Iterated mixed problems for d'Alembertians II

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§1. Introduction and results

The aim of this note is to establish a generalization of the results in the previous paper [2]. Definitions and terminologies in [2] will be used here also.

Let $(t, x)=(t, x', x_n)$ be variables in the $(n+1)$-dimensional Euclidean space $\boldsymbol{R}^{n+1}$ and (τ, σ, λ) $(\tau=\xi-i\gamma)$ the dual variables of (t, x', x_n) respectively. Furthermore let χ be a permutation of m letters $1, \cdots, m$. In the open half space $\boldsymbol{R}_+^{n+1}=\{x_n>0\}$ with boundary $\boldsymbol{R}^n=\{x_n=0\}$, we then consider an iterated mixed problem $(P, {}^\chi B_j)$ for d'Alembertians:

$$
\begin{aligned}
&P(t, x\,;\, D_t, D_x)\, u=f && && \text{in } \boldsymbol{R}_+^{n+1},\\
&{}^\chi B_j(t, x'\,;\, D_t, D_x)\, u=g_j && (j=1, \cdots, m) && \text{on } \boldsymbol{R}^n.
\end{aligned}
$$

Here we shall recall the definitions of P and ${}^\chi B_j$ ([2], §1):

$$
\begin{aligned}
&P^0(t, x\,;\, \tau, \sigma, \lambda)=\prod_{j=1}^{m} P_j^0(t, x\,;\, \tau, \sigma, \lambda),\\
&P_j^0(t, x\,;\, \tau, \sigma, \lambda)=-\tau^2+a_j(t, x)^2(\lambda^2+|\sigma|^2),\\
&0<a_m(t, x)<\cdots<a_1(t, x),\\
&\chi=\begin{pmatrix}1, 2, \cdots\cdots, m\\ k_1, k_2, \cdots, k_m\end{pmatrix},\\
&{}^\chi B_1^0(t, x'\,;\, \tau, \sigma, \lambda)=B_{k_1}^0(t, x'\,;\, \tau, \sigma, \lambda),\\
&{}^\chi B_j^0(t, x'\,;\, \tau, \sigma, \lambda)\\
&\qquad=(B_{k_j}^0)(t, x'\,;\, \tau, \sigma, \lambda)\prod_{h=1}^{j-1} P_{k_h}^0(t, x', 0\,;\, \tau, \sigma, \lambda) \qquad (j\geqq 2),\\
&B_j^0(t, x'\,;\, \tau, \sigma, \lambda)=\lambda-\sum_{k=1}^{n-1} b_{jk}(t, x')\,\sigma_k-c_j(t, x')\,\tau,
\end{aligned}
$$

where b_{jk} and c_j are real valued and Q^0 denotes the principal part of a differential operator Q. We are concerned with L^2-well posed problems ([2], §2) and hence the solution of our problem has zero initial data on $t=0$ provided that $f=0$ and $g_j=0$ in $t<0$.

In order to state the main results we recall a classification of L^2-well posed problems of second order with constant coefficients ([2], §1). We say a problem to be type U if it satisfies uniform Lopatinski condition and say other L^2-well posed problems to be type $\overline{NU}$. Among problems of type $\overline{NU}$, Neumann problem is said to be of type N and other problems are said to be of type NU. Furthermore every L^2-well posed problem is said for the convenience to be of type $\bar{U}$.

THEOREM 1. *Let χ be the unit in the permutation group. Then an iterated mixed problem $(P, {}^{\chi}B_j)$ with constant coefficients is L^2-will posed if and only if the type of an ordered set $((P_1^0, B_1^0), \cdots, (P_m^0, B_m^0))$ of second order problems (P_j^0, B_j^0) becomes one of the following m-types*:

$$\begin{array}{c}(U, \cdots\cdots\cdots\cdots, U, \bar{U}), \\ (U, \cdots\cdots, U, \overline{NU}, N), \\ \vdots \\ (\overline{NU}, N, \cdots\cdots\cdots, N).\end{array} \tag{1.1}$$

Theorem 1 has been proved in [2], Theorem 2 except the sufficiency in the case where a number n of space variables is greater than 2. Regarding (b_{jk}, c_j) $(j=1, \cdots, m\,;\ k=1, \cdots, n-1)$ as real variables in $\boldsymbol{R}^{mn}$, we see from Theorem 1 that all L^2-well posed iterated mixed problems $(P, {}^{\chi}B_j)$ for the unit permutation χ form a semi-algebraic variety with its stratification given by (1. 1). In connection with Theorem 1 we remark the following. Kreiss-Rauch ([8], §6) gave a counter example for the question: Is a limit of L^2-well posed mixed problems with uniform Lopatinski condition also L^2-well posed? Theorem 1 gives in particular such a counter example. For instance, $(P, {}^{\chi}B_j)$ with the unit χ is not L^2-well posed if both types of (P_1, B_1) and (P_2, B_2) tend to NU from $U (m=2)$.

THEOREM 2. *Let χ be the unit in the permutation group. Assume every frozen problem $(P, {}^{\chi}B_j)_{(t_0, x'_0)}$ at boundary point (t_0, x'_0) is L^2-well posed. Furthermore assume that, for such a point (t_0, x'_0), the frozen problems $(P_j^0, B_j^0)_{(t,x')}$ $(j>l)$ are of type N near (t_0, x'_0) if $(P_k^0, B_k^0)_{(t_0, x'_0)}$ $(k<l)$ are of type U and $(P_l^0, B_l^0)_{(t_0, x'_0)}$ is of type $\overline{NU}$ for some l. Then the iterated mixed problem $(P, {}^{\chi}B_j)$ is L^2-well posed.*

In the papers [7], [10] T. Shirota and T. Ohkubo have done recently an attempt to establish a general theory of L^2-well posed mixed problems for systems of first order and obtained many interesting results. For the L^2-well posed mixed problems in Theorem 2, the condition II), β) on Hessian and zeros of Lopatinski determinant in [7], §1 is not always satisfied near a point (t_0, x'_0) in Theorem 2 with $l<m$. However, the method in [7], §7 and 8 deriving a priori estimate is applicable to our case (see the remark after the condition II), γ') in [7], §1. The proof of Theorem 2 is

given in §2.

