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Bochner-Minlos' theorem on infinite dimensional spaces

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§1. Introduction

In [2], Dao-Xing has shown that the following:

THEOREM A. *Let H and G be real separable Hilbert spaces such that H is a linear subspace of G and the inclusion mapping T from H into G is continuous. Let $\mathfrak{B}$ denote the totality of weak Borel sets in G, and $\mathfrak{F}$ the totality of weak Borel sets in the conjugate space H^* of H. Then, the following conditions are equivalent.*

(1) *T is a Hilbert-Schmidt operator from H into G.*

(2) *There exists a H-quasi-invariant finite measure (non-trivial) on $(G, \mathfrak{B})$.*

(3) *For any positive definite continuous function f on G with $f(0)=1$, there exists a unique probability measure μ on $(H^*, \mathfrak{F})$ such that, for any $x \in H$,*

$$f(x)=\int_{H^*} e^{ix^*(x)}\, d\mu(x^*).$$

In [20], the author has proven the following result. This is a generalization of Theorem A.

THEOREM B. *Let Φ be a separable σ-Hilbert space, with the inner products $(\varphi_1, \varphi_2)_n^{\Phi}$, and let Ψ be a linear subspace of Φ, and suppose that Ψ itself is a complete separable σ-Hilbert space with respect to the inner products $(\psi_1, \psi_2)_n^{\psi}$. Also, suppose that the inclusion mapping T from Ψ into Φ is continuous. For each n, let Φ_n denote the completion of Φ with respect to the inner products $(\varphi_1, \varphi_2)_n^{\Phi}$, and Ψ_n denote the completion of ψ with respect to the inner products $(\psi_1, \psi_2)_n^{\psi}$, respectively. Then, the following conditions are equivalent.*

(1) *T is a Hilbert-Schmidt operator from Ψ into Φ in σ-Hilbert spaces. Namely, for any m, there exists n such that T is a Hilbert-Schmidt operator from Ψ_n into Φ_m.*

(2) *For any n, there exists a Ψ-quasi-invariant finite measure (non-trivial) on $(\Phi_n, \mathfrak{B}_n)$.*

(3) *For any positive definite continuous function L on Φ with $L(0)=1$, there exists a unique probability measure μ on $(\Psi^*, \mathfrak{F})$ such that*

$$L(\psi)=\int_{\Psi^*} e^{iF(\psi)}\, d\mu(F) \quad for \quad \psi\in\Psi.$$

In this paper, we shall establish theorems analogous to Theorem A (Theorem B) when G and H (Φ and Ψ) belongs to some suitable class of separable Banach spaces (complete separable σ-normed spaces), respectively. In Theorem A, if the condition (3) is satisfied for G and H, then we shall call that Bochner-Minlos' Theorem is valid for (H, G).

Throughout this paper (except for §2. 1°.), we shall assume that linear spaces are with real coefficients.

§2. Basic definitions and well known results

1°. *p*-absolutely summing operators ($1\leqq p<\infty$)

Let E and F be Banach spaces.

A sequence $\{x_i\}$ with values in E is called weakly p-summable ($l_p(E)$) if for all $x^*\in E^*$, the sequence $\{x^*(x_i)\}\in l_p$.

A sequence $\{x_i\}$ with values in E is called absolutely p-summable, ($l_p\{E\}$) if the sequence $\{\|x_i\|\}\in l_p$.

DEFINITION 2.1.1. *A linear operator T from E into F is called p-absolutely summing if for each $\{x_i\}\subset E$ which is weakly p-summable, $\{T(x_i)\}\subset F$ is absolutely p-summable.*

We shall say "absolutely summing" instead of "1-absolutely summing".

THEOREM 2.1.1. (c.f. [11])

Let a linear operator T from E into F be p-absolutely summing. If $1\leqq p\leqq q<\infty$, then T is q-absolutely summing.

THEOREM 2.1.2. (c.f. [11], [13])

Let H and G be Hilbert spaces and let T be a linear operator from H into G. Then the following conditions are equivalent.

(1) *T is p-absolutely summing.*

(2) *T is a Hilbert-Schmidt operator.*

THEOREM 2.1.3. (c.f. [11])

Let H be a Hilbert space and E be a Banach space. Then the following conditions are equivalent.

