup &\left\{\|(A_1\otimes\cdots\otimes A_n)u\|_\alpha\,;\ u\in X_1\otimes\cdots\otimes X_n,\ \|u\|_\alpha\leq 1\right\}\\ &=\|A_1\|\cdot\|A_2\|\cdots\|A_n\|\,.\end{aligned}$$

If α is a uniform reasonable norm on $X_1\otimes\cdots\otimes X_n$, so is the dual norm α' on $X_1'\otimes\cdots\otimes X_n'$. The greatest and smallest reasonable norms π and ε are uniform.

A reasonable norm $\alpha\geq\varepsilon$ on $X_1\otimes\cdots\otimes X_n$ is said to be *faithful* if the natural linear mapping j_ε^α of $X_1\widehat{\otimes_\alpha\cdots\otimes} X_n$ into $X_1\widehat{\otimes_\varepsilon\cdots\otimes} X_n$, obtained by extending the identity mapping: $X_1\otimes_\alpha\cdots\otimes X_n\to X_1\otimes_\varepsilon\cdots\otimes X_n\subset X_1\widehat{\otimes_\varepsilon\cdots\otimes} X_n$ by continuity to the entire space $X_1\widehat{\otimes_\alpha\cdots\otimes} X_n$ is one-to-one ([4], [14]). In case $\alpha=\pi$, this yields the "problème de biunivocité" ([5]).

For locally convex topological linear spaces X_j, $1\leq j\leq n$, we can define also a tensor product topology on $X_1\otimes\cdots\otimes X_n$ compatible with the structure of the tensor product $X_1\otimes\cdots\otimes X_n$ ([5]).

Finally we refer to the space $\mathcal{O}(K)$ of the germs of functions holomorphic in a neighbourhood of a compact subset K of the Riemann sphere $\boldsymbol{C}^*$.

For an open neighbourhood U of K, let $\mathcal{O}(U)$ be the linear space of the holomorphic functions on U equipped with the topology of compact convergence. $\mathcal{O}(U)$ is a nuclear (F)-space and a commutative topological algebra. $\mathcal{O}(K)$ is defined as the topological inductive limit of the spaces $\mathcal{O}(U)$ when U runs in the directed family of the open neighbourhoods U of K in $\boldsymbol{C}^*$. $\mathcal{O}(K)$ is a complete barrelled (DF)-space (see [5]).

PROPOSITION 1.3. *Let U_j (resp. K_j), $j=1, 2, \cdots, n$, be open (resp. compact) subsets of the Riemann sphere $\boldsymbol{C}^*$. Then it holds*

$$\begin{aligned}\mathcal{O}\left(\prod_{j=1}^n U_j\right)&=\mathcal{O}(U_1)\hat{\otimes}\,\mathcal{O}(U_2)\hat{\otimes}\cdots\hat{\otimes}\,\mathcal{O}(U_n)\\ &=\mathcal{O}(U_1)\widehat{\otimes\cdots\otimes}\,\mathcal{O}(U_n)\end{aligned}$$

$$\mathcal{O}\left(\prod_{j=1}^{n} K_j\right) = \mathcal{O}(K_1) \hat{\otimes} \mathcal{O}(K_2) \hat{\otimes} \cdots \hat{\otimes} \mathcal{O}(K_n)$$
$$= \mathcal{O}(K_1) \widehat{\otimes \cdots \otimes} \mathcal{O}(K_n),$$

where the tensor products are completed with respect to the projective tensor product topology $\mathfrak{T}_\pi$ which amounts in this case to the topology $\mathfrak{T}_\varepsilon$ of uniform convergence on the sets $M_1' \otimes M_2' \otimes \cdots \otimes M_n'$ with M_j' equicontinuous subset of X_j', for $j=1, 2, \cdots, n$.

Proof. We note only some facts. The topologies $\mathfrak{T}_\pi$ and $\mathfrak{T}_\varepsilon$ have the associative character. If both X and Y are (F)-spaces or barrelled (DF)-spaces, the inductive tensor product topology $\mathfrak{T}_i$ on $X \otimes Y$ coincides with the topologies $\mathfrak{T}_\pi$ and $\mathfrak{T}_\varepsilon$. The second relation follows from the first by the properties of the inductive limit of the inductive tensor products ([5] Chap I. §3. N° 1. Prop. 14. p. 76).

2. Operational calculus for Banach algebra elements

2. 1. *Tensor products of Banach algebras*

For $j=1, 2, \cdots, n$, let $\mathfrak{B}_j$ be a complex Banach algebra with unit element e_j, and $\alpha \geq \varepsilon$ be a crossnorm on $\mathfrak{B}_1 \otimes \cdots \otimes \mathfrak{B}_n$ *compatible with multiplication*, i. e. a crossnorm satisfying $\alpha(u \cdot v) \leq \alpha(u) \cdot \alpha(v)$ for $u, v \in \mathfrak{B}_1 \otimes \cdots \otimes \mathfrak{B}_n$. Then the completion $\mathfrak{B}_1 \widehat{\otimes \underset{\alpha}{\cdots} \otimes} \mathfrak{B}_n$ of $\mathfrak{B}_1 \otimes \cdots \otimes \mathfrak{B}_n$ with respect to α turns a complex Banach algebra with unit element $e_1 \otimes \cdots \otimes e_n$. The norm π is known to be compatible with multiplication ([4]).

Given $a=(a_1, \cdots, a_n) \in \prod_{j=1}^{n} \mathfrak{B}_j$, the spectra $\sigma(a_j)$ of a_j are compact subsets of $\boldsymbol{C}$. We shall develop an operational calculus of the germs in $\mathcal{O}\left(\prod_{j=1}^{n} \sigma(a_j)\right)$ for the tensor product $a_1 \otimes \cdots \otimes a_n$, which has some analogy with that due to Waelbroeck ([21], [1]).

By a *morphism* of a complex algebra into another one, we mean a mapping between them which transforms sums to sums, products to products and products by a complex number λ to products by λ.

$\Theta_j(\cdot\,; a_j)$ denotes the continuous morphism of the algebra $\mathcal{O}(\sigma(a_j))$ into the Banach algebra $\mathfrak{B}_j$ which was first investigated by I. Gelfand: if $f(\zeta_j)$ is a holomorphic function in a neighbourhood U_j of $\sigma(a_j)$ and $\tilde{f}$ is the germ of f in $\mathcal{O}(\sigma(a_j))$, we have

$$\Theta_j(\tilde{f}\,; a_j) = f(a_j) = (2\pi i)^{-1} \int_{\Gamma_j} f(\zeta_j)(\zeta_j e_j - a_j)^{-1} d\zeta_j,$$

where the contour Γ_j consists of a finite number of rectifiable, positively oriented Jordan curves lying in $U_j \backslash \sigma(a_j)$ (cf. [3], [8], [15]).

THEOREM 2.1. *Let $\alpha \geq \varepsilon$ be a crossnorm on $\mathfrak{B}_1 \otimes \cdots \otimes \mathfrak{B}_n$ compatible with multiplication. Given $a=(a_1, \cdots, a_n) \in \prod_{j=1}^{n} \mathfrak{B}_j$, there exists a unique continuous morphism $\Theta(\cdot\,; a)$ of the algebra $\mathcal{O}\left(\prod_{j=1}^{n} \sigma(a_j)\right)$ into the Banach algebra $\mathfrak{B}_1 \widehat{\otimes \underset{\alpha}{\cdots} \otimes} \mathfrak{B}_n$ which transforms the germ of the function 1 to $e_1 \otimes \cdots \otimes e_n$ and the germ of the function $(\zeta_1, \cdots, \zeta_n) \to \zeta_j$ to $e_1 \otimes \cdots \otimes e_{j-1} \otimes a_j \otimes e_{j+1} \otimes \cdots \otimes e_n$, for $j=1, 2, \cdots, n$.*

If $f(\zeta_1, \cdots, \zeta_n)$ is a rational function or a polynomial holomorphic in a neighbourhood of $\prod_{j=1}^{n} \sigma(a_j)$, then we have

$$\Theta(\tilde{f}\,; a)=f(a_1 \otimes \cdots \otimes e_n, \cdots, e_1 \otimes \cdots \otimes a_n),$$

where $\tilde{f}$ denotes the germ of f in the neighbourhood of $\prod_{j=1}^{n} \sigma(a_j)$.

Proof. Let U_j be a bounded open neighbourhood of $\sigma(a_j)$, for $j=1, 2, \cdots, n$, and choose a contour Γ_j in $U_j \backslash \sigma(a_j)$ consisting of a finite number of rectifiable, positively oriented Jordan curves.

For $f \in \mathcal{O}\left(\prod_{j=1}^{n} U_j\right)=\mathcal{O}(U_1) \widehat{\otimes \cdots \otimes} \mathcal{O}(U_n)$, the integral

$$(2\pi i)^{-n} \int_{\Gamma_1} \cdots \int_{\Gamma_n} f(\zeta_1, \cdots, \zeta_n)(\zeta_1 e_1-a_1)^{-1} \otimes \cdots \otimes(\zeta_n e_n-a_n)^{-1} d\zeta_1 d\zeta_2 \cdots d\zeta_n$$

defines an element in the Banach algebra $\mathfrak{B}_1 \widehat{\otimes \underset{\alpha}{\cdots} \otimes} \mathfrak{B}_n$, and does not depend upon the choice of Γ_j, $j=1, 2, \cdots, n$, but only upon the germ $\tilde{f}$ of f in the neighbourhood of $\prod_{j=1}^{n} \sigma(a_j)$. By this integral we define $\Theta(\tilde{f}\,; a)$. Theorem 2.1 follows from the properties of $\Theta_j(\cdot\,; a_j)$ using Proposition 1.3. The uniqueness follows from the fact that the rational functions are dense in $\mathcal{O}(U_j)$. $\Theta(\cdot\,; a)$ is exactly the continuous extension of $\Theta_1(\cdot\,; a_1) \otimes \cdots \otimes \Theta_n(\cdot\,; a_n)$ to $\mathcal{O}\left(\prod_{j=1}^{n} \sigma(a_j)\right)=\mathcal{O}(\sigma(a_1)) \widehat{\otimes \cdots \otimes} \mathcal{O}(\sigma(a_n))$.

2.2. *Tensor products of bounded operators*

Applying our result in §2.1 to the operator algebras, we shall prove the spectral mapping theorem.

Thoughout, for $j=1, 2, \cdots, n$, let $L(X_j)$ be the complex Banach algebra of the continuous linear operators on X_j. $L(X_1) \otimes \cdots \otimes L(X_n)$ is considered in the natural way as a complex algebra. The following lemma and theorem are due to Gil de Lamadrid [4] for $n=2$.

LEMMA 2.2. *Let α be a uniform reasonable norm on $X_1\otimes\cdots\otimes X_n$. Then there exists a canonical injective morphism of the algebra $L(X_1)\otimes\cdots\otimes L(X_n)$ into the algebra $L(X_1\widehat{\otimes_\alpha\cdots\otimes} X_n)$.*

Proof. We assign to each $A_1\otimes\cdots\otimes A_n$ of the algebra $L(X_1)\otimes\cdots\otimes L(X_n)$ a linear operator on $X_1\otimes\cdots\otimes X_n$: $u=\sum_{k=1}^m x_1^{(k)}\otimes\cdots\otimes x_n^{(k)}\to(A_1\otimes\cdots\otimes A_n)u$ $=\sum_{k=1}^m A_1x_1^{(k)}\otimes\cdots\otimes A_nx_n^{(k)}$, which is bounded by the uniformness of the norm α. This bounded linear operator can be extended continuously to the entire space $X_1\widehat{\otimes_\alpha\cdots\otimes} X_n$; the continuous extension $A_1\widehat{\otimes_\alpha\cdots\otimes} A_n$ is an element of $L(X_1\widehat{\otimes_\alpha\cdots\otimes} X_n)$. Since all the $A_1\otimes\cdots\otimes A_n$ generate the algebra $L(X_1)\otimes\cdots\otimes L(X_n)$, this mapping can be extended by linearity to the entire algebra $L(X_1)\otimes\cdots\otimes L(X_n)$; we have only to check that the representation

$$\boldsymbol{A}=\sum_{k=1}^m A_1^{(k)}\otimes\cdots\otimes A_n^{(k)}$$

of an element $\boldsymbol{A}\in L(X_1)\otimes\cdots\otimes L(X_n)$ defines the zero operator in $L(X_1\widehat{\otimes_\alpha\cdots\otimes} X_n)$ iff $\boldsymbol{A}=0$.

