On finite groups whose Sylow p-subgroup is a $T.I.$ set.

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1. Introduction. Let G be a finite group whose Sylow p-subgroup is a $T.I.$ set of order p^a and suppose that G has a faithful complex character of degree less than $p^{a/2}$. Then it is conjectured that a Sylow p-subgroup is normal in G [2, 5]. Under some additional assumptions this conjecture was solved by Brauer-Leonard [1] and Leonard [5, 6, 7]. In this paper we prove the following theorem.

THEOREM. *Let G be a finite p-solvable group whose Sylow p-subgroup is a $T.I.$ set of order p^a and suppose that G has a faithful complex character of degree less than $p^a/\varepsilon-1$, where $\varepsilon=1$ if p is odd and $\varepsilon=2$ if p is 2, then a Sylow p-subgroup is normal in G.*

The method of the proof of Theorem is similar to one of Ito [4]. The notation is standard.

2. Proof of Theorem. We use induction on $|G|$, so let G be a minimal counterexample of Theorem and we seek a contradiction. Let P be a Sylow p-subgroup of G and χ be a faithful complex character of degree less than $p^a/\varepsilon-1$.

STEP 1. *$G=PQ$. Q is a q-group for some prime q distinct from p and normal in G.*

PROOF. Since G is p-solvable and P is a $T.I.$ set, it follows that $G/O_{p'}(G) \rhd PO_{p'}(G)/O_{p'}(G)$, that is $G \rhd PO_{p'}(G)$. If $G \neq PO_{p'}(G)$, by the minimality of G $PO_{p'}(G)$ char P and $G \rhd P$. So $G=PO_{p'}(G)$. Next suppose that $\pi(O_{p'}(G))=\{q_1, q_2, \cdots, q_t\}$, $t \geqq 2$. For each prime q_i let Q_i be a P-invariant Sylow q_i-subgroup then $G \neq PQ_i$ and the minimality of G implies $Q_i \subseteqq N_G(P)$ for every i. Thus $G=N_G(P)$, which is a contradiction. So $t=1$.

STEP 2. *We may assume χ is irreducible.*

PROOF. Let ζ be a irreducible constituent of χ and assume Ker $\zeta= H \neq 1$. ζ is a faithful character of G/H. If $G \neq PH$, then $PH \rhd P$ and $G \rhd H$ char $P \cap H$. Since P is a $T.I.$ set $P \cap H=1$ and the order of Sylow p-subgroup of G/H is p^a. Since $|G/H|<|G|$ $G/H \rhd PH/H$, that is

$G \triangleright PH$ and $G \triangleright PH$ char P. So we must have $G=PH$, in particular $H=$ Ker $\zeta \supseteqq Q$. Since above argument is valid for every constituent of χ, if there is no faithful irreducible constituent of χ, Ker $\chi \supseteqq Q$. This is a contradiction.

STEP 3. *Q is nonabelian.*

PROOF. Assume that Q is abelian. By [3, Theorem 5. 2. 3] $Q=[P, Q] \times C_Q(P)$. If $C_Q(P)\neq 1$, then $Q\neq [P, Q]$ and $G\neq P[P, Q]$. The minimality of G shows $P[P, Q] \triangleright P$ and $[[P, Q], P] \subseteqq P\cap Q=1$. Then $[P, Q]\subseteq C_Q(P)$, $Q=C_Q(P)$ and $G=PC_Q(P)\triangleright P$, a contradiction. So $C_Q(P)=1$. If we set $N_G(P)=PQ_0$ $(Q_0\subseteq Q)$, then $G\triangleright Q$ imples that $Q_0\subseteq C_G(P)$. Therefore $N_G(P) =PC_G(P)=P$ and G is a Frobenius group with complement P. The characters of Frobenius group are known and the degrees of the faithful irreducible characters of G are p^a. This contradicts the assumption in Theorem.

STEP 4. *Q is an extra-special q-group of order q^{2m+1}, $m\geqq 1$ and $|N_G(P)|=p^a q$.*

PROOF. Let Q_0 be a P-invariant proper normal subgroup of Q. Then $PQ_0\neq G$ and $PQ_0\triangleright P$. So $[P, Q]\subseteqq P\cap Q_0=1$ and $Q_0\subseteqq C_G(P)$. Therefore P centralizes every P-invariant proper normal subgroup of Q. By [3, Theorem 5. 3. 7] and Step 3 $C_Q(P)=Z(Q)=Q'=\Phi(Q)$ is elementary abelian and $N_G(P)=PC_G(P)=PZ(Q)$. Since by Step 2 G has a faithful irreducible character, $Z(G)=Z(Q)$ is cyclic. So Q is extraspecial, $|N_G(P)|=p^a q$ and we can write $|Q|=q^{2m+1}$ by [3, Theorem 5. 5. 2].

STEP 5. (*final contradiction*).

As Q is extra-special the degrees of the faithful irreducible characters of Q are q^m. Restricting χ to Q we obtain $\chi(1)\geqq q^m$. On the other hand by Step 4 $|G:N_G(P)|=q^{2m}\equiv 1 \pmod{p^a}$ and $p^a|(q^{2m}-1)=(q^m+1)(q^m-1)$. If p is odd, then $p^a\leqq q^m+1\leqq \chi(1)+1<p^a$. If p is 2, then $p^a\leqq 2(q^m+1)\leqq 2(\chi(1)+1)<p^a$. This is a final contradiction.

REMARK. If p is 2, by Suzuki [8] a group with an independent Sylow 2-subgroup is determined and has a well known structure. So our Theorem and direct calculations show that the conjecture discribed in Introduction is true for $p=2$.

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References

[1] R. BRAUER and H. S. LEONARD Jr.: On finite groups with an abelian Sylow group, Canad. J. Math., 14 (1962), 436-450.

[2] W. FEIT,: The current situation in the theory of finite simple groups, Actes. Congrés inter. math., (1970) 55-93.

[3] D. GORENSTEIN,: Finite Groups, Harper and Row, New York-London (1968).

[4] N. ITO,: On a theorem of H. F. Blichfeldt, Nagoya Math. J., 5 (1953), 75-77.

[5] H. S. LEONARD JR.,: On finite groups whose p-Sylow subgroup is a T. I. set, Proc. Amer. Math. Soc., 19 (1968), 667-670.

[6] H. S. LEONRRD JR.,: Finite linear groups having an abelian Sylow subgroup, J. Algebra, 20 (1972), 57-69.

[7] H. S. LEONARD JR.,: Finite linear groups having an abelian Sylow subgroup, II, J. Algebra, 26 (1973), 368-372.

[8] M. SUZUKI,: Finite groups of even order in which Sylow 2-groups are independent, Ann. math., 80 (1964), 58-77.

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