(1) *T is 2-absolutely summing.*

(2) *There exists a Hilbert space G such that*

$$H \xrightarrow[U]{} G \xrightarrow[V]{} E$$

$T=V\circ U$ where U is a Hilbert-Schmidt operator and V is a continuous linear operator.

EXAMPLE 2.1.1. identity operator $I: l_1\to l_2$ is absolutely summing.

EXAMPLE 2.1.2. identity operator $I: l_2\to l_\infty$ is not p-absolutely summing, for $1\leqq p<\infty$.

REMARK 2.1.1. From the Example 2.1.1. and 2.1.2., p-absolutely summing operators are not closed under conjugation.

Generally, p-absolutely summing operators are not necessarily compact (c.f. Ex. 2.1.1).

But a p-absolutely summing operator T from a Hilbert space H into a Banach space E is compact.

Next, we shall introduce the following theorem which plays an important role in the ensuing discussions.

Let X be a set and $\mathfrak{B}$ be a σ-algebra in X, and let μ be a positive measure such that there exist positive constants C_1, C_2 *and pairwise* disjoint measurable subsets $\{X_n\}\subset X$, which satisfy the following conditions:

$$C_1\leqq \mu(X_n)\leqq C_2, \quad \text{for all} \quad n=1, 2, \cdots$$

Let $L_p(X, \mu)$ be a usual Banach space, then l_p (usual sequence space) is a $L_p(X, \mu)$-space which satisfies the above conditions.

We shall denote L_p instead of $L_p(X, \mu)$ in the following theorem.

THEOREM 2.1.4. (c.f. [21])

Let E be a Banach space, and $1\leqq p<\infty$. Then the following conditions are equivalent.

(1) *For all Banach spaces F, if T is a p-absolutely summing operator from E into F, then T^* (conjugate of T) is a p-absolutely summing operator from F^* into E^*.*

(2) *If T is a p-absolutely summing operator from E into L_p, then T^* is a p-absolutely summing operator from L_{p^*} into E^*.*

(3) *For any $\{x_n^*\}\subset E^*$ with $\|x_n^*\|=1$ $(n=1, 2, \cdots)$,*

$$\bigcap_{\alpha} l_p(\rho_{n,\alpha})=l_p$$

where $\rho_{n,\alpha}=\sum_{i=1}^{\infty}|x_n^*(x_i)|^p$, *with* $\{x_i\}\in l_p(E)$.

(4) *For any $\{x_n^*\}\subset E^*$ with $\|x_n^*\|=1$ $(n=1, 2, \cdots)$,*

$$\bigcap_{T\in L(F,E)} l_p(\|T^*x_n^*\|^p)=l_p$$

where the totality of continuous linear operators from F into E is denoted by $L(F, E)$, and F is denoted by the following,

$$F=\begin{cases} l_{p^*} & \text{if } p>1 \\ c_0 & \text{if } p=1 \end{cases} \qquad (1/p+1/p^*=1).$$

In the above theorem, if a Banach space E satisfies the condition (3) (or equivalently (1), (2) and (4)), we shall call that E has the $(*)_p$-conditions.

In this sence, it is easily seen that if E^* is isomorphic to a subspace of l_p, then E has the $(*)_p$-conditions. And also, by Theorem 2. 1. 1. and Theorem 2. 1. 3., if E is isomorphic to a Hilbert space H, then E has the $(*)_p$-conditions $(1\leqq p\leqq 2)$.

More generally, $\mathscr{L}_{p^*,\lambda}$-space has the $(*)_p$-conditions (c. f. [1], [7], [21]). The definition of this space is due to Lindenstrauss and Pelczyński (c. f. [7]), and that is the following:

Let E and F be Banach spaces. The distance $d(E,F)$ between E and F is defined by $d(E,F)=\inf\{\|T\|\circ\|T^{-1}\|\}$, where the infimum is taken over all invertible operators in $L(E,F)$. If no such T exists, i. e., if E and F are not isomorphic, $d(E,F)$ is taken as ∞.