In fact, suppose $\boldsymbol{A}=\sum_{k=1}^m A_1^{(k)}\otimes\cdots\otimes A_n^{(k)}$ defines the zero operator in $L(X_1\widehat{\otimes_\alpha\cdots\otimes} X_n)$. Then for all $(x_1,\cdots,x_n)\in\prod_{j=1}^n X_j$, $(x_1',\cdots,x_n')\in\prod_{j=1}^n X_j'$, we have

$$\sum_{k=1}^m\langle A_1^{(k)}x_1,x_1'\rangle\langle A_2^{(k)}x_2,x_2'\rangle\cdots\langle A_n^{(k)}x_n,x_n'\rangle=0\,.$$

For each pair $(x_j,x_j')\in X_j\times X_j'$, $j=1,2,\cdots,n$, define a continuous linear form μ_j' on the Banach space $L(X_j)$ by $\langle A_j,\mu_j'\rangle=\langle A_jx_j,x_j'\rangle$, $A_j\in L(X_j)$.

Then the above relation is equivalent to

$$\sum_{k=1}^m\langle A_1^{(k)},\mu_1'\rangle\langle A_2^{(k)},\mu_2'\rangle\cdots\langle A_n^{(k)},\mu_n'\rangle=0\,.$$

The set of the continuous linear forms μ_j' with $(x_j,x_j')\in X_j\times X_j'$, is total over $L(X_j)$, for $j=1,2,\cdots,n$, so that its linear hull is dense in the dual space $L(X_j)'$ with respect to the weak topology defined by the dual pair $\langle L(X_j)',L(X_j)\rangle$, for $j=1,2,\cdots,n$.

If follows by the separate weak continuity that

$$\sum_{k=1}^m\langle A_1^{(k)}\otimes A_2^{(k)}\otimes\cdots\otimes A_n^{(k)},\mu'\rangle=0\,,$$

for all $\mu'\in L(X_1)'\otimes\cdots\otimes L(X_n)'$. Since $\langle L(X_1)\otimes\cdots\otimes L(X_n), L(X)'\otimes\cdots\otimes L(X_n)'\rangle$

is a dual pair, we have $\sum_{k=1}^{m} A_1^{(k)} \otimes \cdots \otimes A_n^{(k)} = 0$. It is easily verified that the products of elements in $L(X_1) \otimes \cdots \otimes L(X_n)$ go into the compositions of the corresponding operators in $L(X_1 \widehat{\otimes \cdots_\alpha \otimes} X_n)$. Q. E. D.

From Lemma 2.2, we see that if α is a uniform reasonable norm on $X_1 \otimes \cdots \otimes X_n$, an operator norm can be induced on $L(X_1) \otimes \cdots \otimes L(X_n)$ as a subspace of $L(X_1 \widehat{\otimes \cdots_\alpha \otimes} X_n)$, denoted by $\bar{\alpha}$. Since α' is also uniform reasonable with α on $X_1' \otimes \cdots \otimes X_n'$, $L(X_1' \widehat{\otimes \cdots_{\alpha'} \otimes} X_n')$ induces on its subspace $L(X_1') \otimes \cdots \otimes L(X_n')$ an operator norm $\overline{\alpha'}$.

THEOREM 2.3. *If α is a uniform reasonable norm on $X_1 \otimes \cdots \otimes X_n$, then $\bar{\alpha}$ is a crossnorm on $L(X_1) \otimes \cdots \otimes L(X_n)$ compatible with multiplication. Therefore the completion $L(X_1) \widehat{\otimes \cdots_{\bar{\alpha}} \otimes} L(X_n)$ with respect to $\bar{\alpha}$ is a Banach algebra. The same is true for $\overline{\alpha'}$.*

In virtue of Theorem 2.3, we observe that the same assertion for $\underset{\sim}{A} = (A_1, \cdots, A_n) \in \prod_{j=1}^{n} L(X_j)$ as in Theorem 2.1 is true for $\mathfrak{B}_j$ replaced by $L(X_j)$ and a_j by A_j, $j = 1, 2, \cdots, n$. We denote the continuous morphism of $\mathcal{O}\left(\prod_{j=1}^{n} \sigma(A_j)\right)$ into $L(X_1) \widehat{\otimes \cdots_{\bar{\alpha}} \otimes} L(X_n)$ by $\Theta(\cdot\,; A)$.

We are now in a position to state the spectral mapping theorem for tensor products of bounded linear operators. For $j = 1, 2, \cdots, n$, I_j denotes the identity operator in X_j. For an element a of a Banach algebra $\mathfrak{B}$ with unit element, $\sigma_{\mathfrak{B}}(a)$ denotes the spectrum of a with respect to $\mathfrak{B}$.

THEOREM 2.4. *Let be a uniform reasonable norm on $X_1 \otimes \cdots \otimes X_n$, and $A = (A_1, \cdots, A_n) \in \prod_{j=1}^{n} L(X_j)$. For $\tilde{f} \in \mathcal{O}\left(\prod_{j=1}^{n} \sigma(A_j)\right)$,* we have

$$\sigma_{L(X_1 \widehat{\otimes \cdots_\alpha \otimes} X_n)}\left(\Theta(\tilde{f}\,; A)\right) = f\left(\sigma(A_1), \cdots, \sigma(A_n)\right).$$

Proof. If $\lambda \notin f(\sigma(A_1), \cdots, \sigma(A_n))$, then $(f(\zeta_1, \cdots, \zeta_n) - \lambda)^{-1}$ exists and is holomorphic in a neighbourhood of $\prod_{j=1}^{n} \sigma(A_j)$. Therefore from Theorem 2.1 for $\mathfrak{B}_j = L(X_j)$, $1 \leq j \leq n$, we have

$$\begin{aligned} \Theta(\tilde{f}_\lambda\,; A) \cdot \Theta(\widetilde{f_\lambda^{-1}}\,; A) &= \Theta(\widetilde{f_\lambda^{-1}}\,; A) \cdot \Theta(\tilde{f}_\lambda\,; A), \quad f_\lambda = f - \lambda, \\ &= \boldsymbol{I} \end{aligned}$$

where $\boldsymbol{I}$ means the unit element $I_1 \otimes \cdots \otimes I_n$ of the algebra $L(X_1) \widehat{\otimes \cdots_{\bar{\alpha}} \otimes} (X_n)$, which will also denote the identity operator $I_1 \widehat{\otimes \cdots_\alpha \otimes} I_n$ in $L(X_1 \widehat{\otimes \cdots_\alpha \otimes} X_n)$.

Since $\Theta(\tilde{f}_\lambda;\ A)=\Theta(\tilde{f};\ A)-\lambda \boldsymbol{I}$, the above identity shows that $\Theta(\tilde{f};\ A)-\lambda \boldsymbol{I}$ has a bounded inverse $\Theta(\tilde{f}_\lambda^{-1};\ A)$ in $L(X_1)\widehat{\otimes_\alpha\cdots\otimes} L(X_n)\subset L(X_1\widehat{\otimes_\alpha\cdots\otimes} X_n)$. Thus $\lambda \notin \sigma_{L(X_1\widehat{\otimes}_\alpha\cdots\otimes X_n)}(\Theta(\tilde{f};\ A))$.

We show now the inclusion

$$f\big(\sigma(A_1), \cdots, \sigma(A_n)\big)\subset\sigma_{L(X_1\widehat{\otimes}_\alpha\cdots\otimes X_n)}\big(\Theta(\tilde{f};\ A)\big).$$

We may assume $f\not\equiv$ constant.

Let $(\alpha_1, \cdots, \alpha_n)\in \prod_{j=1}^{n} \sigma(A_j)$. Then there exist $\tilde{g}_j\in \mathcal{O}\left(\prod_{j=1}^{n} \sigma(A_j)\right)$, $1\leq j\leq n$, (see [7]) such that

$$\begin{aligned} &f(\zeta_1, \cdots, \zeta_n)-f(\alpha_1, \cdots, \alpha_n) \\ &\quad=(\zeta_1-\alpha_1)\, g_1(\zeta_1, \cdots, \zeta_n)+\cdots+(\zeta_n-\alpha_n)\, g_n(\zeta_1, \cdots, \zeta_n), \end{aligned}$$

where f and g_j are representatives of $\tilde{f}$, $\tilde{g}_j$, respectively, in some common bounded open neighbourhood $\prod_{j=1}^{n} V_j$ of $\prod_{j=1}^{n} \sigma(A_j)$.

From Theorem 2.1 we have

$$\begin{aligned} &\big[\Theta(\tilde{f};\ A)-f(\alpha_1, \cdots, \alpha_n)(I_1\otimes\cdots\otimes I_n)\big](x_1\otimes\cdots\otimes x_n) \\ &\quad=\Theta(\tilde{g}_1;\ A)\big\{(A_1-\alpha_1 I_1)\, x_1\otimes\cdots\otimes x_n\big\}+\cdots \\ &\quad\cdots+\Theta(\tilde{g}_n;\ A)\big\{x_1\otimes\cdots\otimes(A_n-\alpha_n I_n)\, x_n\big\}, \end{aligned}$$

for $(x_1, \cdots, x_n)\in \prod_{j=1}^{n} X_j$.

The proof is divided into three cases. We shall often use Proposition 1.2. For simplicity we assume $n=3$.

(1) Case: $(\alpha_1, \alpha_2, \alpha_3)\in \prod_{j=1}^{3} \pi(A_j)$. For $j=1, 2, 3$, there exists a sequence of unit vectors $\{x_j^{(\nu)}\}_{\nu=1}^{\infty}\subset X_j$ such that $(A_j-\alpha_j I_j)\, x_j^{(\nu)}\to 0$ as $\nu\to\infty$.

Since α is a reasonable norm on $X_1\otimes X_2\otimes X_3$,

$$\big[\Theta(\tilde{f};\ A)-f(\alpha_1, \alpha_2, \alpha_3)(I_1\otimes I_2\otimes I_3)\big](x_1^{(\nu)}\otimes x_2^{(\nu)}\otimes x_3^{(\nu)})$$

converges to zero in the norm α as $\nu\to\infty$ and

$$\|x_1^{(\nu)}\otimes x_2^{(\nu)}\otimes x_3^{(\nu)}\|_\alpha=\|x_1^{(\nu)}\|\cdot\|x_2^{(\nu)}\|\cdot\|x_3^{(\nu)}\|=1.$$

It follows that $f(\alpha_1,\alpha_2,\alpha_3)\in\pi(\Theta(\tilde{f};\ A))\subset\sigma(\Theta(\tilde{f};\ A))$.