From Theorem 2 we obtain the following corollary which is completed the proof of Theorem 1.

COROLLARY 1. *Let χ be the unit in the permutation group. Furthermore let the types of ordered set* $((P_1^0, B_1^0), \cdots, (P_m^0, B_m^0))_{(t,x')}$ *of frozen problems be uniformly in* (t, x') *one of* m *types stated in Theorem* 1. *Then the iterated mixed problem* $(P, {}^{\chi}B_j)$ *is* L^2*-well posed.*

In §3 we present an example which satisfies the assumptions of Theorem 2, but does not enjoy the one of Corollary 1, the iterated mixed problems for permutations χ different from the unit and the precision of Kreiss- Rauch example.

§2. Proof of Theorem 2

The proof is mainly based on the method developed in [7]. We reduce near zeros of Lopatinski determinant our problem $(P, {}^{\chi}B_j)$ to problem for $2m\times 2m$ system of pseud-differential operators of first order and show here a process to the stage applicable thier method to our problem. In this section we assume that χ is the unit.

In order to carry out the above reduction we use the following finite partition of unity in the cotangent space

$$(2.1)\qquad \sum_{j,k\geq 1}\psi_j(\xi,\sigma,\gamma)\,\phi_k(t,x')\,\phi_0(x_n)+\big(1-\phi_0(x_n)\big)=1$$

constructed in the following manner. Let

$$S=\big\{(\tau,\sigma)=(\xi-i\gamma,\sigma)\,;\ |\tau|^2+|\sigma|^2=1,\ \gamma\geqq 0\big\}$$

and let $\{\psi_j'\}$ be a finite partition of unity in the compact set S. Then the ψ_j are defined by $\psi_j(\xi,\sigma,\gamma)=\psi_j'(\xi\Lambda^{-1},\sigma\Lambda^{-1},\gamma\Lambda^{-1})$ where $\Lambda=(|\tau|^2+|\sigma|)^{1/2}$. Furthermore let $\{\phi_k\}$ be a finite partition of unity in $\boldsymbol{R}^n$ such that $\phi_1=1$ in $|t|+|x'|\geqq R>0$ and $\phi_0(x_n)=1$ near the boundary $x_n=0$. Hereafter we denote a representative of terms in (2.1) by $\beta(t,x\,;\xi,\sigma,\gamma)$ and put $\beta^0(t,x'\,;\xi,\sigma,\gamma)=\beta(t,x',0\,;\xi,\sigma,\gamma)$.

Let a point (t_0, x_0') be arbitrary but fixed. Furthermore assume that the frozen problems $(P_{l+1}^0, B_{l+1}^0)_{(t,x')}, \cdots, (P_m^0, B_m^0)_{(t,x')}$ are of (type N near (t_0, x_0') if $(P_1^0, B_1^0)_{(t_0,x'_0)}, \cdots, (P_{l-1}^0, B_{l-1}^0)_{(t_0,x'_0)}$ are of type U and $(P^0, B^0)_{(t_0,x'_0)}$ is of type NU for some l. Then we see from [2], Lemma 4.1 and the fact $0<a_m<\cdots<a_1$ that Lopatinski determinant ${}^{\chi}R(t_0,x_0'\,;\tau,\sigma)$ for $(P,{}^{\chi}B_j)$ has zeros only on each sheet $\xi^2=a_j(t,x',0)|\sigma|^2$ $(j=l,\cdots,m)$. In fact, we have from [2], (3.2) that

$$
\begin{aligned}
{}^{\chi}R(t_0, x_0'; \tau, \sigma) &= \prod_{j=1}^{m} R_j(t_0, x_0'; \tau, \sigma) \prod_{j>k} (\lambda_j^+ + \lambda_k^+)(t_0, x_0'; \tau, \sigma) \\
&= \prod_{j=l}^{m} R_j(t_0, x_0'; \tau, \sigma) \times (\text{non-zero factor}).
\end{aligned}
$$

Here R_j is Lopatinski determinant for (P_j, B_j) of second order:

$$
\begin{aligned}
R_j(t, x'; \tau, \sigma) &= B_j^0\big(t, x'; \tau, \sigma, \lambda_j^+(t, x', 0; \tau, \sigma)\big) \\
&= \lambda_j^+(t, x', 0; \tau, \sigma) - \sum_{k=1}^{n-1} b_{jk}(t, x')\sigma_k - c_j(t, x')\tau \qquad (2.2) \\
&= \lambda_j^+(t, x', 0; \tau, \sigma) - \alpha_j(t, x'; \tau, \sigma)
\end{aligned}
$$

and λ_j^+ (resp. λ_j^-) is a root of $P_j^0(t, x; \tau, \sigma, \lambda)=0$ in λ which has positive (resp. negative) imaginary part for $\operatorname{Im}\tau = -\gamma < 0$. More precisely it follows from [2], Corollary 4.2 that $R_l(t_0, x_0'; \xi_0, \sigma_0)=0$ is equivalent to

$$
\alpha_l(t_0, x_0'; \xi_0, \sigma_0) = 0 \quad \text{and} \quad \xi_0^2 = a_l(t_0, x_0', 0)\,|\sigma_0|^2. \qquad (2.3)
$$

Furthermore since $(P_j^0, B_j^0)_{(t_0 x'_0)}$ $(j>l)$ is of type N we see that $R_j(t_0, x_0'; \xi_0, \sigma_0)=0$ is equivalent to

$$
\xi_0^2 = a_j(t_0, x_0', 0)\,|\sigma_0|^2. \qquad (2.4)
$$

for each $j>l$.