DEFINITION 2. 1. 2. *Let $1\leqq p\leqq\infty$, and $1\leqq\lambda<\infty$. A Banach space E is called an $\mathscr{L}_{p,\lambda}$-space if for all finite dimensional subspaces $M\subset E$ there exists a finite dimensional subspace N containing M such that $d(N,l_p^n)\leqq\lambda$, where $n=\dim(N)$.*

It can be shown (c. f. [7]) that every $L_p(\mu)$ space is an $\mathscr{L}_{p,\lambda}$-space for all $\lambda>1$ and every space of type $C(K)$, where K is a compact Hausdorff space, is an $\mathscr{L}_{\infty,\lambda}$-space for all $\lambda>1$. More generally, every Banach space whose dual is isometric to an $L_1(\mu)$-space (e. g. every M-space in the sense of Kakutani [5]) is an $\mathscr{L}_{\infty,\lambda}$-space for every $\lambda>1$ (c. f. [8]).

2°. Cylinder sets and Cylinder measure

In this subsection, we describe certain σ-algebras which will often be used in the ensuing discussion, and examine the relations between them.

DEFINITION 2. 2. 1. *Let E be a real linar topological space and E^* be a conjugate space of E. If A is a Borel set in real n-dimensional space R_n, and $x_1, x_2, \cdots, x_n\in E$, the set*

$$\left\{x^*\middle|\left(x^*(x_1), \cdots, x^*(x_n)\right)\in A,\ x^*\in E^*\right\}$$

will be called the Borel cylinder with baes A corresponding to $x_1, \cdots, x_n$.

If the elements $x_1, \cdots, x_n$ generate the linear subspace M of E, then we also call the above set a Borel cylinder corresponding to M, or a Borel M-cylinder. The totality of Borel cylinders corresponding to a fixed M form a σ-algebra, which we denote by $S(M)$, and the totality of all Borel *cylinders forms an algebra S. Let $\mathfrak{F}$ denote the smallest σ-algebra containing S; we call the elements of $\mathfrak{F}$ weak Borel sets.*

Similarly, let $\mathfrak{F}$ be the smallest σ-algebra of subsets of E which contains all sets of the form

$$\left\{x \mid x^*(x) < a\right\} \quad -\infty < a < \infty,\ x^* \in E^*.$$

The elements of $\mathfrak{F}$ will be called weak Borel sets.

The following lemma shows that the weak Borel sets constitute a sufficiently wide class of sets.

LEMMA 2.2.1. (c.f. [3], [9])

If E is a separable σ-normed space, then every open (or closed) subset of E is a weak Borel set.

LEMMA 2.2.2. (c.f. [3], [9])

Let E be a separable σ-normed space, with the norm sequence $\{\|x_n\|\}$. Then, $S_{-n}(R)=\{\|x^*\|_{-n}\leqq R\}$ is a weak Borel set in E^*.

By this lemma, we can conclude that E_n^* is a weak Borel set in E^*.

DEFINITION 2.2.2. *Let E be a linear topological space, and let S be the algebra of all Borel cylinders in E^*. Suppose that P is a set function on S having the following property: if M is any finite dimensional linear subspace of E, and S(M) is the σ-algebra of Borel cylinders corresponding to M, then the restriction of P to S(M) is a probabtility measure. Then we call P a cylinder measure on E^*. Clearly, any cylinder measure P also has the following properties:*

(1) $0\leqq P(Z)\leqq 1$ *for all* $Z\in S$

(2) $P(E^*)=1$

(3) *P is finitely additive.*

However, P is not generally σ-additive.

But if it happens that P is σ-additive, then, using well-known technique, we can extend P to a probability measure on the σ-algebra generated by S.

Next, we shall show the continuity of cylinder measures.

DEFINITION 2.2.3. *Let E be a linear topological space, and let P be a cylinder measure on E^*. Suppose that, given any positive number ε, there exists a neighborhood V of zero in E such that*

$$P\left(\left\{x^* \,\middle|\, |x^*(x)|>1,\ x^*\in E^*\right\}\right)<\varepsilon$$

where $x\in V$. Then we say that P is continuous.

LEMMA 2.2.3. (c.f. [2], [3])

Let E be a linear topological space and let P be a cylinder measure on E^*. Then the function

$$L(x)=\int_{E^*} e^{ix^*(x)}\, dP(x^*) \quad for \quad x \in E$$

is continuous iff P is continuous.

