Hokkaido Mathematical Journal Vol. 6 (1977) p. 52-55

Remarks on characterization of locally compact abelian groups

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(Received March 24, 1976)

§1. Introduction

In this note G is a locally compact Abelian group with ordered dual Γ. This means that there exists a closed semigroup P such that $P\cap(-P)=\{0\}$ and $P\cup(-P)=\Gamma$.

Let $M(G)$ denote the usual Banach algebra of all complex measures on G and $\hat{\mu}(\gamma)$ the Fourier-Stieltjes transform of measure $\mu\in M(G)$. $L^{1}(G)$ is the set of all functions integrable on G. H. Helson and D. Lowdenslager proved the following generalized F. and M. Riesz theorem.

THEOREM 0. *Suppose G is compact, μ belongs to $M(G)$ and μ is of analytic type. If $d\mu=d\mu_{s}+f(x)dx$, where μ_{s} is singular with respect to Haar measure dx of G and f belongs to $L^{1}(G)$.*

Then, (1) *both μ_{s} and f are of analytic type,*
(2) $\hat{\mu}_{s}(0)=0$

In general, however, the conclusion of theorem 0 cannot be strengthened to "$\mu_{s}=0$". Indeed, we can see such an example in Rudin's book (1).

In theorem 1 and 2 below we shall show that the necessary and sufficient condition that $\mu\in L^{1}(G)$ if μ is of analytic type is $G=T$ or R.

§2. Compact case

THEOREM 1. *Suppose G is compact. Then, the following statements* (A) *and* (B) *are equivalent.*

(A) *Let $\mu\in M(G)$ be of analytic type, then $\mu\in L^{1}(G)$.*

(B) $G=T$.

We shall show some lemmas before we prove theorem 1.

LEMMA 1. [(1)]. *Suppose G is compact. If there exists a non-zero singular measure $\mu_{s}\in M(G)$ which is of analytic type, $E_{\mu}=\{\gamma\in\Gamma;\ \hat{\mu}(\gamma)\neq 0, \gamma>0\}$ has no minimum elememt.*

LEMMA 2. *Suppose G is compact, then the following statemets* (A)' *and* (B)' *are equivalent.*

(A)' *there exists a measure* $\mu \in M(G)$ *such that* $\mu \notin L^1(G)$ *and* μ *is of analytic type.*

(B)' *there exists a positive element* $\gamma_0 \in \Gamma$ *such that* $\{\gamma \in \Gamma\ ;\ 0 < \gamma < \gamma_0\}$ *is an infinite set.*

[proof of lemma 2]

(B)'$\Longrightarrow$(A)'

(case 1) We suppose that Γ is an Archimedean ordered group. Then, we may assume that Γ is a discrete subgroup of R. [(1): 8. 1. 2.]. We define a function $f(x)$ be $\max(|2\gamma_0| - x, 0)$. Then, $f(x)$ is a positive definite function on R. If ϕ is the restriction function of f to Γ, it follows that ϕ is positive definite on Γ.

Hence, by Bochner's theorem, there exists a measure $\mu \in M(G)$ such that $\hat{\mu}(\gamma) = \phi(\gamma)$ on Γ. We define a measure $\mu \in M(G)$ by $\hat{\mu}(\gamma) = \hat{\mu}(\gamma - 2\gamma_0)$. Clearly, μ is of analytic type.

By the hypothesis, $\{\gamma \in \Gamma\ ;\ \hat{\mu}(\gamma) > \delta\}$ is an infinite set for some positive number δ. Hence, by Riemann-Lebesgue's lemma, μ is not absolutely continuous with respect to the Haar measure on G.

(case 2) We suppose that Γ is not an Archimedean ordered group. Then, there exists some positive elements $\gamma_1, \gamma_2 \in \Gamma$ such that $n\gamma_1 < \gamma_2$ for any $n \in Z$.

We put $\Lambda = \{n\gamma_1\ ;\ n \in Z\}$. Since Λ is a subgroup of Γ, there existe a measure $\mu \in M(G)$ such that $\hat{\mu}(\gamma) = \chi_{\gamma_2 - \Lambda}(\gamma)$.

Where $\chi_{\gamma_2 - \Lambda}$ is a characteristic function of $\gamma_2 - \Lambda$.