Remark that if $(P_l^0, B_l^0)_{(t_0, x'_0)}$ is of type NU then the points (ξ_0, σ_0) satisfying (2.3) form 1-dimensional manifold for fixed (t_0, x_0') (see [2], p. 114).

Lopatinski determinant is different from zero except points satisfying (2.3) or (2.4). Hence it follows from the method developed in [3], [4], [9] and a sharp form of Gårding inequality that if $\beta(t, x'; \xi, \sigma, \gamma)$ has the support outside such points then there exist positive constants C and γ_0 such that

$$
\begin{aligned}
&\gamma^2 \|\beta u\|_{2m-1,\gamma}^2 + \gamma \sum_{j=0}^{2m-1} \langle\!\langle \beta^0 D_n^j u \rangle\!\rangle_{2m-1-j,\gamma}^2 \\
&\leqq C\Big(\|Pu\|_{0,\gamma}^2 + \gamma \sum_{j=1}^{m} \langle\!\langle {}^{\chi}B_j u \rangle\!\rangle_{2m-2j,\gamma}^2 \\
&\qquad + \gamma \|u\|_{2m-1,\gamma}^2 + \gamma \sum_{j=1}^{2m-1} \langle\!\langle D_n^j u \rangle\!\rangle_{2m-1-j-1/2,\gamma}^2\Big)
\end{aligned}
$$

for any $u \in H_{2m,\gamma}(\boldsymbol{R}_+^{n+1})$ and $\gamma \geqq \gamma_0$. Here we use the same notations as in [2], §2 for Hilbert spaces of functions and their norms. Furthermore recall the definition of pseudo-differential operator $a = a(t, x; D_t', D_{x'}; \gamma)$ with its symbol $a(t, x; \xi, \sigma, \gamma)$ depending parameters $\gamma > 0$ and $x_n \geqq 0$:

$$
\begin{aligned}
au(t, x) &= a(t, x; D_t', D_{x'}, \gamma)\, u(t, x) \\
&= (2\pi)^{-n} \int_{R^n} e^{i(\tau t + \sigma x')}\, a(t, x; \xi, \sigma, \gamma)\, \hat{u}(\tau, \sigma, x_n)\, d\xi\, d\sigma,
\end{aligned}
$$

where

$$\hat{u}(\tau, \sigma, x_n) = \int_{R^n} e^{-i(\tau t + \sigma x')} u(t, x', x_n)\, dt dx',$$

$$D'_t = D_t + i\gamma.$$

Therefore, in order to prove a priori estemate

$$\gamma^2 \|u\|^2_{2m-1,\gamma} \leqq C(\|Pu\|^2_{0,\gamma} + \sum_{j=1}^{m} \langle\!\langle {}^{\chi}B_j u \rangle\!\rangle^2_{2m-2j+1/2,\gamma}) \tag{2.5}$$

it suffice to show the estemate

$$\begin{aligned} &\gamma^2 \|\beta u\|^2_{2m-1,\gamma} \\ &\leq C(\|Pu\|^2_{0,\gamma} + \sum_{j=1}^{m} \langle\!\langle {}^{\chi}B_j u \rangle\!\rangle^2_{2m-2j+1/2,\gamma} \\ &\quad + \gamma \|u\|^2_{2m-1,\gamma} \\ &\quad + \gamma \sum_{j=0}^{2m-1} \langle\!\langle D_n^j u \rangle\!\rangle^2_{2m-1-j-1/2,\gamma}), \end{aligned} \tag{2.6}$$

for any $u \in H_{2m,\gamma}(\boldsymbol{R}^{n+1}_+)$ and $\gamma \geqq \gamma_0 > 0$ where the support of β is contained in a neighbourhood of a point $(t_0, x'_0, 0\,;\, \xi_0, \sigma_0)$ satisfying (2.3) or (2.4). Here for $(1-\phi_0(x_n))u$ we use the hyperbolic a priori estemate without boundary conditions. The existence of solutions is proved by (2.5) and a priori estemate for the dual problem (for instance see [5] or [2], Proposition 7.2)

Now we reduce in a neighbourhood of a point satisfying (2.3) or (2.4) our problem to one for $2m \times 2m$ system of first order. First we consider the equation $Pu=f$. Since $a_j(t, x) \neq 0$ $(j=1, \cdots, m)$, the principal symbol of P_j may be replaced by the following:

$$\begin{aligned} P_j^0(t, x\,;\, \tau, \sigma, \lambda) &= \lambda^2 + |\sigma|^2 - a_j(t, x)^{-2}\tau^2 \\ &= \lambda^2 - p_j(t, x\,;\, \tau, \sigma)\Lambda^2 \\ &= \left(\lambda - \lambda_j^+(t, x\,;\, \tau, \sigma)\Lambda\right)\left(\lambda - \lambda_j^-(t, x\,;\, \tau, \sigma)\Lambda\right) \end{aligned} \tag{2.7}$$

where

$$p_j(t, x\,;\, \tau, \sigma) = a_j(t, x)^{-2}\tau^2\Lambda^{-2} - |\sigma|^2\Lambda^{-2},$$

$$\Lambda = \Lambda(\tau, \sigma) = \left(|\tau|^2 + |\sigma|^2\right)^{1/2}, \quad \tau = \xi - i\gamma,$$

and we consider here $\lambda_j^{\pm}$ as symbols of order zero.