LEMMA 2.2.4. (c.f. [2], [3])

Let E be a linear topological space and let $L(x)$ be a continuous positive definite function on E with $L(0)=1$. Then, there is a unique continuous cylinder measure P on (E^*, S), such that

$$L(x)=\int_{E^*} e^{ix^*(x)}\, dP(x^*) \quad for \quad x \in E.$$

REMARK 2.2.1. In Lemma 2.2.4., if E is a nuclear space, then P is a probability measure on $(E^*, \mathfrak{F})$.

If E is a σ-Hilbert space and $L(x)$ is continuous relative to the nuclear topology, then also P is a probability measure.

(For details, c.f. [2], [3], [9], [19], [22])

3°. Minlos' Theorem and Sazonov's Theorem

THEOREM 2.3.1. (c.f. [9])

In order that every continuous cylinder measure, defined in a space E^* conjugate to a σ-Hilbert space E, be extendable to a σ-additive measure in E^*, it is necessary and sufficient that E be a nuclear space.

REMARK 2.3.1. In Theorem 2.3.1., if E is a nuclear (not necessarily metrizable), then the sufficiency is valid (c.f. [22]).

In our sense (c.f. §1), if E is a nuclear space, then we can say that Bochner-Minlos' Theorem is valid for (E, E).

THEOREM 2.3.2. (c.f. [3])

Let H and G be Hilbert spaces, and let T be a continuous linear operator from H into G. Then the following conditions are equivalent.

(1) *T is a Hilbert-Schmidt operator from H into G.*

(2) *Let μ_G be the Gaussian measure, defined in G^* by $(x, y)_G$, then the measure $T^*\mu_G$ in H^* is σ-additive.*

(3) *For any continuous cylinder measure μ in G^*, the measure $T^*\mu$ in H^* is σ-additive.*

REMARK 2.3.2. In Theorem 2.3.2., if H and G be σ-Hilbert spaces, then the condition (1), (2) and (3) are equivalent (c.f. [19]).

However, if H is a Banach space and G is a Hilbert space, then the condition (1), (2) and (3) are not necessarily equivalent.

The counter example shall be given in the next section.

THEOREM 2. 3. 3. (c. f. [14])

In order that a cylinder measure μ *in the Hilbert space* H *be* σ-*additive, it is necessary and sufficient that* μ *be continuous relative to the topology in* H *defined by some sequence* $B_1, B_2, \cdots$ *of positive-definite nuclear operators.*

The continuity of μ *means the following: For any* $\varepsilon>0$ *there exists a* $\delta>0$ *and* n *such that the inequality* $(B_n x, x)\leqq\delta$ *implies that* $\mu(\Gamma_x)\leqq\varepsilon$, *where* Γ_x *denotes the strip defined by* $|(x, y)|\geqq 1$.

We shall call that the topology defined in the above theorem is a nuclear topology.

REMARK 2. 3. 3. Theorem 2. 3. 3. is due to V. Sazonov, and σ-Hilbert case is due to the author and Dao-Xing (c. f. [2], [19]), and more general case is due to Badrikian (c. f. [16]).

Throughout this subsection, we shall assume that linear spaces are separable with real coefficients.

4°. Theorems for the existence of quasi-invariant measures

DEFINITION 2. 4. 1. *Let* E *be a linear space,* F *be a linear subspace of* E, *and* $\mathfrak{B}$ *be a* σ-*algebra in* E, *which is invariant under translations. A measure* μ *on* $(E, \mathfrak{B})$ *is called* F-*quasi-invariant if*

$$\mu(B)=0 \quad \textit{implies} \quad \mu(B+x)=0 \quad \textit{for every} \quad x\in F, B\in\mathfrak{B}.$$

DEFINITION 2. 4. 2. *Let* E *be a linear topological space,* E^* *be a conjugate space of* E, *and let* $\|x\|_H$ *be a continuous Hilbertian norm on* E. *It is easily seen that the following* $L(x)$ *is continuous positive definite function on* E.

$$L(x)=e^{-\frac{\|x\|_H^2}{2}}$$

The corresponding cylinder measure on E^* (*by Lemma* 2. 2. 4.) *is called a Gaussian measure.* (*mean zero, variance* 1)

THEOREM 2. 4. 1. (c. f. [3], [22])

Let E *be a nuclear space, and* $\|x\|_H$ *be a continuous Hilbertian norm on* E. *Then, the corresponding Gaussian measure* μ_H *on* E^* *is* σ-*additive and* E-*quasi-invariant.*

$$(E\subset H\cong H^*\subset E^*)$$

Next, we shall introduce a theorem which gives a necessary condition for the existence of quasi-invariant measures.