(2) Case: $(\alpha_1, \alpha_2, \alpha_3)\in \prod_{j=1}^{3} \gamma(A_j)$. This implies $(\alpha_1, \alpha_2, \alpha_3)\in \prod_{j=1}^{3} P_\sigma(A_j')\subset \prod_{j=1}^{3} \pi(A_j')$ in virtue of Proposition 1.2. Since α' is also a reasonable norm on $X_1'\otimes X_2'\otimes X_3'$, we obtain in a similar way to (1)

$$f(\alpha_1, \alpha_2, \alpha_3) \in P_\sigma\left(\Theta(\tilde{f}\,;\, A')\right) \subset \pi\left(\Theta(\tilde{f}\,;\, A')\right),$$

where $\Theta(\cdot\,;\, A')$, $A'=(A'_1, A'_2, A'_3)$, is the continuous morphism of $\mathcal{O}\left(\prod_{j=1}^{3} \sigma(A'_j)\right)$ into $L(X'_1)\widehat{\otimes L(X'_2)\otimes}_{\alpha'} L(X'_3)$. The restriction of $\Theta(\tilde{f}\,;\, A')$ considered as an operator on $X'_1\widehat{\otimes X'_2\otimes}_{\alpha'} X'_3$ to the dense subspace $X'_1\otimes_{\alpha'} X'_2\otimes X'_3$ is given with the relations $\sigma(A_j)=\sigma(A'_j)$, $j=1, 2, 3$, by

$$\begin{aligned}
&\Theta(\tilde{f}\,;\, A')(x'_1\otimes x'_2\otimes x'_3)\\
&= (2\pi i)^{-3}\int_{\Gamma_1}\int_{\Gamma_2}\int_{\Gamma_3} f(\zeta_1, \zeta_2, \zeta_3)\, R(\zeta_1\,;\, A'_1)\, x'_1\otimes R(\zeta_2\,;\, A'_2)\, x'_2\\
&\otimes R(\zeta_3\,;\, A'_3)\, x'_3 d\zeta_1 d\zeta_2 d\zeta_3,\\
&= (2\pi i)^{-3}\int_{\Gamma_1}\int_{\Gamma_2}\int_{\Gamma_3} f(\zeta_1,\zeta_2,\zeta_3)\, R(\zeta_1\,;\, A_1)'\, x'_1\otimes R(\zeta_2\,;\, A_2)'\, x'_2\\
&\otimes R(\zeta_3\,;\, A_3)'\, x'_3 d\zeta_1 d\zeta_2 d\zeta_3,
\end{aligned}$$

where $R(\zeta_j\,;\, A_j)=(\zeta_j I_j - A_j)^{-1}\in L(X_j)$, $R(\zeta_j\,;\, A'_j)=(\zeta_j I'_j - A'_j)^{-1}\in L(X'_j)$, $1\le j \le 3$, and for $j=1, 2, 3$, each contour Γ_j consists of a finite number of rectifiable, positively oriented Jordan curves lying in $V_j\backslash\sigma(A_j)$.

The densely defined bounded operator $R(\zeta_1;\, A_1)'\otimes R(\zeta_2;\, A_2)'\otimes R(\zeta_3;\, A_3)'$ in $X'_1\widehat{\otimes X'_2\otimes}_{\alpha'} X'_3$ can be considered also as an operator in the Banach space $(X_1\widehat{\otimes X_2\otimes}_{\alpha} X_3)'=(X_1\otimes_{\alpha} X_2\otimes X_3)'$ with the same domain $X'_1\otimes_{\alpha'} X'_2\otimes X'_3$. The adjoint of the densely defined bounded operator $R(\zeta_1;\, A_1)\otimes R(\zeta_2;\, A_2)\otimes R(\zeta_3;\, A_3)$ in $X_1\widehat{\otimes X_2\otimes}_{\alpha} X_3$ is a bounded operator on $(X_1\widehat{\otimes X_2\otimes}_{\alpha} X_3)'=(X_1\otimes_{\alpha} X_2\otimes X_3)'$ which is clearly an extension of the operator $R(\zeta_1;\, A_1)'\otimes R(\zeta_2;\, A_3)'\otimes R(\zeta_3;\, A_3)'$.

Thus we see that the adjoint operator $\Theta(\tilde{f};\, A)'$ on $(X_1\widehat{\otimes X_2\otimes}_{\alpha} X_3)'$ of $\Theta(\tilde{f};\, A)$ is an extension of $\Theta(\tilde{f};\, A')$. In virtue of Lemma 1.1 and Proposition 1.2 we obtain

$$\begin{aligned}
f(\alpha_1, \alpha_2, \alpha_3) &\in \pi\left(\Theta(\tilde{f};\, A')\right)\\
&\subset \pi\left(\Theta(\tilde{f};\, A)'\right)\\
&\subset \sigma\left(\Theta(\tilde{f};\, A)'\right) = \sigma\left(\Theta(\tilde{f};\, A)\right)
\end{aligned}$$

(3) Case: $(\alpha_{s_1}, \alpha_{s_2}, \alpha_{s_3})\in \pi(A_{s_1})\times \gamma(A_{s_2})\times \gamma(A_{s_3})$ or $(\alpha_{s_1}, \alpha_{s_2}, \alpha_{s_3})\in \pi(A_{s_1})\times \pi(A_{s_2})\times \gamma(A_{s_3})$, where $s=\begin{pmatrix}1 & 2 & 3\\ s_1 & s_2 & s_3\end{pmatrix}$ is an arbitrary permutation. We shall see

that we have only to consider the latter case, to which the former case is reduced. Without loss of generality, we may assume that s is the identity permutation :

$$\text{(a)}: \quad (\alpha_1, \alpha_2, \alpha_3) \in \pi(A_1) \times \gamma(A_2) \times \gamma(A_3) \text{ or}$$

$$\text{(b)}: \quad (\alpha_1, \alpha_2, \alpha_3) \in \pi(A_1) \times \pi(A_2) \times \gamma(A_3).$$

We need the following lemma due to Remmert-Stein on analytic sets in a bounded domain of C^n ([16] 2. Satz 7, p. 288).

LEMMA 2.5. *Let $G \subset C^n (n > 1)$ be a bounded domain (open but not necessarily connected) and M be an analytic set in G. Then if M is not pure zero-dimensional, there exists a boundary point of G which is an accumulation point of M.*

If $g(\zeta_1, \cdots, \zeta_n)$ is holomorphic in $G \subset C^n (n > 1)$, then the set $M(g) \equiv \{\zeta = (\zeta_1, \cdots, \zeta_n) \in G\,;\ g(\zeta) = 0\}$ is a principal analytic set; principal analytic sets in C^n are pure $(n-1)$-dimensional, therefore not pure zero-dimensional.

End of Proof of Theorem 2.4. We assume (a). Set $f_0(\zeta_1, \zeta_2, \zeta_3) = f(\zeta_1, \zeta_2, \zeta_3) - f(\alpha_1, \alpha_2, \alpha_3)$. $\gamma(A_2)$ and $\gamma(A_3)$ are bounded open subsets in C and the boundary of $\gamma(A_j)$ is contained in that of $\pi(A_j)$, since the boundary of $\sigma(A_j)$ is a subset of the boundary of $\pi(A_j)$. Since $\tilde{f}_0$ belongs to $\mathcal{O}\left(\prod_{j=1}^{3} \sigma(A_j)\right)$, the function $g(\zeta_2, \zeta_3) = f_0(\alpha_1, \zeta_2, \zeta_3)$ is holomorphic in an open neighbourhood of $\overline{\gamma(A_2)} \times \overline{\gamma(A_3)} \subset C^2$, where the $\overline{\gamma(A_j)}$, $j=2, 3$, are the closures of $\gamma(A_j)$ which are compact subsets of $\sigma(A_j)$. Set $M(g) = \{(\zeta_2, \zeta_3) \in \gamma(A_2) \times \gamma(A_3)\,;\ g(\zeta_2, \zeta_3) = 0\}$. $M(g)$ is a prinicipal analytic set of pure one-dimension in the bounded domain $\gamma(A_2) \times \gamma(A_3)$. In virtue of Lemma 2.5, there exists a boundary point (α_2^0, α_3^0) of the domain $\gamma(A_2) \times \gamma(A_3)$ which is an accumulation point of $M(g)$: $\alpha_2^0 \in \pi(A_2)$ *or* $\alpha_3^0 \in \pi(A_3)$.

By the continuity of g up to the boundary, we obtain

$$g(\alpha_2^0, \alpha_3^0) = f_0(\alpha_1, \alpha_2^0, \alpha_3^0) = 0\,,$$

so that

$$f(\alpha_1, \alpha_2, \alpha_3) = f(\alpha_1, \alpha_2^0, \alpha_3^0)\,.$$

Thus, considering $(\alpha_1, \alpha_2^0, \alpha_3^0)$ instead of $(\alpha_1, \alpha_2, \alpha_3)$, we can reduce the case (a) to the case that two of the numbers $\alpha_1, \alpha_2, \alpha_3$ belong to the approximate point spectra of the corresponding operators.

Thus, we have now only to consider the case (b). For the pair (α_1, α_2) and the approximate point spectra of the adjoints of the corresponding operators, we shall make an argument similar to before.

If $(\alpha_1, \alpha_2) \in \pi(A_1') \times \pi(A_2')$, then $(\alpha_1, \alpha_2, \alpha_3) \in \prod_{j=1}^{3} \pi(A_j')$ by Proposition 1.2. In this case the required assertion follows from (2).

Otherwise, we have (a'): $(\alpha_1, \alpha_2) \in \gamma(A_1') \times \gamma(A_2')$ or (b'): $(\alpha_1, \alpha_2) \in \pi(A_1') \times \gamma(A_2')$ or (c'): $(\alpha_1, \alpha_2) \in \gamma(A_1') \times \pi(A_2')$. We show the case (a') is reduced to the case (b') or (c').

Assume $(\alpha_1, \alpha_2) \in \gamma(A_1') \times \gamma(A_2')$, i.e.

$$(\alpha_1, \alpha_2, \alpha_3) \in \pi(A_1) \times \pi(A_2) \times \gamma(A_3) \cap \gamma(A_1') \times \gamma(A_2') \times \boldsymbol{C}.$$

Set $g(\zeta_1, \zeta_2) = f(\zeta_1, \zeta_2, \alpha_3) - f(\alpha_1, \alpha_2, \alpha_3)$. Then $g(\alpha_1, \alpha_2) = 0$. $\gamma(A_1')$ and $\gamma(A_2')$ are bounded open subsets of $\boldsymbol{C}$ and the boundary of $\gamma(A_j')$ is contained in that of $\pi(A_j')$. $g(\zeta_1, \zeta_2)$ is holomorphic in a neighbourhood of $\overline{\gamma(A_1')} \times \overline{\gamma(A_2')} \subset \boldsymbol{C}^2$. The set $M(g) = \{(\zeta_1, \zeta_2) \in \gamma(A_1') \times \gamma(A_2')\,;\ g(\zeta_1, \zeta_2) = 0\}$ is a principal analytic set of pure one-dimension in the bounded domain $\gamma(A_1') \times \gamma(A_2')$. In virtue of Lemma 2.5, there exist a boundary point (α_1^0, α_2^0) of the domain $\gamma(A_1') \times \gamma(A_2')$ and a sequence $(\alpha_1^{(\nu)}, \alpha_2^{(\nu)})_{\nu=1}^{\infty}$ in $M(g)$ converging to (α_1^0, α_2^0): $\alpha_1^0 \in \pi(A_1')$ *or* $\alpha_2^0 \in \pi(A_2')$. By the continuity of g, we have

$$g(\alpha_1^0, \alpha_2^0) = g(\alpha_1^{(\nu)}, \alpha_2^{(\nu)}) = 0\,, \qquad \nu = 1, 2, \cdots,$$

so that under this procedure the value of f is invariant:

$$\begin{aligned} f(\alpha_1^0, \alpha_2^0, \alpha_3) &= f(\alpha_1^{(\nu)}, \alpha_2^{(\nu)}, \alpha_3) \qquad (\nu = 1, 2, \cdots) \\ &= f(\alpha_1, \alpha_2, \alpha_3)\,. \end{aligned}$$

Since $(\alpha_1^{(\nu)}, \alpha_2^{(\nu)}) \in \gamma(A_1') \times \gamma(A_2') \subset \pi(A_1) \times \pi(A_2)$, $\nu = 1, 2, \cdots$, in virtue of Proposition 1.2 and the approximate point spectrum is closed in $\boldsymbol{C}$, we have, by tending $\nu \to \infty$, $(\alpha_1^0, \alpha_2^0) \in \pi(A_1) \times \pi(A_2)$.

Thus, considering $(\alpha_1^0, \alpha_2^0, \alpha_3)$ instead of $(\alpha_1, \alpha_2, \alpha_3)$, we can reduce the case (a') to either the case (b') or (c').