Put

$$V = {}^t(\Lambda^{2m-1}u, \Lambda^{2m-2}D_n u, \cdots, D_n^{2m-1}u),$$

where ${}^tM$ stands for the transposed matrix of a matrix M. Furthermore let A be the pseudo-differential operator of order 0 whose symbol is

$$A(t, x\,;\,\tau, \sigma)=\begin{pmatrix} 0, & 1 & & & & \\ & 0, & 1 & & 0 & \\ & & \ddots & \ddots & & \\ & 0 & & & 1 & \\ & & & & 0 & 1 \\ -q_m, & 0, & \cdots & & -q_1, & 0 \end{pmatrix}.$$

Here the symbols q_j of order zero are determined by the relation:

$$\prod_{j=1}^{m} P_j^0=\lambda^{2m}+q_1 \Lambda^2 \lambda^{2m-2}+\cdots+q_m \Lambda^{2m}.$$

Then we see that the quation $Pu=f$ becomes

$$D_n V-A\Lambda V+(l.\,o.\,0.)\,V=F \tag{2.8}$$

where $F={}^t(0, \cdots, 0, f)$ and $(l.\,o.\,0.)$ means 'lower order operator'.

Let $S^j(t, x\,;\,\tau, \sigma)$ be the $2m\times 2m$ homogeneous matrix of order zero arising from replacing $(2j-1)$-th and $2j$-th columns in the matrix below by $V_{2j-1}={}^t(1, 0, p_j, 0, \cdots, p_j^{m-1}, 0)$ and $V_{2j}={}^t(0, 1, 0, p_j, \cdots, 0, p_j^{m-1})$:

$$\begin{pmatrix} 1 & 1 & \cdots\cdots & 1 & 1 \\ \lambda_1^+ & \lambda_1^- & \cdots\cdots & \lambda_m^+ & \lambda_m^- \\ \vdots & \vdots & & \vdots & \vdots \\ (\lambda_1^+)^{2m-1} & (\lambda_1^+)^{2m-1} & \cdots & (\lambda_m^+)^{2m-1} & (\lambda_m^-)^{2m-1} \end{pmatrix} \tag{2.9}$$

From the definitions of $p_j, q_j, \lambda_j^{\pm}$ and the fact that $AV_{2j}=V_{2j-1}$, $AV_{2j-1}={}^t(1, 0, p_j, 0, \cdots, 0, p_j^{m-1}, 0, -q_m-p_j q_{m-1}-\cdots-p_j^{m-1} q_1)$, we have

$$S^j M^j(t, x\,;\,\tau, \sigma)=AS^j(t, x\,;\,\tau, \sigma) \qquad (j=1, \cdots, m). \tag{2.10}$$

Here M^j is defined by

$$M^j=\begin{pmatrix} \lambda_1^+ & & & & & & \\ & \lambda_1^- & & & & 0 & \\ & & \ddots & & & & \\ & & & 0 & 1 & & \\ & & & p_j & 0 & & \\ & 0 & & & & \ddots & & \\ & & & & & & \lambda_m^+ & \\ & & & & & & & \lambda_m^- \end{pmatrix} \tag{2.11}$$

Put $U_j=S^j V$. Then it follows from (2.8) and (2.10) that

$$D_n \beta U_j-M^j \Lambda \beta U_j+(l.\,o.\,0.)\,U_j=(\phi S^j)^{-1}\beta F \qquad (j=1, \cdots, m). \tag{2.12}$$

Here the support of symbol of β is contained in a neighbourhood of a point $(t_0, x_0', 0\,;\, \xi_0, \sigma_0)$ satisfying (2. 3) or (2. 4). Moreover, the symbol of $\tilde{\Phi}$ is equal to 1 on the support of β.

Second we shall get the boundary condition for (2. 12) from the original one ${}^{\chi}B_j u = g_j (j=1, \cdots, m)$. Clearly we have

$$\Lambda^{2m-2j}\, {}^{\chi}B_j u = \Lambda^{2m-2j} g_j . \tag{2. 13}$$

Let $q_{jk}(t, x'\,;\, \tau, \sigma)$ $(j=2, \cdots, m\,;\, k=1, \cdots, j-1)$ be defined by the relations:

$$\begin{aligned} Q_j^0(t, x'\,;\, \tau, \sigma, \lambda) &= \prod_{h=1}^{j-1} P_h^0(t, x', 0\,;\, \tau, \sigma, \lambda) \\ &= \lambda^{2j-2} + q_{j1}\Lambda^2\lambda^{2j-4} + \cdots + q_{j\overline{j-1}}\Lambda^{2j-2}, \end{aligned} \tag{2. 14}$$

where the first equality is the definition. Furthermore we consider here α_j defined in (2. 2) as symbol of order zero, i.e.,

$$B_j^0(t, x'\,;\, \tau, \sigma, \lambda) = \lambda - \alpha_j(t, x'\,;\, \tau, \sigma)\,\Lambda . \tag{2. 15}$$

Then it follows from (2. 13), (2. 14), (2. 15) and the definition of V that the boundary condition for (2. 8) becomes

$$DV + (l.\, o.\, t.)\, V = G \tag{2. 16}$$

where

$$G = {}^t(\Lambda^{2m-2} g_1, \cdots, \Lambda^2 g_{m-1}, g_m)$$

and $D(t, x'\,;\, \tau, \sigma)$ is the following $m\times 2m$ matrix:

$$\begin{pmatrix} -\alpha_1 & 1 & 0 & 0 & & & \cdots\cdots & 0 & 0 & 0 & 0 \\ -\alpha_2 q_{21} & q_{21} & -\alpha_2 & 1 & 0 & 0 & \cdots\cdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & & 0 & 0 & & \\ & & & & & & & -\alpha_{m-1} & 1 & 0 & 0 \\ -\alpha_m q_{m1} & q_{m1} & & \cdots\cdots & & & & -\alpha_m q_{m\overline{m-1}} & q_{m\overline{m-1}} & -\alpha_m & 1 \end{pmatrix}$$