THEOREM 2.4.2. (c.f. [20])

Let F be a Banach space, E be a linear subspace of F, and suppose that E itself is a complete σ-normed space with the norm sequence $\|x\|_n$ $(n=1, 2, \cdots)$. Also, suppose that the inclusion mapping T from E into F is continuous.

Then, the existence of a E-quasi-invariant finite measure (non-trivial) μ on $(F, \mathfrak{F})$ implies that, there exists n_0 such that

(1) *T^* is absolutely summing* $(T^*: F^* \to E^*_{n_0})$

(2) *T^* is compact* $(T^*: F^* \to E^*_{n_0})$.

REMARK 2.4.1. In the above theorem, $\mathfrak{F}$ is a σ-algebra in F which is invariant under translations and contrains all cylinder sets.

In virtue of Theorem 2.4.2., we obtain the following theorem which gives a necessary and sufficient condition for the existence of quasi-invariant measures.

THEOREM 2.4.3. (c.f. [20])

Let H be a separable Hilbert space, and let $\mathfrak{F}$ be the totality of weak Borel sets in H. Let E be a linear subspace of H, and suppose that E itself is a complete σ-normed space with the norm sequence $\{\|x\|_n\}$.

Also, suppose that the inclusion mapping T from E into H is continuous.

Then, the following conditions are equivalent.

(1) *There exists a E-quasi-invariant finite measure (non-trivial) on $(H, \mathfrak{F})$.*

(2) *There exists n such that the conjugate operator T^* from H^* into E^*_n is absolutely summing.*

(3) *There exists a separable Hilbert space H_1 such that*

$$E \underset{J}{\subset} H_1 \underset{K}{\subset} H$$

$T = K \circ J$ where injection map J is continuous and K is a Hilbert-Schmidt operator respectively.

REMARK 2.4.2. Theorem 2.4.3. is due to the author, and that is the generalization of the Dao-Xing's theorem (c.f. [2]).

Finally, we shall introduce a theorem due to Dao-Xing, which gives a sufficient condition for the validity of Bochner-Minlos' Theorem.

THEOREM 2.4.4. (c.f. [2])

Let F be a linear topological space, E be a linear subspace of F, and *suppose that E itself is a linear topological space of the second category. Also, suppose that the inclusion mapping from E into F is continuous. Let* $\mathfrak{B}$ *be the* σ*-algebra generated by the totality of closed subsets of F, and suppose that there exists a E-quasi-invariant regular finite measure* μ *on* $(G, \mathfrak{B})$. *Then, for each continuous positive definite function* $L(x)$ *on F with* $L(0)=1$, *there is a unique probability measure P on* $(E^*, \mathfrak{F})$, *such that*

$$L(x)=\int_{E^*} e^{ix^*(x)}\, dP(x^*), \qquad for \quad x\in E.$$

REMARK 2. 4. 3. In the above theorem, let $\mathfrak{F}$ denote the totality of weak Borel sets in E^*. If the assumptions of Theorem 2. 4. 4. is satisfied, then, in our sense, we can say that Bochner-Minlos' Theorem is valid for (E, F). However, if E is a nuclear space, then Bochner-Minlos' Theorem is valid for (E, E), but the assumptions of Theorem 2. 4. 4. is not satisfied (c. f. [17], [22]).

§ 3. Main theorems and other results

Throughout this section, we shall assume that linear spaces are separable with real coefficient. However, by the similar manner, we can discuss for non-separable cases.

1°. General cases

In this subsection, we shall establish theorems analogous to Theorem A (Theorem B, Theorem 2. 3. 1., Theorem 2. 3. 2., Theorem 2. 3. 3., etc.) for complete σ-normed spaces.