We may consider now the case (b'); (c') will be treated similarly. Thus, the remaining case to treat turns altogether

$$(\alpha_1, \alpha_2, \alpha_3) \in \pi(A_1) \times \pi(A_2) \times \gamma(A_3) \cap \pi(A_1') \times \gamma(A_2') \times \boldsymbol{C}.$$

Set $g(\zeta_2, \zeta_3) = f(\alpha_1, \zeta_2, \zeta_3) - f(\alpha_1, \alpha_2, \alpha_3)$. $\gamma(A_2')$ and $\gamma(A_3)$ are bounded open in $\boldsymbol{C}$. Set $M(g) = \{(\zeta_2, \zeta_3) \in \gamma(A_2') \times \gamma(A_3)\,;\ g(\zeta_2, \zeta_3) = 0\}$. Making a similar argument and using Lemma 2.5, we find a boundary point (α_2^0, α_3^0) of the bounded domain $\gamma(A_2') \times \gamma(A_3)$ which is the limit of a sequence in $M(g)$. It follows as before that

$$\alpha_2^0 \in \pi(A_2) \text{ and } \alpha_3^0 \in \pi(A_3')$$

and that

$$\alpha_2^0 \in \pi(A_2') \textit{ or } \alpha_3^0 \in \pi(A_3)\,.$$

By the continuity of g up to the boundary we have $f(\alpha_1, \alpha_2^0, \alpha_3^0) = f(\alpha_1, \alpha_2, \alpha_3)$.

If $\alpha_2^0 \in \pi(A_2')$, then $(\alpha_1, \alpha_2^0, \alpha_3^0) \in \prod_{j=1}^{3} \pi(A_j')$ in virtue of Proposition 1.2. The assertion follows from (2).

If $\alpha_3^0 \in \pi(A_3)$, then $(\alpha_1, \alpha_3^0, \alpha_3^0) \in \prod_{j=1}^{3} \pi(A_j)$. The assertion follows from (1).

Q. E. D.

REMARK 2.6. From the proof of Theorem 2.4, we see

$$\sigma_{L(X_1 \widehat{\otimes_\alpha \cdots \otimes} X_n)}\big(\Theta(\tilde{f};\, A)\big) = \sigma_{L(X_1) \widehat{\otimes_{\tilde{\alpha}} \cdots \otimes} L(X_n)}\big(\Theta(\tilde{f};\, A)\big).$$

$$= f\big(\sigma(A_1), \cdots, \sigma(A_n)\big).$$

By the aid of Proposition 1.2 we have

COROLLARY 2.7. *Under the same assumptions as in Theorem 2.4 we have*

$$f\big(\pi(A_1), \cdots, \pi(A_n)\big) \subset \pi\big(\Theta(\tilde{f};\, A)\big),$$

$$f\big(P_\sigma(A_1), \cdots, P_\sigma(A_n)\big) \subset P_\sigma\big(\Theta(\tilde{f};\, A)\big),$$

$$\begin{aligned} f\big(\Gamma(A_1), \cdots, \Gamma(A_n)\big) &= f\big(P_\sigma(A_1'), \cdots, P_\sigma(A_n')\big) \\ &\subset P_\sigma\big(\Theta(\tilde{f};\, A')\big) \\ &\subset P_\sigma\big(\Theta(\tilde{f};\, A)'\big) \\ &= \Gamma\big(\Theta(\tilde{f};\, A)\big). \end{aligned}$$

If $f = \zeta_1 \cdot \zeta_2 \cdot \cdots \cdot \zeta_n$, the fact that the spectrum does not change under the continuous extension yields ([9], cf. [2])

COROLLARY 2.8. *Let α be a uniform reasonable norm on $X_1 \otimes \cdots \otimes X_n$. For $j = 1, 2, \cdots, n$, let A_j be a densely defined bounded linear operator in X_j. Then we have*

$$\sigma(A_1) \cdot \sigma(A_2) \cdot \cdots \cdot \sigma(A_n) = \sigma(A_1 \otimes \cdots \otimes A_n) = \sigma(A_1 \widehat{\otimes_\alpha \cdots \otimes} A_n),$$

where $A_1 \otimes \cdots \otimes A_n$ is considered as an operator in $X_1 \widehat{\otimes_\alpha \cdots \otimes} X_n$ and $A_1 \widehat{\otimes_\alpha \cdots \otimes} A_n$ its continuous extension to $X_1 \widehat{\otimes_\alpha \cdots \otimes} X_n$.

REMARK 2.9. In Theorem 2.4, in particular, if f is a polynomial, it may be considered as an extension of Stéphanos' results for A_j being matrices (see [13] VII Th. 43. 8). Brown and Pearcy [2] proved Corollary 2.8 for bounded operators defined on the same Hilbert space. Schechter [19] proved Theorem 2.4 for polynomials.

2.3. *The spectral mapping theorem for tensor products of Banach algebras*

On the basis of Remark 2.6, it is expected to show the validity of the spectral mapping theorem not only for the tensor product of operator algebras but also for the tensor product of general Banach algebras.

For a complex Banach algebra $\mathfrak{B}$ with unit element e and $a \in \mathfrak{B}$, we consider the left regular representation A corresponding to a of $\mathfrak{B}$ into the Banach algebra $L(\mathfrak{B})$ of the continuous linear operators on $\mathfrak{B}$ considered as a Banach space (cf. [15], [12]).

Then we have

LEMMA 2.10. $\quad \sigma_{\mathfrak{B}}(a) = \sigma_{L(\mathfrak{B})}(A) \qquad$ ([12]).

THEOREM 2.11. *Under the same assumptions and notations as in Theorem* 2.1, *we have for* $\tilde{f} \in \mathcal{O}\left(\prod_{j=1}^{n} \sigma(a_j)\right)$

$$\sigma_{\mathfrak{B}_1 \widehat{\otimes}_\alpha \cdots \otimes \mathfrak{B}_n}\left(\Theta(\tilde{f};\, a)\right) = f\left(\sigma(a_1), \cdots, \sigma(a_n)\right).$$

Proof. Clear from Lemma 2.10 and Theorem 2.4.

3. Operational calculus for tensor products of closed operators

3.1. *The spectral mapping theorem for tensor products of closed operators*

We extend our previous results in § 2.2 and develop an operational calculus for tensor products of closed operators in an analogous way to that by Taylor ([20], [3], [8]).

For $j=1, 2, \cdots, n$, let X_j be a complex Banach space and $A_j: D[A_j] \subset X_j \to X_j$ be a closed linear operator with domain $D[A_j]$ in X_j with non-empty resolvent set $\rho(A_j)$. Set $A=(A_1, \cdots, A_n)$.

$\Theta_j(\cdot\,;A_j)$ denotes the continuous morphism of the algebra $\mathcal{O}(\sigma_e(A_j))$ into the Banach algebra $L(X_j)$ which was investigated by Dunford and Taylor: if $f(\zeta_j)$ is a holomorphic function in an open neighbourhood U_j of the extended spectrum $\sigma_e(A_j)$ in $\boldsymbol{C}^*$ and $\tilde{f}$ is the germ of f in $\mathcal{O}(\sigma_e(A_j))$, we have

$$\Theta_j(\tilde{f};\, A_j) = f(A_j) = f(\infty)\, I_j + (2\pi i)^{-1} \int_{\Gamma_j} f(\zeta_j)\, R(\zeta_j;\, A_j)\, d\zeta_j\,,$$

where the contour Γ_j consists of a finite number of rectifiable, positively oriented Jordan curves lying in $U_j \backslash \sigma_e(A_j)$. We assume $f(\infty)=0$, if $\sigma_e(A_j)=\sigma(A_j)$.

THEOREM 3.1. *Let α be a uniform reasonable norm on $X_1\otimes\cdots\otimes X_n$. Given a closed linear operator A_j in X_j with non-empty resolvent set $\rho(A_j)$, for $j=1, 2, \cdots, n$, there exists a unique continuous morphism $\Theta(\cdot\,; A)$ of $\mathcal{O}\left(\prod_{j=1}^{n}\sigma_e(A_j)\right)$ into the Banach algebra $L(X_1)\widehat{\otimes_{\tilde{\alpha}}\cdots\otimes} L(X_n)$ which transforms the germ of the function 1 to $I_1\otimes\cdots\otimes I_n$ and the germ of the function $(\zeta_1, \cdots, \zeta_n)\to(\beta_j-\zeta_j)^{-1}$ for some fixed $(\beta_1, \cdots, \beta_n)\in\prod_{j=1}^{n}\rho(A_j)$ to $I_1\otimes\cdots\otimes I_{j-1}\otimes R(\beta_j; A_j)\otimes I_{j+1}\otimes\cdots\otimes I_n$ for $j=1, 2, \cdots, n$.*

In virtue of Proposition 1.3 we see that $\Theta(\cdot\,; A)$ is the continuous extension of $\Theta_1(\cdot\,; A_1)\otimes\cdots\otimes\Theta_n(\cdot\,; A_n)$ to the space

$$\mathcal{O}\left(\prod_{j=1}^{n}\sigma_e(A_j)\right)=\mathcal{O}\left(\sigma_e(A_1)\right)\widehat{\otimes\cdots\otimes}\,\mathcal{O}\left(\sigma_e(A_n)\right).$$

Now we state the spectral mapping theorem for tensor products of closed operators. Its proof will be reduced to Theorem 2.4 just as the operational calculus for closed operators was reduced to that for bounded operators, by the device of considering the resolvent of the resolvent of A_j for a fixed value of the parameter in the latter resolvent.

THEOREM 3.2. *Under the same assumptions and notations as in Theorem 3.1 we have for $\tilde{f}\in\mathcal{O}\left(\prod_{j=1}^{n}\sigma_e(A_j)\right)$*

$$\begin{aligned}\sigma_{L(X_1\widehat{\otimes_{\alpha}\cdots\otimes}X_n)}\left(\Theta(\tilde{f}; A)\right)&=\sigma_{L(X_1)\widehat{\otimes_{\tilde{\alpha}}\cdots\otimes}L(X_n)}\left(\Theta(\tilde{f}; A)\right)\\&=f\left(\sigma_e(A_1), \cdots, \sigma_e(A_n)\right).\end{aligned}$$

REMARK 3.3. It will be shown that Corollary 2.7 is also valid under the same assumption as in Theorem 3.2.

3.2. *Polynomial operators and their closability*

There are in $\mathcal{O}\left(\prod_{j=1}^{n}\sigma_e(A_j)\right)$ no germs of polynomials $P(\zeta_1, \cdots, \zeta_n)$ of degree ≥ 1 in some ζ_j, if $\sigma_e(A_j)$ contains ∞, or if $A\notin L(X_j)$. For a comprehensive operational calculus, it is important to include a theory of polynomials.

For $j=1, 2, \cdots, n$, let X_j be a complex Banach space and $A_j: D[A_j]\subset X_j\to X_j$ be a closable linear operator with resolvent set $\rho(A_j)$. Let

$$P(\zeta_1, \cdots, \zeta_n)=\sum_{k}a_k\zeta^k\equiv\sum_{k_1,\cdots,k_n}a_{k_1,\cdots,k_n}\zeta_1^{k_1}\cdot\zeta_2^{k_1}\cdot\cdots\cdot\zeta_n^{k_n},$$
$$a_k\in\boldsymbol{C},\quad k=(k_1, \cdots, k_n),\quad \zeta=(\zeta_1, \cdots, \zeta_n),$$

be a polynomial of degrees m_1 in ζ_1, m_2 in $\zeta_2, \cdots, m_n$ in ζ_n. For P, we define a polynormial operator $P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)$ in $X_1\widehat{\otimes_{\alpha}\cdots\otimes} X_n$ (with a reasonable norm α) with domain $D[A_1^{m_1}]\otimes D[A_2^{m_2}]\otimes\cdots\otimes D[A_n^{m_n}]$ as

$$\begin{aligned}&P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)\\&=\sum_{k_1,\cdots,k_n} a_{k_1,\cdots,k_n} A_1^{k_1}\otimes A_2^{k_2}\otimes\cdots\otimes A_n^{k_n}.\end{aligned}$$

By a property of algebraic tensor products (e.g. [9] Lemma 4.20), it is verified that this domain coincides with $\bigcap_{\substack{k\\ a_k\neq 0}} (D[A_1^{k_1}]\otimes D[A_2^{k_2}]\otimes\cdots\otimes D[A_n^{k_n}])$.