Since $U_j = S^j V$, the boundary condition (2. 16) becomes

$$DS^j \beta U_j + (l.\, o.\, o.)\, U_j = \beta G \qquad (j=1, \cdots, m). \tag{2. 17}$$

Here DS^j is the $m\times 2m$ matrix of order zero arising from replacing $(2j-1)$-th and $2j$-th columns in the matrix below by ${}^t(-\alpha_1, -\alpha_2 Q_2^0(\lambda_j^-), \cdots, -\alpha_j Q_j^0(\lambda_j^-), 0, \cdots, 0)$ and ${}^t(1, Q_2^0(\lambda_j^+), \cdots, Q_j^0(\lambda_j^+), 0, \cdots, 0)$ respectively:

$$\begin{pmatrix} {}^{\chi}B_1^0(\lambda_1^+) & {}^{\chi}B_1^0(\lambda_1^-) & {}^{\chi}B_1^0(\lambda_2^+) & {}^{\chi}B_1^0(\lambda_2^-) & \cdots & {}^{\chi}B_1^0(\lambda_m^+) & {}^{\chi}B_1^0(\lambda_m^-) \\ 0 & 0 & {}^{\chi}B_2^0(\lambda_2^+) & {}^{\chi}B_2^0(\lambda_2^-) & \cdots & {}^{\chi}B_2^0(\lambda_m^+) & {}^{\chi}B_2^0(\lambda_m^-) \\ & & 0 & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & 0 \quad 0 & 0 & {}^{\chi}B_m^0(\lambda_m^+) & {}^{\chi}B_m^0(\lambda_m^-) \end{pmatrix} \tag{2. 18}$$

$$Q_j^0(\lambda_k^\pm) = (\lambda_k^\pm)^{2j-1} + q_{j1}(\lambda_k^\pm)^{2j-4} + \cdots + q_{jj-1},$$
$$B_j^0(\lambda_k^\pm) = \lambda_k^\pm - \alpha_j, \qquad {}^{\chi}B_j^0 = (B_j^0 Q_j^0)(\lambda_k^\pm).$$

In order to relate the boundary conditions (2.17) to Lopatinski determinant and reflection coefficients, put

$$B^\pm(t, x'; \tau, \sigma) = \begin{pmatrix} {}^{\chi}B_1^0(\lambda_1^\pm) & \cdots & {}^{\chi}B_1^0(\lambda_m^\pm) \\ & \ddots & \vdots \\ 0 & & {}^{\chi}B_m^0(\lambda_m^\pm) \end{pmatrix}. \tag{2.19}$$

Then we see from [2], p. 111 that $\det B^+$ is nothing but the numerator of Lopatinski determinant ${}^{\chi}R$ for $(P, {}^{\chi}B_j)$ and (j, k)-components of the matrix $(B^+)^{-1}B^-$ are also the reflection coefficients ${}^{\chi}C_{jk}$ for $(P, {}^{\chi}B_j)$. Let $B^{j\prime}$ and $B^{j\prime\prime}$ be the matrices resulting from replacing j-th column in B^+ and B^- by ${}^t(1, Q_2^0(\lambda_j^+), \cdots, Q_j^0(\lambda_j^+), 0, \cdots, 0)$ and ${}^t(-\alpha_1, -\alpha_2, Q_2^0(\lambda_j^-), \cdots, -\alpha_j Q_j^0(\lambda_j^-), 0, \cdots, 0)$ decrived above respectively. Then we have

$$\det B^+ = R_j \det B^{j\prime} \tag{2.20}$$

where $R_j = B_j^0(\lambda_j^+)$ is Lopatinski determinant for (P_j, B_j) of second order. Using the above notations we can replace (2.18) by

$$B^{j\prime}\beta^0 U_j' + B^{j\prime\prime}\beta^0 U_j'' + (l.\,o.\,0.)\,U_j = \beta^0 G \qquad (j=1, \cdots, m) \tag{2.21}$$

where β is the same one as in (2.12) and

$$\begin{aligned} U &= {}^t(u_1^+, u_1^-, \cdots, u_m^+, u_m^-), \\ U_j &= {}^t(u_1^+, u_1^-, \cdots, u_{j-1}^-, u'', u', u_{j+1}^+, \cdots, u_m^+, u_m^-), \\ U_j' &= {}^t(u_1^+, \cdots, u_{j-1}^+, u', u_{j+1}^+, \cdots, u_m^+), \\ U_j'' &= {}^t(u_1^-, \cdots, u_{j-1}^-, u'', u_{j+1}^-, \cdots, u_m^-). \end{aligned}$$

From (2.3), (2.4) and (2.20) we may assume that the matrix $B^{j\prime}$ is non-singular on the support of β. Consequently our iterated mixed problem is micro-locally reduced to one for $2m \times 2m$ system of first order:

$$\begin{aligned} & D_n\beta U_j - M^j \Lambda \beta U_j + (l.\,o.\,0.)\,U_j = (\tilde{\Phi} S^j)^{-1}\beta F && \text{in } \boldsymbol{R}_+^{n+1}, \\ & \beta^0 U_j' + K^j \beta^0 U_j'' + (l.\,o.\,0.)\,U_j = (\tilde{\Phi}^0 B^{j\prime})^{-1}\beta^0 G && \text{on } \boldsymbol{R}^n \end{aligned} \tag{2.22}$$

for $j=1, \cdots, m$. Here

$$K^j = (B^{j\prime})^{-1} B^{j\prime\prime} \tag{2.23}$$

and β, $\tilde{\Phi}$ are the same as in (2.12). The components of the matrix K^j are called the coupling coefficients of $(P, {}^{\chi}B_j)$ in [7].