LEMMA 3. 1. 1. *Let E be a* σ*-normed space with the norm sequence* $\{\|x\|_n\}$, *and* E^* *be a conjugate space of E. For each n, let* E_n *denote the completion of E with respect to the norm* $\|x\|_n$. *Then, if a cylinder measure* μ *in* E^* *is* σ*-additive,* μ *is continuous relative to the absolutely summing topology.*

The continuity of μ *means the following: There exists the sequence of continuous seminorms* $\{p_n\}$ *in E such that the natural injection from* E_n *into* E_{p_n} *is absolutely summing, and* μ *is continuous relative to the seminorms* $\{p_n\}$; *namely, for any* $\varepsilon>0$ *there exists n and* $\delta>0$, *such that the inequality* $p_n(x)\leqq\delta$ *implies that* $\mu(\Gamma_x)\leqq\varepsilon$, *where* Γ_x *denotes the strip defined by* $|x^*(x)|\geqq 1$.

PROOF. Since μ is σ-additive, hence by Lemma 2. 2. 2., $S_{-n}(n)=\{\|x^*\|_{-n}\leqq n\}$ is μ-measurable.

We define p_n by setting

$$p_n(x)=\int_{S_{-n}(n)}\left|x^*(x)\right|d\mu(x^*) \qquad for \quad x\in E.$$

Then, obviously p_n is a continuous seminorm on E.

CLAIM (a): *The natural injection from E_n into E_{p_n} is absolutely summing.*

For each $\{x_i\}\subset E_n$ which is weakly summable, it is easily seen that we have the following;

$$C=\sup_{\|x^*\|_n\leqq 1}\left\{\sum_{i=1}^{\infty}\left|x^*(x_i)\right|\right\}<\infty.$$

Hence, we have

$$\sum_{i=1}^{\infty}p_n(x_i)=\sum_{i=1}^{\infty}\int_{S_{-n}(n)}\left|x^*(x_i)\right|d\mu(x^*)$$
$$\leqq nC\mu\left(S_{-n}(n)\right)<\infty.$$

Thus, we have the assertion.

CLAIM (b): *μ is continuous relative to the seminorms $\{p_n\}$.*

Without loss of generality, we may assume that $\|x\|_1\leqq\|x\|_2\leqq\cdots\cdots$ Hence, we have

$$E^*=\bigcup_{n=1}^{\infty}S_{-n}(n)$$
$$S_{-n}(n)\subset S_{-(n+1)}(n+1) \qquad (n=1, 2, \cdots).$$

Since μ is σ-additive and $\mu(E^*)=1$, for any $\varepsilon>0$ there exists n such that the complemet of $S_{-n}(n)$ has measure less than $\varepsilon/2$.

Now consider any element x in E such that

$$p_n(x)\leqq\varepsilon/2,$$

and let us estimate the measure of the strip Γ_x defined by $|x^*(x)|\geqq 1$. Obviously,

$$\mu(\Gamma_x)=\mu(\Gamma_x')+\mu(\Gamma_x'')$$

where Γ_x' is that part of Γ_x contained in the ball $S_{-n}(n)$, and Γ_x'' is that part lying outside $S_{-n}(n)$. In view of the choice of $S_{-n}(n)$ we have $\mu(\Gamma_x'')\leqq\varepsilon/2$. On the other hand, from the inequalitty $|x^*(x)|\geqq 1$, which holds for all $x^*\in\Gamma_x$ and therefore for all $x^*\in\Gamma_x'$, it follows that

$$\mu(\Gamma_x')=\int_{\Gamma_x'}d\mu(x^*)\leqq\int_{\Gamma_x'}\left|x^*(x)\right|d\mu(x^*)$$
$$\leqq\int_{S_{-n}(n)}\left|x^*(x)\right|d\mu(x^*)=p_n(x)\leqq\varepsilon/2.$$

Hence $\mu(\Gamma_x) \leqq \varepsilon$.

Thus we have the assertion. That completes the proof.

REMARK 3.1.1. For a cylinder measure μ in E^*, Fourier transform of μ is defined by

$$\hat{\mu}(x) = \int_{E^*} e^{ix^*(x)}\, d\mu(x^*), \qquad \text{for} \quad x \in E.$$

Then, from Lemma 2.2.3., we can say that if a cylinder measure μ is σ-additive, $\hat{\mu}(x)$ is continuous relative to the absolutely summing topology.

LEMMA 3.1.2. *Let E and F be Banach spaces, and T be a continuous linear operator from E into F. Then, the following condition* (1) *implies the condition* (2).