We denote by $\Theta(P; A)$ a maximal extension of $P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)$ in $X_1\widehat{\otimes_{\alpha}\cdots\otimes} X_n$. If the closures of the $A_j^{\nu_j}$ coincide with $(\tilde{A}_j)^{\nu_j}$, $1\leq\nu_j\leq m_j$, $1\leq j\leq n$, $\Theta(P; A)$ is also a maximal extension $\Theta(P; \tilde{A})$ of $P(\tilde{A}_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes\tilde{A}_n)$, where $\tilde{A}_j$ denotes the closure of A_j for $j=1, 2, \cdots, n$.

Let $\beta=(\beta_1, \cdots, \beta_n)\in\prod_{j=1}^{n}\rho(A_j)$ and $\varphi_\beta(\zeta)=\prod_{j=1}^{n}(\zeta_j-\beta_j)^{-m_j}$. If $\Theta(\varphi_\beta^{-1}; A) =(A_1-\beta_1 I_1)^{m_1}\widehat{\otimes_{\alpha}\cdots\otimes}(A_n-\beta_n I_n)^{m_n}$ is a maximal extension of $(A_1-\beta_1 I_1)^{m_1}\otimes\cdots\otimes(A_n-\beta_n I_n)^{m_n}$ in $X_1\widehat{\otimes_{\alpha}\cdots\otimes} X_n$, it has a bounded inverse and its domain $D[\Theta(\varphi_\beta^{-1}; A)]$ is independent of the choice of $\beta\in\prod_{j=1}^{n}\rho(A_j)$. In particular, if $(\tilde{A}_1-\beta_1 I_1)^{m_1}\otimes\cdots\otimes(\tilde{A}_n-\beta_n I_n)^{m_n}$ is closable in $X_1\widehat{\otimes_{\alpha}\cdots\otimes} X_n$, the domain of its closure coincides with the range of the continuous extension $\Theta(\tilde{\varphi}_\beta; A)$ of $(A_1-\beta_1 I_1)^{-m_1}\otimes\cdots\otimes(A_n-\beta_n I_n)^{-m_n}$ to the entire space $X_1\widehat{\otimes_{\alpha}\cdots\otimes} X_n$.

In general, the domain of $\Theta(\varphi_\beta^{-1}; A)$ is contained in the domain of $\Theta(P; A)$.

We show that the closability of operators is one of the properties of permanence for tensor products equipped with a faithful reasonable norm (cf. [9]).

THEOREM 3.4. *Let α be a faithful reasonable norm on $X_1\otimes\cdots\otimes X_n$. If for $j=1, 2, \cdots, n$, A_j is a closable linear operator in X_j with non-empty resolvent set $\rho(A_j)$, so is $P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)$ closable in $X_1\widehat{\otimes_{\alpha}\cdots\otimes} X_\alpha$. In particular, if all A_j are closed, so is $P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)$ closable in $X_1\widehat{\otimes_{\alpha}\cdots\otimes} X_n$.*

Proof. If for $j=1, 2, \cdots, n$, $\tilde{A}_j$ is the closure of A_j, $P(\tilde{A}_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes\tilde{A}_n)$ is an extension of $P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)$. So we may assume all A_j closed. We shall give two proofs of Theorem 3.4, one of which is made with the additional condition that α is uniform.

(1) Recall that, X and Y being Banach spaces, a densely defined linear transformation of $D[T]\subset Y$ into X is closable iff $D[T']$ is total over X, or equivalently, iff $D[T']$ is dense in X' with respect to the weak topology defined by the dual pair $\langle X', X\rangle$. The space X' equipped with this weak topology is denoted by X'_s.

Set $Y_j=\overline{D[A_j]}$, $j=1, 2, \cdots, n$. Y_j are Banach spaces. Let $Y_1\widehat{\otimes_\alpha\cdots\otimes} Y_n$ be the closure of $Y_1\otimes\cdots\otimes Y_n$ in $X_1\widehat{\otimes_\alpha\cdots\otimes} X_n$, which is a Banach space. In virtue of the Hahn-Banach theorem, we see that for $j=1, 2, \cdots, n$ and $\nu=1, 2, \cdots$, A_j^ν is a densely defined closed linear transformation of $D[A_j^\nu]\subset Y$ into X. The domian $D[(A'_j)^\nu]$ is dense in $(X'_j)_s$.

Since the natural linear mapping j_ι^α of $X_1\widehat{\otimes_\alpha\cdots\otimes} X_n$ into $X_1\widehat{\otimes_\iota\cdots\otimes} X_n$ is one-to-one by the faithfulness of α and every element of $X_1\widehat{\otimes_\iota\cdots\otimes} X_n$ is considered to be a separately continuous multilinear form on $(X'_1)_s\times\cdots\times(X'_n)_s$, the domain of the operator $P(A'_1\otimes\cdots\otimes I'_n, \cdots, I'_1\otimes\cdots\otimes A'_n)$ in $(X_1\widehat{\otimes_\alpha\cdots\otimes} X_n)'$, i.e. $D[(A'_1)^{m_1}]\otimes\cdots\otimes D[(A'_n)^{m_n}]$, is total over $X_1\widehat{\otimes_\alpha\cdots\otimes} X_n$. $P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)$ with domain $D[A_1^{m_1}]\otimes\cdots\otimes D[A_n^{m_n}]$ is densely defined in $Y_1\widehat{\otimes_\alpha\cdots\otimes} Y_n$, and so the adjoint $P(A_1\otimes\cdots\otimes I_n, \cdots)'$ in $(X_1\widehat{\otimes_\alpha\cdots\otimes} X_n)'$ is well-defined and an extension of $P(A'_1\otimes\cdots\otimes I'_n, \cdots)$. It follows that the domain of $P(A_1\otimes\cdots\otimes I_n, \cdots)'$ is total over $X_1\widehat{\otimes_\alpha\cdots\otimes} X_n$. Thus $P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)$ is closable in $Y_1\widehat{\otimes_\alpha\cdots\otimes} Y_n$, therefore in $X_1\widehat{\otimes_\alpha\cdots\otimes} X_n$.

(2) We give now another proof of Theorem 3.4, when α is a uniform faithful reasonable norm.

For $(\beta_1, \cdots, \beta_n)\in\prod_{j=1}^n \rho(A_j)$ rewrite

$$P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)=\sum b_{k_1,\cdots,k_n}(A_1-\beta_1 I_1)^{k_1}\otimes\cdots\otimes(A_n-\beta_n I_n)^{k_n}.$$

Let $u_\nu\in D[A_1^{m_1}]\otimes\cdots\otimes D[A_n^{m_n}]$, $u_\nu\to 0$ and $P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)\,u_\nu\to v$ in the norm α as $\nu\to\infty$. We shall show $v=0$. Applying the continuous operator $\Theta(\tilde{\varphi}_\beta; A)$, $\varphi_\beta(\zeta)=\prod_{j=1}^n(\zeta_j-\beta_j)^{-m_j}$, which is the continuous extension of $(A_1-\beta_1 I_1)^{-m_1}\otimes\cdots\otimes(A_n-\beta_n I_n)^{-m_n}$ to the entire space $X_1\widehat{\otimes_\alpha\cdots\otimes} X_n$, we have by continuity

$$\begin{aligned}&\Theta(\tilde{\varphi}_\beta; A)\,v\\ &\quad=\Theta(\widetilde{\zeta_{1\beta}^{-1}}; A)^{m_1}\cdot\Theta(\widetilde{\zeta_{2\beta}^{-1}}; A)^{m_2}\cdot\cdots\cdot\Theta(\widetilde{\zeta_{n\beta}^{-1}}; A)^{m_n}v=0\,,\quad \zeta_{j\beta}=\zeta_j-\beta_j\,,\end{aligned}$$

where $\Theta(\widetilde{\zeta_{1\beta}^{-1}}; A), \cdots$, are the continuous extensions of $(A_1-\beta_1 I_1)^{-1}\otimes I_2\otimes\cdots\otimes I_n, \cdots$, to the entire space $X_1\widehat{\otimes_\alpha\cdots\otimes} X_n$.

If $\alpha=\varepsilon$, the operators $\Theta(\widetilde{\zeta_{j\beta}^{-1}}; A)$ are one-to-one. Moreover, if α is faithful, they are also one-to-one (see [9]). Thus $v=0$, which implies the closability of $P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)$.

Q. E. D.

When $P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)$ is closable in $\widetilde{X_1\otimes_\alpha\cdots\otimes X_n}$, $\Theta(P; A)$ is nothing but the closure of $P(A_1\otimes\cdots\otimes I_n, \cdots)$. Its domain $D[\Theta(P; A)]$ is the completion of $D[A_1^{m_1}]\otimes\cdots\otimes D[A_n^{m_n}]$ with respect to the graph norm of $P(A_1\otimes\cdots\otimes I_n, \cdots)$.

3. 3. *The spectral mapping theorem for polynomial operators*

We shall determine how the spectra of a polynomial operator $P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)$ and of its maximal extension $\Theta(P; A)$ are related to the spectra of the $A_1, \cdots, A_n$.

Given n subsets G_j of $\boldsymbol{C}$, $1\leq j\leq n$, we can define the set $P(G_1, \cdots, G_n)$ in an obvious way if $P(\zeta_1, \cdots, \zeta_n)$ is a polynomial independent of those variables ζ_j for which the G_j are empty, and otherwise we set $P(G_1, \cdots, G_n)=\phi$.

On the other hand, given n non-empty subsets G_j of $\boldsymbol{C}^*$, $1\leq j\leq n$, we define $P(G_1, \cdots, G_n)=P(G_1\backslash\{\infty\}, \cdots, G_n\backslash\{\infty\})$ if $P(\zeta_1, \cdots, \zeta_n)$ is independent of those variables ζ_j for which the G_j contain ∞, and otherwise we set $P(G_1, \cdots, G_n)=P(G_1\backslash\{\infty\}, \cdots, G_n\backslash\{\infty\})\cup\{\infty\}$.

We show first that the set $P(\sigma(A_1), \cdots, \sigma(A_n))$ is contained in the spectrum of $P(A_1\otimes\cdots\otimes I_n, \cdots)$. They do not coincide in general without any additional conditions on polynomials P(cf. [9]).

We need

LEMMA 3. 5. *Let T: $D[T]\subset X\rightarrow X$ be a closed linear operator in a Banach space X with non-empty resolvent set $\rho(T)$. If $\lambda\in\pi(T)$, and m is a positive integer, there exists a sequence $\{x_\nu\}_{\nu=1}^\infty\subset D[T^m]$ with $\|x_\nu\|=1$ such that the sequence $\{S(T)x_\nu\}$ is bounded for any polynomial $S(\zeta)$ of degree $\leq m$ and such that $Q(T)(T-\lambda I)x_\nu$ converges to zero as $\nu\rightarrow\infty$ for all polynomials $Q(\zeta)$ of degree $\leq m-1$.*

Proof. By assumption there exists a sequence $\{y_\nu\}_{\nu=1}^\infty\subset D[T]$, $\|y_\nu\|=1$, $\nu=1, 2, \cdots$, such that $(T-\lambda I)y_\nu$ converges to zero. For $\mu\in\rho(T)$ fixed, $(T-\mu I)^{-1}\in L(X)$. Then $\|(T-\mu I)^{-(m-1)}y_\nu\|$ is bounded away from zero. Set

$$x_\nu=\|(T-\mu I)^{-(m-1)}y_\nu\|^{-1}(T-\mu I)^{-(m-1)}y_\nu, \qquad \nu=1, 2, \cdots.$$

The sequence $\{x_\nu\}$ is a required one.

THEOREM 3. 6. *For $j=1, 2, \cdots, n$, let A_j: $D[A_j]\subset X_j\rightarrow X_j$ be a densely defined closed linear operator in a complex Banach space X_j with non-empty* *resolvent set* $\rho(A_j)$. *Let* α *be a reasonable norm on* $X_1\otimes\cdots\otimes X_n$.