It is easy to verify that μ is of analytic type. Since Λ is an infinite subgroup, by Riemann-Lebesgue's lemma, μ is not absolutely continuous with respect to the Haar measure on G.

(A)'$\Longrightarrow$(B)'. Suppose that there exists a measure $\mu \in M(G)$ such that μ is of analytic type, but does not belong to $L^1(G)$.

By theorem 0, we may assume that μ is singular with respect to the Haar measure on G. Since $\mu \neq 0$, there exists a positive element $\gamma_0 \in \Gamma$ such htat $\hat{\mu}(\gamma_0) \neq 0$. Hence, by lemma 1, $\{\gamma \in \Gamma\ ;\ 0 < \gamma < \gamma_0\}$ is an infinite set. q. e. d.

[proof of theorem 1]

(B)$\Longrightarrow$(A) trivial

(A)$\Longrightarrow$(B) By lemma 2, $\{\gamma \in \Gamma\ ;\ 0 < \gamma < \gamma_0\}$ is a finite set for any positive element $\gamma_0 \in \Gamma$. Hence, Γ is an Archimedean ordered group. So, Γ is a subgroup of R and $\{\gamma \in \Gamma\ ;\ \gamma > 0\}$ has a minimal element $\gamma_0 \in \Gamma$. Hence, $G = T$ because of $\Gamma = \{n\gamma_0\ ;\ n \in Z\}$. q. e. d.

§ 3. Non-compact case

THEOREM 2. We suppose that G is not compact.

If any measure $\mu \in M(G)$ which is of analytic type belongs to $L^1(G)$, then $G=R$.

[proof of theorem 2]

$\Gamma=G$ is a non-discrete ordered group. Hence, by (1) 8.1.5, $\Gamma=R\oplus D$. Where D is a discrete group.

We suppose $D\neq\{0\}$. Let P denote the closed semi-group which induces the order into Γ.

claim 1. There exists a non-zero element $d_0\in D$ suth that $(R+d_0)\cap P\neq\phi$.

Because, we assume $(R+d)\cap P=\phi$ for any $d\in D\backslash\{0\}=C$.

Then $(\bigcup_{d\in C}(R+d))\cap P=\phi$.

Since $P\cup(-P)=\Gamma$, $\bigcup_{d\in C}(R+d)\subset -P$. We fixe a non-zero element $-d_0$ $\in D$. Then, $R-d_0\subset -P$. Hence $R+d_0\subset P$.

We have a contradiction.

claim 2. $R+d_0\subset P$ for d_0 of claim 1.

Because, by claim 1, $(R+d_0)\cap P\neq\phi$. Since $0\notin R+d_0$, $R+d_0=(R+d_0)\cap(P\backslash\{0\})\cup((R+d_0)\cap P^c)$.

Since $(R+d_0)\cap(P\backslash\{0\})\neq\phi$ and $R+d_0$ is connected, We have $(R+d_0)\cap P^c=\phi$. Hence, $R+d_0\subset P$. Now, let denote δ_0 a dirac measure at 0 in R and m a Haar measure on $\hat{D}$. We define a measure $\mu\in M(G)$ by

$$d\mu(s, x)=d\delta_0(s)\times d\lambda(x)$$

Where $d\lambda(x)=(x, d_0)\,dm(x)$. $(s\in R,\ x\in\hat{D})$.

Since
$$\begin{aligned}\hat{\mu}(q, d)&=\int_{R\times\hat{D}}(-s, q)(-x, d)\,d\delta_0(s)\times d\lambda(x)\\&=\hat{m}(d-d_0)\\&=\chi_{R\times\{d_0\}}(q, d)\qquad(q\in R,\ d\in D)\end{aligned}$$

Hence, supp $(\hat{\mu})\subset R+d_0\subset P$. Therefore, μ is of analytic type. But, by Riemann-Lebesgue's lemma, μ does not belong to $L^1(G)$. This is contrary to the hypothesis. q. e. d.

REMARK. Let P be a closed semi-group of R such that (i) $P\cup-(P)=R$ and (ii) $P\cap(-P)=\{0\}$. Then, P is $[0,\infty)$ or $(-\infty,0]$. Hence, the converse of theorem 2 is the *F.* and *M.* Riesz theorem on R.

References

[1] W. RUDIN: Fourier analysis on groups. New York interscience, 1962.

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