(1) *For any continuous cylinder measure μ in F^*, the cylinder measure $T^*\mu$ in E^* is σ-additive.*

(2) *Let $1 \leqq p \leqq 2$. Then, for each $\{x_i\} \in l_p(E)$, and $\{y_n^*\} \in l_p(F^*)$, we have*

$$\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \left| \langle y_n^*, Tx_i \rangle \right|^p < \infty.$$

PROOF. If $1 \leqq p \leqq 2$, the function

$$\exp\left(-|t|^p\right), \quad -\infty < t < \infty,$$

is a positive definite continuous function on R (c.f. [2]).
Therefore, it is easily seen that for each $\{y_n^*\} \in l_p(F^*)$, the function $L(x)$ defined by

$$L(x) = \exp\left(-\sum_{n=1}^{\infty} \left| \langle y_n^*, x \rangle \right|^p\right), \qquad x \in F,$$

is a positive definite continuous function on F.

From Lemma 2.2.4., there exists a unique continuous cylinder measure μ on F^* such that

$$L(x) = \hat{\mu}(x), \qquad x \in F.$$

Now, let suppose that the condition (1) is hold, then the measure $T^*\mu$ on E^* is σ-additive. Hence, by the remark of Lemma 3.1.1., Fourier transform of the measure $T^*\mu$ is continuous relative to the absolutely summing topology.

On the other hand, by easy calculations, we have

$$\widehat{T^*\mu}(x) = \hat{\mu}(Tx) = \exp\left(-\sum_{n=1}^{\infty} \left| \langle y_n^*, Tx \rangle \right|^p\right).$$

Next, we shall define the seminorm $p(x)$ by

$$p(x)=\left(\sum_{n=1}^{\infty}\left|\langle y_n^*,\ Tx\rangle\right|^p\right)^{1/p}, \qquad x\in E,$$

then it is easily seen that the seminorm $p(x)$ is continuous relative to the seminorms $\{p_n\}$ (c. f. Lemma 3. 1. 1.).
Hence we have that there exists a positive constant C and n, such that

$$p(x)\leqq Cp_n(x), \qquad \text{for} \quad x\in E.$$

Since the natural mapping from E into E_{p_n} is absolutely summing, by Theorem 2. 1. 1., it is p-absolutely summing. Therefore, we have that for any $\{x_i\}\in l_p(E)$,

$$\sum_{i=1}^{\infty} p(x_i)^p \leqq C^p \sum_{i=1}^{\infty} p_n(x_i)^p<\infty .$$

This shows that the condition (2) is hold.

LEMMA 3. 1. 3. *Let E be a Banach space, F be a σ-normed space with the norm sequence $\{\|x\|_n\}$, and let T be a continuous linear operator from E into F. For each n, let F_n denote the completion of F with respect to the norm $\|x\|_n$. Then, if for any continuous cylinder measure μ in F^*, the measure $T^*\mu$ in E^* is σ-additive, we have that the followings;*

(1) *If a Banach space E has the $(*)_p$-conditions $(1\leqq p\leqq 2)$, then, for each n, the conjugate operator T^* from F_n^* into E^* is p-absolutely summing.*

(2) *If for each n, a Banach space F_n^* (dual of F_n) has the $(*)_p$-conditions $(1\leqq p\leqq 2)$, then, for each n, the operator T from E into F_n is p-absolutely summing.*

PROOF of (1). For each n, from Lemma 3. 1. 2., it is easily seen that for each $\{x_i\}\in l_p(E)$, and $\{y_j^*\}\in l_p(F_n^*)$, we have that the following;

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\langle y_j^*,\ Tx_i\rangle\right|^p<\infty .$$

Without loss of generality, we may assume that $\|T^*y_j^*\|$ is not equal to zero $(j=1, 2, \cdots)$. Thus we have

$$\sum_{j=1}^{\infty}\|T^*y_j^*\|^p\left(\sum_{i=1}^{\infty}\left|\langle x_j^*,\ x_i\rangle\right|^p\right)<\infty ,$$

where $x_j^*=T^*y_j^*/\|T^*y_j^*\|$.

Since $\|x_j^*\|=1$ $(j=1, 2, \cdots)$, and a Banach space E has the $(*)_p$-conditions (c. f. Theorem 2. 1. 4.), thus we have the following;

$$\sum_{j=1}^{\infty}\|T^*y_j^*\|^p<\infty .$$

This shows that the operator T^* from F_n^* into E^* is p-absolutely summing.