If $P(\zeta_1, \cdots, \zeta_n)$ *is a polynomial of degrees* m_1 *in* $\zeta_1, \cdots, m_n$ *in* ζ_n, *then we have for the spectra of the operator* $P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)$ *and its maximal extension* $\Theta(P; A)$ *in* $X_1\widehat{\otimes_\alpha\cdots\otimes} X_n$

$$P\Big(\sigma(A_1), \cdots, \sigma(A_n)\Big)\subset\sigma\Big(P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)\Big)$$
$$=\sigma\Big(\Theta(P; A)\Big),$$

provided that none of the $\sigma(A_1), \cdots, \sigma(A_n)$ *are empty.*

Proof. The fact that the spectrum does not change under maximal extensions yields the last equality.

We may assume $P(\zeta_1, \cdots, \zeta_n)\not\equiv$ constant.

Let $(\alpha_1, \cdots, \alpha_n)\in \prod_{j=1}^{n} \sigma(A_j)$. Then there exist polynomials $P_j(\zeta_1, \cdots, \zeta_n)$ of degrees $\leq m_i$ in ζ_i, $i\neq j$ and $\leq m_j-1$ in ζ_j, $1\leq j\leq n$, such that

$$P(\zeta_1, \cdots, \zeta_n)-P(\alpha_1, \cdots, \alpha_n)$$
$$=(\zeta_1-\alpha)P_1(\zeta_1, \cdots, \zeta_n)+\cdots+(\zeta_n-\alpha_n)P_n(\zeta_1, \cdots, \zeta_n),$$

to which corresponds by definition the operator

$$P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)-P(\alpha_1, \cdots, \alpha_n)(I_1\otimes\cdots\otimes I_n)$$
$$=P_1(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)\Big\{(A_1-\alpha_1 I_1)\otimes\cdots\otimes I_n\Big\}+\cdots$$
$$\cdots+P_n(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)\Big\{I_1\otimes\cdots\otimes(A_n-\alpha_n I_n)\Big\}.$$

Our proof will proceed in an analogous way to that of Theorem 2.4, by the aid of Proposition 1.2. The proof is divided into three cases. For simplicity we assume $n=3$ again.

(1) Case: $(\alpha_1, \alpha_2, \alpha_3)\in \prod_{j=1}^{3} \pi(A_j)$. In virtue of Lemma 3.5, there exists sequences of unit vectors $\{x_j^{(\nu)}\}_{\nu=1}^{\infty}\subset D[A_j^{m_j}]$, $1\leq j\leq 3$, such that for each j $\|Q_j(A_j)(A_j-\alpha_j I_j)x_j^{(\nu)}\|\to 0$ as $\nu\to\infty$, for all polynomials $Q_j(\zeta_j)$ of degree $\leq m_j-1$, and $\|S_j(A_j)x_j^{(\nu)}\|$ is bounded for any polynomial $S_j(\zeta_j)$ of degree $\leq m_j$.

By the reasonableness of α, we obtain

$$\|x_1^{(\nu)}\otimes x_2^{(\nu)}\otimes x_3^{(\nu)}\|_\alpha=\|x_1^{(\nu)}\|\cdot\|x_2^{(\nu)}\|\cdot\|x_3^{(\nu)}\|=1, \qquad \nu=1, 2, \cdots,$$

and

$$\Big\{P(A_1\otimes I_2\otimes I_3, I_1\otimes A_2\otimes I_3, I_1\otimes I_2\otimes A_3)-P(\alpha_1, \alpha_2, \alpha_3)(I_1\otimes I_2\otimes I_3)\Big\}(x_1^{(\nu)}\otimes x_2^{(\nu)}\otimes x_3^{(\nu)})$$

converges to zero in the norm α as $\nu\to\infty$, since $P_j(\zeta_1, \zeta_2, \zeta_3)$ is of degree $\leq m_i$ in ζ_i, $i\neq j$, and $\leq m_j-1$ in ζ_j. It follows that

$$P(\alpha_1, \alpha_2, \alpha_3)\in\pi\left(P(A_1\otimes I_2\otimes I_3, I_1\otimes A_2\otimes I_3, I_1\otimes I_2\otimes A_3)\right)$$
$$\subset\sigma\left(P(A_1\otimes I_2\otimes I_3, I_1\otimes A_2\otimes I_3, I_1\otimes I_2\otimes A_3)\right).$$

(2) Case: $(\alpha_1, \alpha_2, \alpha_3)\in\prod_{j=1}^{3}\gamma(A_j)$. This implies $(\alpha_1, \alpha_2, \alpha_3)\in\prod_{j=1}^{3}P_\sigma(A_j')\subset\prod_{j=1}^{3}\pi(A_j')$ by Proposition 1.2. The dual norm α' is reasonable on $X_1'\otimes X_2'\otimes X_3'$, so we obtain in a similar way to (1)

$$P(\alpha_1, \alpha_2, \alpha_3)\in P_\sigma\left(P(A_1'\otimes I_2'\otimes I_3', I_1'\otimes A_2'\otimes I_3', I_1'\otimes I_2'\otimes A_2')\right)$$
$$\subset\pi\left(P(A_1'\otimes I_2'\otimes I_3', I_1'\otimes A_2'\otimes I_3', I_1'\otimes I_2'\otimes A_2')\right)$$

If the A_j are densely defined closed in X_j with non-empty resolvent sets, then the $D[A_j^{m_j}]$ are also dense in X_j, $1\leq j\leq 3$, so that $D[A_1^{m_1}]\otimes\cdots\otimes D[A_n^{m_n}]$ is dense in $X_1\widehat{\otimes_\alpha X_2}\otimes X_3$. Then the adjoint of $P(A_1\otimes I_2\otimes I_3, I_1\otimes A_2\otimes I_3, I_1\otimes I_2\otimes A_3)$ is well-defined in $(X_1\widehat{\otimes_\alpha X_2\otimes} X_3)'$, which is an extension of $P(A_1'\otimes I_2'\otimes I_3, \cdots)$, if the latter is considered as an operator in $(X_1\widehat{\otimes_\alpha X_2\otimes} X_3)'$ with domain $D[(A_1')^{m_1}]\otimes D[(A_2')^{m_2}]\otimes D[(A_3')^{m_3}]$. In virtue of Lemma 1.1 and Proposition 1.2, we obtain

$$P(\alpha_1, \alpha_2, \alpha_3)\in\pi\left(P(A_1\otimes I_2\otimes I_3, I_1\otimes A_2\otimes I_3, I_1\otimes I_2\otimes A_3)'\right)$$
$$\subset\sigma\left(P(A_1\otimes I_2\otimes I_3, I_1\otimes A_2\otimes I_3, I_1\otimes I_2\otimes A_3)'\right)$$
$$=\sigma\left(P(A_1\otimes I_2\otimes I_3, I_1\otimes A_2\otimes I_3, I_1\otimes I_2\otimes A_3)\right).$$

(3) Case: $(\alpha_{s_1}, \alpha_{s_2}, \alpha_{s_3})\in\pi(A_{s_1})\times\gamma(A_{s_2})\times\gamma(A_{s_3})$ or $(\alpha_{s_1}, \alpha_{s_2}, \alpha_{s_3})\in\pi(A_{s_1})\times\pi(A_{s_2})\times\gamma(A_{s_3})$, where $s=\begin{pmatrix}1 & 2 & 3\\ s_1 & s_2 & s_3\end{pmatrix}$ is an arbitrary permutation.

We shall be able to treat this case just in the same way as in the proof (3) of Theorem 2.4, making use of the following lemma instead of Lemma 2.5.

LEMMA 3.7. *Let $G_1, G_2\subsetneqq\boldsymbol{C}$ be open sets and suppose the boundary ∂G_1 or ∂G_2 has an infinite number of points. Let $P(\zeta_1, \zeta_2)$ be a polynomial in ζ_1, ζ_2 and $P(\alpha_1, \alpha_2)=0$ for some $(\alpha_1, \alpha_2)\in G_1\times G_2$. Then there exist a boundary point (α_1^0, α_2^0) of $G_1\times G_2\subset\boldsymbol{C}^2$ and a continuous contour $(\alpha_1(t), \alpha_2(t))$, $0\leq t\leq 1$,* *such that* $(\alpha_1(t), \alpha_2(t)) \in G_1 \times G_2$ *for* $0 \leq t < 1$, $\alpha_1(0) = \alpha_1$, $\alpha_2(0) = \alpha_2$; $\alpha_1(1) = \alpha_1^0$, $\alpha_2(1) = \alpha_2^0$, *and such that* $P(\alpha_1(t), \alpha_2(t)) = 0$ *for* $0 \leq t \leq 1$.

Proof. Suppose ∂G_1 has an infinite number of points. Without loss of generality, we may assume G_1 and G_2 connected. Let $\eta = \eta(\xi)$ be a root of $P(\xi, \eta) = 0$ with the property $\eta(\alpha_1) = \alpha_2$. Since $\eta(\xi)$ is defined for all ξ and has only a finite number of poles in $\boldsymbol{C}$, that subset G_1' of G_1 which excludes all these poles is an open connected set with non-empty boundary and $(\alpha_1, \alpha_2) \in (G_1' \cap G_1) \times G_2$. There exists a boundary point α_1^0 of G_1' which is also a boundary point of G_1, but which is no pole of $\eta = \eta(\xi)$, and a continuous contour $\alpha_1(t)$, $0 \leq t \leq 1$, such that $\alpha_1(t) \in G_1$, $0 \leq t < 1$, and $\alpha_1(0) = \alpha_1$, $\alpha_1(1) = \alpha_1^0$. Set $\alpha_2(t) = \eta(\alpha_1(t))$. Then $\alpha_2(t)$ is obviously continuous for $0 \leq t \leq 1$. We have by continuity $P(\alpha_1(t), \alpha_2(t)) = P(\alpha_1(t), \eta(\alpha_1(t))) = 0$, $0 \leq t \leq 1$. There exists $0 < t_0 \leq 1$ such that $(\alpha_1(t), \alpha_2(t)) \in G_1 \times G_2$ for $0 \leq t < t_0$, and such that $(\alpha_1(t_0), \alpha_2(t_0))$ is a boundary point of $G_1 \times G_2$. We have only to make a suitable transformation of the parameter t.

Q. E. D.

End of Proof of Theorem 3.6. We shall give only the proof of reducing the case $(\alpha_1, \alpha_2, \alpha_3) \in \pi(A_1) \times \Upsilon(A_2) \times \Upsilon(A_3)$ to the case $(\alpha_1, \alpha_2, \alpha_3) \in \pi(A_1) \times \pi(A_2) \times \Upsilon(A_3)$ or $(\alpha_1, \alpha_2, \alpha_3) \in \pi(A_1) \times \Upsilon(A_2) \times \pi(A_3)$. $\Upsilon(A_2)$ and $\Upsilon(A_3)$ are open subsets of $\boldsymbol{C}$, the boundary of $\Upsilon(A_j)$ is contained in that of $\pi(A_j)$ and the polynomial

$$Q(\zeta_2, \zeta_3) = P(\alpha_1, \zeta_2, \zeta_3) - P(\alpha_1, \alpha_2, \alpha_3)$$

in ζ_2, ζ_3 satisfies the condition of Lemma 3.7. So there exist a boundary point (α_2^0, α_3^0) of $\Upsilon(A_2) \times \Upsilon(A_3)$ such that $Q(\alpha_2^0, \alpha_3^0) = 0$ and such that $\alpha_2^0 \in \pi(A_2)$ *or* $\alpha_3^0 \in \pi(A_3)$. Thus, considering $(\alpha_1, \alpha_2^0, \alpha_3^0)$ instead of $(\alpha_1, \alpha_2, \alpha_3)$ yields the desired reduction.

We omit the remaining proof which will proceed just in the same way as in the proof (3) of Theorem 2.4, by the aid of Lemma 3.7 like above.