Now we ready for the proof of a priori estemate (2.6). To apply the method in [7] to our case, we first introduce a new variable ζ in a neighbourhood of a point $(t_0, x'_0, 0\ ;\ \xi_0, \sigma_0)$ satisfying (2.3):

$$\zeta=\left(\tau-a_l(t, x', 0)|\sigma|\right)\Lambda^{-1} \qquad \text{if } \xi_0>0 \tag{2.24}$$

or

$$\zeta=\left(\tau+a_l(t, x', 0)|\sigma|\right)\Lambda^{-1} \qquad \text{if } \xi_0<0 \tag{2.25}$$

and rewrite α_l as

$$\alpha_l(t, x'\ ;\ \tau, \sigma)=\alpha_{l0}(t, x'\ ;\ \sigma)+c_l(t, x')\zeta\,.$$

Then we see from the proof of [2], Lemma 4.1 that, in a neighbourhood of $(t_0, x'_0\ ;\ \xi_0, \sigma_0)$,

$$\alpha_{l0}(t, x'\ ;\ \sigma)\geqq 0 \quad \text{or} \quad \leqq 0 \tag{2.26}$$

corresponding to (2.24) or (2.25) respectively. In fact, if $\alpha_{l0}(t, x'\ ;\ \sigma)<0$ in the case (2.24) than $R_l(t, x'\ ;\ \xi, \sigma)=0$ for some ξ with $\xi>a_l(t, x', 0)|\sigma|$, which contradicts L^2-well posedness of frozen problems (also see [7], Lemma 6.4). The symbol α_l play the same role as $-Q$ in [7], §6. Since $(P_j^0, B_j^0)_{(t,x')}$ $(j>l)$ are of type N near (t_0, x'_0), the symbols α_j are identically equal to zero. Hence we have (2.26) for $j>l$.

Secondly we rewrite, near a point satisfying (2.3), the coupling coefficients K^l as a simple form. It follows from (2.23) and definitions of $B^{l'}$, $B^{l''}$ that, denoting k_{jk}^l by (j, k)-component of K^l,

$$\begin{aligned} &k_{ll}^l(t, x'\ ;\ \tau, \sigma)=-\alpha_l(t, x'\ ;\ \tau, \sigma), \\ &k_{jk}(t, x'\ ;\ \tau, \sigma)=0 \qquad (j>k) \end{aligned} \tag{2.27}$$

and for $j>l$

$$\begin{aligned} &k_{lj}^l(t, x'\ ;\ \tau, \sigma) \\ &=\begin{vmatrix} B_l^0Q_l^0(\lambda_j^-) & B_l^0Q_l^0(\lambda_{l+1}^+), \cdots, & B_l^0Q_l^0(\lambda_j^+) \\ B_{l+1}^0Q_{l+1}^0(\lambda_j^-) & B_{l+1}^0Q_{l+1}^0(\lambda_{l+1}^+), & \cdots, \ B_{l+1}^0Q_{l+1}^0(\lambda_j^+) \\ \vdots & 0 & \\ & \vdots \quad 0 \quad \ddots & \\ B_j^0Q_j^0(\lambda_j^-) & 0 & B_j^0Q_j^0(\lambda_j^+) \end{vmatrix} \\ &\quad \Big/ \prod_{k=l}^{j} Q_k^0(\lambda_k^+) \prod_{k=l+1}^{j} B_k^0(\lambda_k^+)\,. \end{aligned} \tag{2.28}$$

Remark that α_j is identically equal to zero near $t_0, x'_0)$ for $j>l$. Therefore, by adding the last column to the first column in the determinant (2.28), we obtain that

$$k_{lj}^l(t, x'\ ;\ \tau, \sigma)=-2\alpha_l(t, x'\ ;\ \tau, \sigma) \qquad (j>l). \tag{2.29}$$

To derive (2. 6) this plays an important role. Consequently, it follows from (2. 27), (2. 28) and (2. 29) that

$$K^l=\begin{pmatrix} K_{11} & & K'' & K_{12} \\ 0 \cdots 0 & & -\alpha_l & K_2 \\ & & 0 & \\ 0 & & \vdots & K_{22} \\ & & 0 & \end{pmatrix} l. \tag{2. 30}$$

where K_2 is the $1\times(m-l)$ matrix:

$$(-2\alpha_l, \cdots, -2\alpha_l), \tag{2. 31}$$

K'' is an $l\times 1$ matrix and K_{11}, K_{12} are triangular. The matrix K^l plays the same role as K in [7], Lemma 6. 7 and we have

$$K^l=\begin{pmatrix} K_{III\,III} & K_{III\,II} & K_{III\,I} \\ K_{II\,III} & K_{II\,II} & K_{II\,I} \\ K_{I\,III} & K_{I\,II} & K_{I\,I} \end{pmatrix}$$

with the notations in [7]. Hence we see from (3. 31) that the conclusion (6. 7. 1) of [7], Lemma 6. 7 is valid, i. e., near a point satisfying (2. 3),

$$|K_2|^2=4|\alpha_l|^2\leqq 4\alpha_l(\zeta=0). \tag{2. 32}$$

On the other hand, since α_j $(j>l)$ vanish in a neighbourhood of a point satisfying (2. 4), the corresponding K_2 vanish there. Hence (2. 32) hold for these cases.

Therefore a priori estemate (2. 6) is obtained from (2. 26) and (2. 32) in the same way as in [7], §7 and 8.