Since the approximate point spectrum and compression spectrum do not change under maximal extensions, we have from the proof of Theorem 3.6 and Proposition 1.2

COROLLARY 3.8. *Under the same assumptions as in Theorem* 3.6, *the following relations hold*

$$P\big(P_\sigma(A_1), \cdots, P_\sigma(A_n)\big) \subset P_\sigma\big(P(A_1 \otimes \cdots \otimes I_n, \cdots, I_1 \otimes \cdots \otimes A_n)\big)$$
$$\subset P_\sigma\big(\Theta(P; A)\big);$$

$$P\big(\pi(A_1), \cdots, \pi(A_n)\big) \subset \pi\big(P(A_1 \otimes \cdots \otimes I_n, \cdots, I_1 \otimes \cdots \otimes A_n)\big) = \pi\big(\Theta(P; A)\big);$$

$$P\big(\Gamma(A_1), \cdots, \Gamma(A_n)\big) = P\big(P_\sigma(A'_1), \cdots, P_\sigma(A'_n)\big)$$
$$\subset P_\sigma\big(P(A'_1\otimes\cdots\otimes I'_n, \cdots, I'_1\otimes\cdots\otimes A'_n)\big)$$
$$\subset P_\sigma\big(P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)'\big)$$
$$= \Gamma\big(P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)\big)$$
$$= \Gamma\big(\Theta(P; A)\big).$$

REMARK 3.9. Since the spectrum and the approximate point spectrum are closed, we see further from Theorem 3.6 and Corollary 3.8 that the closure of the set $P(\sigma(A_1), \cdots, \sigma(A_n))$ is contained in $\sigma(P(A_1\otimes\cdots\otimes I_n, \cdots)$ and the closure of the set $P(\pi(A_1), \cdots, \pi(A_n))$ in $\pi(P(A_1\otimes\cdots\otimes I_n, \cdots))$.

The following Lemma and Corollary are concerned with the existence of inverse operators. We assume α is a uniform reasonable norm on $X_1\otimes\cdots\otimes X_n$.

LEMMA 3.10. *For $j=1, 2, \cdots, n$, let A_j: $D[A_j]\subset X_j\rightarrow X_j$ be a closed linear operator with non-empty resolvent set $\rho(A_j)$. Let $\tilde{f}\in\mathcal{O}\Big(\prod_{j=1}^{n}\sigma_e(A_j)\Big)$ and $F=\varphi_\beta^{-1}\cdot f$, $\varphi_\beta(\zeta)=\prod_{j=1}^{n}(\zeta_j-\beta_j)^{-m_j}$ for a fixed $\beta=(\beta_1, \cdots, \beta_n)\in\prod_{j=1}^{n}\rho(A_j)$. If F and F^{-1} belong to $\mathcal{O}\Big(\prod_{j=1}^{n}\sigma_e(A_j)\Big)$, then each maximal extension $\Theta(\varphi_\beta^{-1}; A)$ of $(A_1-\beta_1I_1)^{m_1}\otimes\cdots\otimes(A_n-\beta_nI_n)^{m_n}$ in $X_1\widehat{\otimes_\alpha\cdots\otimes} X_n$ determines a maximal operator $\boldsymbol{F}$; $D[\boldsymbol{F}]\subset X_1\widehat{\otimes_\alpha\cdots\otimes}X_n\rightarrow X_1\widehat{\otimes_\alpha\cdots\otimes}X_n$ with domain $D[\boldsymbol{F}]=D[\Theta(\varphi_\beta^{-1}; A)]$ and dense range $R[\boldsymbol{F}]$ such that*

$$\Theta(\tilde{f}; A)\cdot\boldsymbol{F}u=u, \qquad u\in D[\boldsymbol{F}].$$

Proof. The continuous extension $\Theta(\tilde{\varphi}_\beta; A)$ of $(A_1-\beta_1I_1)^{-m_1}\otimes\cdots\otimes$ $\otimes(A_n-\beta_nI_n)^{-m_n}$ to $X_1\widehat{\otimes_\alpha\cdots\otimes}X_n$ may not in general be invertible (cf. Proof (2) of Theorem 3.4). However, by the properties of maximal extensions, we have, if $\Theta(\varphi_\beta^{-1}; A)$ is a maximal extension of $(A_1-\beta_1I_1)^{m_1}\otimes\cdots\otimes(A_n-\beta_nI_n)^{m_n}$ in $X_1\widehat{\otimes_\alpha\cdots\otimes}X_n$,

$$\Theta(\tilde{\varphi}_\beta; A)\cdot\Theta(\varphi_\beta^{-1}; A)\,u=u, \qquad u\in D\big[\Theta(\varphi_\beta^{-1}; A)\big]$$
$$\Theta(\varphi_\beta^{-1}; A)\cdot\Theta(\tilde{\varphi}_\beta; A)\,v=v, \qquad v\in R\big[\Theta(\varphi_\beta^{-1}; A)\big]$$

Here we note that all the maximal extensions $\Theta(\varphi_\beta^{-1}; A)$ have the same domain $D[\Theta(\varphi_\beta^{-1}; A)]$ which is independent of $\beta\in\prod_{j=1}^{n}\rho(A_j)$, but they may have in general different ranges $R[\Theta(\varphi_\beta^{-1}; A)]$.

In virtue of Theorem 3.1, we obtain

$$\Theta(\tilde{F}; A)\cdot\Theta(\widetilde{F^{-1}}; A)=\Theta(\widetilde{F^{-1}}; A)\cdot\Theta(\tilde{F}; A)=\boldsymbol{I}$$

and

$$\Theta(\tilde{f}; A)=\Theta(\tilde{\varphi}_\beta; A)\cdot\Theta(\tilde{F}; A)=\Theta(\tilde{F}; A)\cdot\Theta(\tilde{\varphi}_\beta; A),$$

where all the elements are considered as ones of the algebra $\widehat{L(X_1)\otimes\cdots_{\alpha}\cdots\otimes L(X_n)}$.

Set $\boldsymbol{F}=\Theta(\widetilde{F^{-1}}; A)\cdot\Theta(\varphi_\beta^{-1}; A)$. Then $\boldsymbol{F}$ is maximal, since $\Theta(\varphi_\beta^{-1}; A)$ is maximal and $\Theta(\widetilde{F^{-1}}; A)$ is an automorphism of $\widehat{X_1\otimes\cdots_{\alpha}\cdots\otimes X_n}$. The range $R[\boldsymbol{F}]$ of $\boldsymbol{F}$ is dense, since $R[\Theta(\varphi_\beta^{-1}; A)]$ is dense.

It is clear that for $u\in D[\boldsymbol{F}]=D[\Theta(\varphi_\beta^{-1}; A)]$

$$\begin{aligned}\Theta(\tilde{f}; A)\cdot\boldsymbol{F}u&=\left[\Theta(\tilde{f}; A)\cdot\Theta(\widetilde{F^{-1}}; A)\right]\cdot\Theta(\varphi_\beta^{-1}; A)\,u\\&=\Theta(\tilde{\varphi}_\beta; A)\cdot\Theta(\varphi_\beta^{-1}; A)\,u\\&=u.\end{aligned}$$

Q. E. D.

COROLLARY 3.11. *Let A_j be as in Lemma 3.10. Suppose $P(\zeta)$ is a polynomial of degrees m_j in ζ_j, $1\leq j\leq n$, such that $P(\zeta)^{-1}$ exists and is holomorphic in an open neighbourhood of $\prod_{j=1}^{n}\sigma_e(A_j)$ in $\boldsymbol{C}^{*n}$, Then every maximal extension $\Theta(P; A)$ of $P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)$ in $\widehat{X_1\otimes\cdots_{\alpha}\cdots\otimes X_n}$ has the same domain with any maximal extension $\Theta(\varphi_\beta^{-1}; A)$ of $(A_1-\beta_1 I_1)^{m_1}\otimes\cdots\otimes(A_n-\beta_n I_n)^{m_n}$ in $\widehat{X_1\otimes\cdots_{\alpha}\cdots\otimes X_n}$ and admits a densely defined bounded inverse.*

Proof. Among maximal extensions of $(A_1-\beta_1 I_1)^{m_1}\otimes\cdots\otimes(A_n-\beta_n I_n)^{m_n}$ in $\widehat{X_1\otimes\cdots_{\alpha}\cdots\otimes X_n}$, we take an arbitrary one and denote it by $\Theta(\varphi_\beta^{-1}; A)$.

Since $P(\zeta)^{-1}$ satisfies the condition of f in Lemma 3.10, $\boldsymbol{F}=\Theta(\widetilde{F^{-1}}; A)\cdot\Theta(\varphi_\beta^{-1}; A)$ where $F=\varphi_\beta^{-1}\cdot P^{-1}$ is a maximal operator in $\widehat{X_1\otimes\cdots_{\alpha}\cdots\otimes X_n}$ with dense range $R[\boldsymbol{F}]$ which has a densely defined bounded inverse $\Theta(\widetilde{P^{-1}}; A)|R[\boldsymbol{F}]$.

Since we have with $P(\zeta)=\sum b_k(\zeta-\beta)^k$,

$$\begin{aligned}&P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)\\&\quad=\left\{\sum b_k(A_1-\beta_1 I_1)^{-(m_1-k_1)}\otimes\cdots\otimes(A_n-\beta_n I_n)^{-(m_n-k_n)}\right\}\\&\qquad\cdot\left\{(A_1-\beta_1 I_1)^{m_1}\otimes\cdots\otimes(A_n-\beta_n I_n)^{m_n}\right\},\end{aligned}$$

$\boldsymbol{F}$ is an extension of $P(A_1\otimes\cdots\otimes I_n, \cdots)$. By the maximality, it follows that $\boldsymbol{F}$ is a maximal extension of $P(A_1\otimes\cdots\otimes I_n, \cdots)$ in $X_1\widehat{\otimes\underset{\alpha}{\cdots}\otimes} X_n$, which we denote by $\Theta(P; A)$. Thus $\Theta(P; A)$ has a densely defined bounded inverse $\Theta(\widetilde{P^{-1}}; A)|R[\boldsymbol{F}]$. By the properties of maximal extensions this is true for another maximal extension $\Theta(P; A)$ of $P(A_1\otimes\cdots\otimes I_n, \cdots)$.

Q. E. D.

We shall state the spectral mapping theorem for polynomial operators continuous at the product of the extended spectra.

DEFINITION 3.12. Given n non-empty compact subsets K_j of $\boldsymbol{C}^*$, $1\leq j\leq n$, a polynomial $P(\zeta)$ is said to be *continuous at* $\prod_{j=1}^{n} K_j$ *(as a mapping of* $\boldsymbol{C}^{*n}$ *into* $\boldsymbol{C}^*$*)*, if for any open neighbourhood V in $\boldsymbol{C}^*$ of the closure of $P(K_1, \cdots, K_n)$ there exist open neighbourhoods U_j is $\boldsymbol{C}^*$ of K_j, $1\leq j\leq n$, such that $P(U_1, \cdots, U_n)$ is contained in V.

It is seen that $P(\zeta)$ is continuous at $\prod_{j=1}^{n} K_j$ iff $P(\zeta)$ is continuous at every point ζ of $\prod_{j=1}^{n} K_j$. Consequently, $P(K_1, \cdots, K_n)$ is compact in $\boldsymbol{C}^*$ or $P(K_1, \cdots, K_n)\backslash\{\infty\}$ is closed in $\boldsymbol{C}$.

THEOREM 3.13. *For $j=1, 2, \cdots, n$, let A_j: $D[A_j]\subset X_j\to X_j$ be a densely defined closed linear operator in a complex Banach space X_j with non-empty resolvent set $\rho(A_j)$. Let α be a uniform reasonable norm on $X_1\otimes\cdots\otimes X_n$. If $P(\zeta)$ is a polynomial continuous at $\prod_{j=1}^{n} \sigma_e(A_j)$ (as a mapping of $\boldsymbol{C}^{*n}$ into $\boldsymbol{C}^*$), then we have for the spectra of $P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)$ and of its maximal extension $\Theta(P; A)$ in $X_1\widehat{\otimes\underset{\alpha}{\cdots}\otimes} X_n$*

$$P\big(\sigma(A_1), \cdots, \sigma(A_n)\big)=\sigma\big(P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)\big)$$
$$=\sigma\big(\Theta(P; A)\big),$$

provided none of the $\sigma(A_j)$ are empty. The emptiness of at least one of the $\sigma(A_j)$ is equivalent to the emptiness of $\sigma(P(A_1\otimes\cdots\otimes I_n, \cdots))$ and of $\sigma(\Theta(P; A))$.