§3. Examples

3. 1. First we present an example which satisfies the assumption of Theorem 2, but does not enjoy the assumption in Corollary 1. Consider the following mixed problems of second order in the two space variables $(y, x)=(x_1, x_2)$:

$$\begin{cases} P_1^0=-D_t^2+D_y^2+D_x^2, \\ B_1^0=D_x-b_1(y)\,D_y-c_1(y)\,D_t, \end{cases} \qquad \begin{cases} P_2^0=-D_t^2+4^{-1}(D_y^2+D_x^2), \\ B_2^0=D_x-b_2(y)\,D_y-c_2(y)\,D_t, \end{cases}$$

where

$$c_1(y)\geqq b_1(y)>0, \qquad c_1(y)=b_1(y) \text{ in } |y|\leqq 1,$$

$$c_2(y)\geqq 2b_2(y)>0, \qquad c_2(y)=b_2(y)=0 \text{ in } |y|\leqq 1+\varepsilon \quad (\varepsilon>0),$$

$b_1(y)$, $b_2(y)$, $c_1(y)$, $c_2(y)$ are constant for large $|y|$.

Then the iterated mixed problem $(P, {}^{\chi}B_j)$ (χ = the unit) is L^2-well posed and in a neighbourhood of ± 1 the type of frozen problems $(P_1^0, B_1^0)_y$ vary from NU to U.

3. 2 Secondly we present a characterization of iterated mixed problems $(P, {}^{\chi}B_j)$ of 6th order with constant coefficients. As announced in [2], § 1, there appears a more restricted type than one in Theorem 1.

PROPOSITION 3. 1. An iterated mixed problem $(P, {}^{\chi}B_j)$ of 6th order with constant coefficients is L^2-well posed if and only if the type of an ordered set $((P_1^0, B_1^0), (P_2^0, B_2^0), (P_3^0, B_3^0))$ of second order problems becomes one of the following three types:

(i) $$\chi = \begin{pmatrix}1, 2, 3\\1, 2, 3\end{pmatrix}, \quad \begin{pmatrix}1, 2, 3\\3, 2, 1\end{pmatrix}, \quad \begin{pmatrix}1, 2, 3\\1, 3, 2\end{pmatrix},$$

$\begin{pmatrix}1, 2, 3\\2, 3, 1\end{pmatrix}$, three types are

$$(U, U, \bar{U}), \quad (U, \overline{NU}, N), \quad (\overline{NU}, N, N),$$

(ii) $$\chi = \begin{pmatrix}1, 2, 3\\2, 1, 3\end{pmatrix}, \quad \begin{pmatrix}1, 2, 3\\3, 1, 2\end{pmatrix}, \text{ three types are}$$

$$(U, U, \bar{U}), \quad (U_2, \overline{NU}, N), \quad (\overline{NU}, N, N) \qquad (n=2),$$

$$(U, U, \bar{U}), \quad (U_2, NU, N), \quad (\overline{NU}, N, N) \qquad (n>2).$$

Here we explain a type U_2 depending on (P_2^0, B_2^0). A mixed problem (P_1^0, B_1^0) of second order is said to be type U_2 if

$$c_1 \neq 0 \quad \text{and} \quad c_1 a_2 = |b_{11}| \qquad (n=2),$$

$$b_2 \neq 0, \ c_1 a_2 |b_2| = |(b_1, b_2)| \quad \text{and} \quad c_1 a_1 > |b_1| \qquad (n>2)$$

where $b_j = (b_{j1}, \cdots, b_{j\overline{n-1}})$ $(j=1, 2)$, $(b_1, b_2) = \sum_{k=1}^{n-1} b_{1k} b_{2k}$ and $|b_j| = (b_j, b_j)^{1/2}$. Recall that the type U of (P_1^0, B_1^0) means $c_1 a_1 > |b_1|$. Then all problems of type U_2 form a subset of problems of type U. Here we use $a_2 < a_1$.

The proof of Proposition 3. 1 is based on a characterization of an L^2-well posed mixed problem with constant coefficients by the reflection coefficients (see the proof of Theorem 1 in [2]). Furthermore we use the fact that the zeros of Lopatinski determinant R_j for (P_j^0, B_j^0) has of dimension 1 if (P_j^0, B_j^0) is of type NU (see [2], p. 114).

3. 3. Finally we give the precision of Kreiss-Rauch example. Consider the mixed problem (P, B) for 3×3 system of first order with constant coefficients in two space variables $(x_1, x_2) = (y, x)$:

$$P=D_t+\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & s\end{pmatrix}D_x+\begin{pmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}D_y,$$

$$B=\begin{pmatrix}1, & \alpha, & 0\\ a, & b, & -1\end{pmatrix}.$$

Here $0<s<1$ and a, b, α are complex constants. Then we have the following

PROPOSITION 3.2. The mixed problem (P, B) mentioned above is L^2-well posed if and only if either $|\alpha|<1$ or $|\alpha|=1$ and $a\alpha-b=0$.

The mixed problem (P, B) satisfies the uniform Lopatinski condition if and only if $|\alpha|<1$. Let α, a, b be satisfied $|\alpha|=1$ and $a\alpha-b\neq 0$. Then the corresponding problem (P, B) is not L^2-well posed, although it is a limit of L^2-well posed problems with uniform Lopatinski condition. H. O. Kreiss-J. Rauch proved implicitly the existence of such α, a, b (see [8], §6).

PROOF OF PROPOSITION 3.2. The roots of characteristic equation $\det P(\tau, \sigma, \lambda)=0$ are

$$\lambda_1^{\pm}(\tau, \sigma)=\pm i\sqrt{\sigma^2-\tau^2}\,,\ \lambda_2^{+}=-s^{-1}\tau$$

and the corresponding right eigenvectors are

$$e_1^{\pm}(\tau, \sigma)={}^t(\lambda_1^{\pm}-\tau,\ -\sigma,\ 0),$$
$$e_2^{+}={}^t(0, 0, 1),$$

respectively. Here $\operatorname{Im}\lambda_j^{+}>0$ $(j=1, 2)$ and $\operatorname{Im}\lambda_1^{-}<0$ if $\operatorname{Im}\tau=-\gamma<0$. Hence Lopatinski determinant $R(\tau, \sigma)$ for (P, B) is as follows:

$$R(\tau, \sigma)=\begin{vmatrix}(b_1, e_1^{+}) & (b_1, e_2^{+})\\ (b_2, e_1^{+}) & (b_2, e_2^{+})\end{vmatrix}=-(\lambda_1^{+}-\tau-\alpha\sigma), \tag{3.1}$$

where $b_1=(1, \alpha, 0)$ and $b_2=(a, b, -1)$. Furthermore, the reflection coefficients C_{11} and C_{21} for (P, B) are as follows:

$$C_{11}(\tau, \sigma)=\begin{vmatrix}(b_1, e_1^{-}) & (b_1, e_2^{+})\\ (b_2, e_1^{-}) & (b_2, e_2^{+})\end{vmatrix}\Big/R(\tau, \sigma)=\frac{\lambda_1^{-}-\tau-\alpha\sigma}{\lambda_1^{+}-\tau-\alpha\sigma}, \tag{3.2}$$

$$C_{21}(\tau, \sigma)=\begin{vmatrix}(b_1, e_1^{+}) & (b_1, e_1^{-})\\ (b_2, e_1^{+}) & (b_2, e_1^{-})\end{vmatrix}\Big/R(\tau, \sigma)=\frac{2\lambda_1^{+}(b-a\alpha)\sigma}{\lambda_1^{+}-\tau-\alpha\sigma} \tag{3.3}$$

We see from the same method obtained [2], Lemma 3.1 that (P, B) is L^2-well posed if and only if

$$R(\tau, \sigma)\neq 0 \quad \text{for} \quad \operatorname{Im}\tau=-\gamma<0 \tag{3.4}$$

and, in a neighbourhood in $S=\{(\tau, \sigma),\ |\tau|^2+|\sigma|^2=1\}$ of (ξ_0, σ_0) satisfying $R(\xi_0, \sigma_0)=0$,

$$\begin{aligned} &\Big|C_{11}(\tau, \sigma)\Big| \\ &\leqq C(\xi_0, \sigma_0)\Big|\mathrm{Im}\,\lambda_1^+(\tau, \sigma)\,\mathrm{Im}\,\lambda_1^-(\tau, \sigma)\Big|^{1/2} \\ &\quad\times\left|\left(\frac{\partial}{\partial\lambda}\det P\right)\left(\tau, \sigma, \lambda_1^-(\tau, \sigma)\right)\right|\gamma^{-1}. \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\Big|C_{21}(\tau, \sigma)\Big| \\ &\leqq C(\xi_0, \sigma_0)\Big|\mathrm{Im}\,\lambda_2^+(\tau, \sigma)\,\mathrm{Im}\,\lambda_1^-(\tau, \sigma)\Big|^{1/2} \\ &\quad\times\left|\left(\frac{\partial}{\partial\lambda}\det P\right)\left(\tau, \sigma, \lambda_1^-(\tau, \sigma)\right)\right|\gamma^{-1}. \end{aligned} \tag{3.6}$$

Since $R(\tau, \sigma)$ and $C_{11}(\tau, \sigma)$ mentioned above are Lopatinski determinant and the reflection coefficient for the following problem of second order:

$$\begin{aligned} &(-D_t^2+D_y^2+D_x^2)\,u=f && (x>0), \\ &(D_x-\alpha D_y-D_t)\,u=0 && (x=0), \end{aligned}$$

it follows from [1], §5, Example 2 that the conditions (5.11) and (3.5) are equivalent to $|\alpha|\leqq 1$. When $|\alpha|<1$ the problem (P, B) satisfyies uniform Lopatinski condition; that is, $R(\tau, \sigma)\neq 0$ for $\mathrm{Im}\,\tau=-\gamma\leqq 0$. When $\alpha=\pm 1$, there exist a point (ξ_0, σ_0) such that

$$\xi_0^2=\sigma_0^2 \quad\text{and}\quad R(\xi_0, \sigma_0)=0\,. \tag{3.7}$$

Furthermore, when $|\alpha|=1$ and $\alpha\neq\pm 1$, there exists a point (ξ_0, σ_0) such that

$$\xi_0^2<\sigma_0^2 \quad\text{and}\quad R(\xi_0, \sigma_0)=0 \tag{3.8}$$

First assume $\alpha=\pm 1$ and $b-a\alpha\neq 0$. Then we see from (3.3) and (3.7) that

$$\Big|C_{21}(\xi_0-i\gamma, \sigma_0)\Big|\geqq C(\xi_0, \sigma_0), \tag{3.9}$$

$$\Big|D_{21}(\xi_0-i\gamma, \sigma_0)\Big|\leqq C(\xi_0, \sigma_0)\,\gamma^{1/4} \tag{3.10}$$

for small $\gamma>0$ and some constant $C(\xi_0, \sigma_0)>0$. Here $D_{21}(\tau, \sigma)$ stands for the right hand side of inequality (3.6). Second assume $|\alpha|=1$, $\alpha\neq\pm 1$ and $b-a\alpha\neq 0$. Then we obtain from (3.3) and (3.8) that

$$\Big|C_{21}(\xi_0-i\gamma, \sigma_0)\Big|\geqq C(\xi_0, \sigma_0)\,\gamma^{-1}, \tag{3.11}$$

$$\Big|D_{21}(\xi_0-i\gamma, \sigma_0)\Big|\leqq C(\xi_0, \sigma_0)\,\gamma^{-1/2} \tag{3.12}$$

for small $\gamma>0$ and some constant $C(\xi_0, \sigma_0)>0$. Therefore, if $b-a\alpha\neq 0$ for

$|\alpha|=1$, then (3.9), (3.10), (3.11) and (3.12) show contradiction for (3.6); that is, for L^2-well posedness of (P, B). On the other hand, it is proved by the same method as in I, §6 that the fact $|\alpha|=1$ and $b-a\alpha=0$ implies (3.6).

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