Proof. If all A_j are bounded, Theorem 3.13 follows from Theorem 2.4. By Theorem 3.6, we have only to show the other inclusion, assuming $P(\zeta)$ is not identically constant and $P(\sigma(A_1), \cdots, \sigma(A_n))\neq\boldsymbol{C}$.

Suppose $\lambda\notin P(\sigma(A_1), \cdots, \sigma(A_n))$. Set $P_\lambda(\zeta)=P(\zeta)-\lambda$. Then the continuity of $P(\zeta)$ at $\prod_{j=1}^{n} \sigma_e(A_j)$ implies that $P_\lambda(\zeta)^{-1}$ is holomorphic in an open neighbourhood of $\prod_{j=1}^{n} \sigma_e(A_j)$ in $\boldsymbol{C}^{*n}$, so that $P_\lambda(\zeta)$ satisfies the condition of P in Corollary 3.11. It follows that $\Theta(P_\lambda; A)=\Theta(P; A)-\lambda\boldsymbol{I}$ admits a densely defined bounded inverse $\Theta(\widetilde{P_\lambda^{-1}})|R[\Theta(P_\lambda; A)]$ in $X_1\widehat{\otimes\underset{\alpha}{\cdots}\otimes} X_n$. Thus $\lambda \notin \sigma(\Theta(P; A))$.

Q. E. D.

REMARK 3.14. Since the spectrum, approximate point spectrum and compression spectrum do not change under maximal extensions, Theorem 3.6, Corollary 3.8 and Theorem 3.13 are valid for densely defined closable linear operators A_j in X_j, $1\leq j\leq n$, if the closures of A_j^ν coincide with $(\tilde{A}_j)^\nu$ for $j=1, 2, \cdots, n$, and $\nu=1, 2, \cdots$, so that any maximal extension of $P(A_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes A_n)$ in $X_1\widehat{\otimes\underset{\alpha}{\cdots}\otimes} X_n$ is at the same time a maximal extension of $P(\tilde{A}_1\otimes\cdots\otimes I_n, \cdots, I_1\otimes\cdots\otimes\tilde{A}_n)$ in $X_1\widehat{\otimes\underset{\alpha}{\cdots}\otimes} X_n$.

Finally we note that all the considerations would be simpler in virtue of Theorem 3.4, if the norm α is in addition faithful.

3.4. *The tensor product* $A\otimes B$

Theorem 3.13 with Remark 3.14 and Corollary 2.8 give now a precise knowledge of the spectrum $\sigma(A\otimes B)$ for the tensor product $A\otimes B$ of densely defined *closable* operators A and B, if we take the polynomial $P(\zeta_1, \zeta_2)=\zeta_1\cdot\zeta_2$ (cf. [9]).

THEOREM 3.15. *Let X and Y be complex Banach spaces and α be a uniform reasonable norm on $X\otimes Y$. Let A(resp. B) be a densely defined closable linear operator in X(resp. Y) with spectrum $\sigma(A)$ (resp. $\sigma(B)$) and extended spectrum $\sigma_e(A)$ (resp. $\sigma_e(B)$). Denote by $A\otimes B$ the tensor product of A and B and by $A\hat{\otimes}_\alpha B$ its maximal extension in $X\hat{\otimes}_\alpha Y$.*

Then among the following four assertions:

(1) *It is not the case that one of the extended spectra $\sigma_e(A)$ and $\sigma_e(B)$ contains 0 and the other contains ∞;*

(1)′ *$\sigma_e(A)\sigma_e(B)=\{\alpha\beta; (\alpha, \beta)\in\sigma_e(A)\times\sigma_e(B)\}$ is a well-defined (closed) subset of the extended complex plane $\boldsymbol{C}^*$, where $\alpha\cdot\beta$ is defined except* $\alpha=0$, $\beta=\infty$ or $\alpha=\infty$, $\beta=0$;

(1)″ (i) *A and B are bounded in X, Y, respectively, so that $\sigma_e(A)=\sigma(A)$ and $\sigma_e(B)=\sigma(B)$ are compact in $\boldsymbol{C}$;*

(ii) *A and B have densely defined bounded inverses, so that $0\in\rho(A)$ and $0\in\rho(B)$;*

(iii) *One of A and B is bounded and has a densely defined bounded inverse, while the other is arbitrary;*

(2) *$\sigma(A)\sigma(B)=\sigma(A\otimes B)=\sigma(A\hat{\otimes}_\alpha B)$, if none of $\sigma(A)$ and $\sigma(B)$ are empty, while $\sigma(A\otimes B)=\sigma(A\hat{\otimes}_\alpha B)=\phi$ is equivalent to $\sigma(A)=\phi$ or $\sigma(B)=\phi$;*

we have the implication (1)=(1)′=(1)″⇒(2) *and if* (1) *is not satisfied,* (2) *is not always true* ([9]); *it may be true, for instance, note that the spectrum of the tensor product* $A\otimes B$ *of the zero operator* $A=O$ *and an arbitrary operator* B *is* $\{0\}$.

4. An application

We consider the characteristic initial value problem for the partial differential equation

$$L[u] = \sum_{\substack{0\leq j\leq m\\ 0\leq k\leq n}} c_{jk}\left(\frac{\partial}{\partial x}\right)^j\left(\frac{\partial}{\partial y}\right)^k u(x, y) = f(x, y),$$

with constant coefficients c_{jk}, $c_{mn}\neq 0$, under the initial condition

$$u(0, y) = \frac{\partial}{\partial x}u(0, y) = \cdots = \left(\frac{\partial}{\partial x}\right)^{m-1}u(0, y) = 0,$$

$$u(x, 0) = \frac{\partial}{\partial y}u(x, 0) = \cdots = \left(\frac{\partial}{\partial y}\right)^{n-1}u(x, 0) = 0,$$

in the square $I\times I$, $I=[0, 1]\subset \boldsymbol{R}$.

We denote the operator L with this initial condition also by the same L.

Denote by $\mathfrak{R}$ one of the following function spaces on $I\times I$:

$C(I\times I)$=Banach space of the continuous functions on $I\times I$;

$B(I\times I)$=Banach space of the functions continuous on $I\times I$, analytic in the interior of $I\times I$;

$L^2(I\times I)$=Hilbert space of the square integrable functions on $I\times I$.

For each $\mathfrak{R}$, let

$$\mathfrak{D} = \left\{f;\ \left(\frac{\partial}{\partial x}\right)^j\left(\frac{\partial}{\partial y}\right)^k f(x, y)\in\mathfrak{R},\ \begin{matrix}0\leq j\leq m\\ 0\leq k\leq n\end{matrix},\ u(0, y) = \frac{\partial}{\partial x}u(0, y) = \cdots\right.$$
$$\left.\cdots = \left(\frac{\partial}{\partial x}\right)^{m-1}u(0, y) = 0,\quad u(x, 0) = \frac{\partial}{\partial y}u(x, 0) = \cdots = \left(\frac{\partial}{\partial y}\right)^{n-1}u(x, 0) = 0\right\}.$$

THEOREM 4.1. *L^{-1} exists and is a continuous operator of $\mathfrak{R}$ onto $\mathfrak{D}$.*

Proof. Denote by $\mathfrak{x}$ one of the function spaces on I:

$C(I)$=Banach space of the continuous functions on I;

$B(I)$=Banach space of the functions continuous on I, analytic in the interior of I;

$L^2(I)$=Hilbert space of the square integrable functions on I.

Then we have

$C(I\times I)=C(I)\hat{\otimes}_\epsilon C(I)$, $B(I\times I)=B(I)\hat{\otimes}_\epsilon B(I)$, and $L^2(I\times I)=L^2(I)\hat{\otimes} L^2(I)$, where all norms are faithful.

For each $\mathfrak{x}$, let

$$\mathfrak{d}=\left\{\varphi\,;\ \varphi(t)\in\mathfrak{x},\quad \frac{d}{dt}\varphi(t)\in\mathfrak{x},\quad \varphi(0)=0\right\}.$$

Consider the ordinary differential operators $A=\frac{d}{dx}$, $B=\frac{d}{dy}$ in $\mathfrak{x}$ with domains $D[A]=D[B]=\mathfrak{d}$. A and B are densely defined closed operators in $\mathfrak{x}$ with these domains, and their spectra $\sigma(A)$ and $\sigma(B)$ are empty.

We can verify that the operator $L: \mathfrak{D}\subset\mathfrak{R}\to\mathfrak{R}$ coincides with the closure of the operator $\sum_{j,k} c_{jk}A^j\otimes B^k$ in $\mathfrak{R}$. Then, in virtue of Theorem 3.13, the spectrum of L is empty, which yields the assertion.

Mathematical Institute
Nagoya University
Nagoya, Japan

Present address. Department of Mathematics
Hokkaido University
Sapporo, Japan

Bibliography

[1] BOURBAKI, N.: Théories Spectrales. Act. Sci. et Ind., No. 1332. Paris: Hermann 1967.

[2] BROWN, A. and PEARCY, C.: Spectra of tensor products of operators. Proc. Am. Math. Soc. 17 (1966), 162–166.

[3] DUNFORD, N. and SCHWARTZ, J. T.: Linear Operators I. New York: Interscience 1958.

[4] GIL DE LAMADRID, J.: Uniform crossnorms and tensor products of Banach algebras. Duke Math. J. 32 (1965), 359–369.

[5] GROTHENDIECK, A.: Produits tensoriels topologiques et espaces nucléaires. Memoirs Am. Math. Soc. 16, 1955.

[6] GROTHENDIECK, A.: Resumé de la théorie métrique des produits tensoriels topologiques. Boletim da sociedade de matemática de São Paulo 8 (1956).

[7] HEFER, H.: Zur Funktionentheorie mehrerer Veränderlichen. Über eine Zerlegung analytischer Funktionen und die Weilsche Integraldarstellung. Math. Ann. 122 (1950), 276–278.

[8] HILLE, E. and PHILLIPS, R, S.: Functional Analysis and Semi-Groups. Revised Ed. Providence: Am. Math. Soc. Colloq. Publ. 31, 1957.

[9] ICHINOSE, T.: On the spectra of tensor products of linear operators in Banach spaces. J. reine angew. Math. 244 (1970), 119–153.

[10] KÖTHE, G.: Topologische Lineare Räume I. Berlin-Heidelberg-New York: Springer 1966.

[11] KÖTHE, G.: General linear transformations of locally convex spaces. Math. Ann. 159 (1965), 309–328.

[12] LUMER, G. and ROSENBLUM, M.: Linear operator equations. Proc. Am. Math. Soc. 10 (1959), 32–41.

[13] MACDUFFEE, C. C.: The Theory of Matrices. Ergebnisse der Math. u. ihrer Grenzgebiete. Berlin: Springer 1933.

[14] MALLIOS, A.: Heredity of tensor products of topological algebras. Math. Ann. 162 (1966), 246–257.

[15] NAIMARK, M. A.: Normed Rings. Groningen: P. Noordhoff N. V. 1959.

[16] REMMERT, R. and STEIN, K.: Über die wesentlichen Singularitäten analytischer Mengen. Math. Ann. 126 (1953), 263–306.

[17] RUSTON, A. F.: Direct products of Banach spaces and linear functional equations. Proc. London Math. Soc. (3) 1 (1951), 327–384.

[18] SCHATTEN, R.: A Theory of Cross-spaces. Princeton University Press 1950.

[19] SCHECHTER, M.: On the spectra of operators on tensor products. J. Functional Analysis 4 (1969), 95–99.

[20] TAYLOR, A. E.: Spectral theory of closed distributive operators. Acta Math. 84 (1951), 189–224.

[21] WAELBROECK, L.: Le calcul symbolique dans les algèbres commutatives. J. math. pures appl. 33 (1954), 147–186.

(Received November 18, 1974)