PROOF of (2). For each n, by the similar arguments for the proof of (1), we have that the following;
for each $\{x_i\}\in l_p(E)$, and $\{y_j^*\}\in l_p(F_n^*)$,

$$\sum_{i=1}^{\infty}\|Tx_i\|_n^p\left(\sum_{j=1}^{\infty}\left|\langle y_j^*, z_i\rangle\right|^p\right)<\infty,$$

where $z_i=Tx_i/\|Tx_i\|_n$.

Since a Banach space F_n is isometric to a subspace of F_n^{**}, $\|z_i\|_n=1$ $(i=1, 2, \cdots)$, and a Banach space F_n^* has the $(*)_p$-conditions, thus we have the following;

$$\sum_{i=1}^{\infty}\|Tx_i\|_n^p<\infty.$$

This shows that the operator T from E into F_n is p-absolutely summing. Thus, we complete the proof.

REMARK 3.1.1. Examples of Banach spaces which satisfy the $(*)_p$-conditions, were given in Section 2.

LEMMA 3.1.4. *Let E and F be σ-normed spaces, and for each n, let E_n denote the completion of E with respect to the norm $\|x\|_n^E$, F_n denote the completion of F with respect to the norm $\|x\|_n^F$, respectively. Also, suppose that T is a continuous linear operator from E into F. If a σ-normed space F satisfies the following codition* $(*)$, *then the following condition* (1) *implies the condition* (2).

$(*)$ *For each m, there exists a positive definite continuous function $L_m(x)$ on F_m (with $L(0)=1$), which satisfies that the following;*
For any $\varepsilon>0$, there exists $\delta>0$, such that the inequality

$$\left|L_m(x)-1\right|<\delta \quad \textit{implies that } \|x\|_m^F<\varepsilon.$$

(1) *For any continuous cylinder measure μ in F^*, the cylinder measure $T^*\mu$ in E^* is σ-additive.*

(2) *For any m, there exists n such that the operator T can be extended to an absolutely summing operator from E_n into F_m.*

PROOF. Since a σ-normed space F satisfies the condition $(*)$, by the similar arguments for the proof of Lemma 3.1.2., it is easily seen that the following; for any m there exists a positive constant C and n, such that

$$\|Tx\|_m^F\leqq Cp_n(x), \qquad \text{for} \quad x\in E,$$

where $p_n(x)$ is a continuous seminorm on E, and the natural mapping from E_n into E_{p_n} is absolutely summing (c.f. Lemma 3.1.1.).

From this, it is easily seen that the operator T can be extended to an absolutely summing operator from E_n into F_m.

REMARK 3.1.2. Let F be a σ-normed space, which satisfies that the following; for any m, a Banach space F_m is isomorphic to a subspace of l_p $(1 \leqq p \leqq 2)$. Then it is easily seen that a σ-normed space F satisfies the above condition $(*)$. In particular, if F is a Köthe space defined by

$$F = \bigcap_{n=1}^{\infty} l_p(a_{m,n}), \quad 1 \leqq p \leqq 2, \quad 0 < a_{m,n} \leqq a_{m,n+1} < \infty, \qquad (m, n = 1, 2, \cdots),$$

then a σ-normed space F satisfies the above condition $(*)$.
And also, if F is a σ-Hilbert space, then F satisfies the above condition $(*)$.

Now, we shall apply these lemmas in the ensuing discussions.

COROLLARY 3.1.1. *Let E be a Banach space, which satisfies one of the following three conditions;*

(1) *A Banach space E has the $(*)_p$-conditions* $(1 \leqq p \leqq 2)$.

(2) *A Banach space E^* (dual of E) has the $(*)_p$-conditions* $(1 \leqq p \leqq 2)$.

(3) *A Banach space E satisfies the condition $(*)$ in Lemma* 3.1.4.
Then, in order that every continuous cylinder measure μ in E^* be σ-additive, it is necessary and sufficient that E be a finite dimensional space.

PROOF. First we prove the necessity of the condition. Suppose that every continuous cylinder measure in E^* be σ-additive.

If a Banach space E satisfies the condition (1), then by Lemma 3.1.3., the identity operator from E^* into E^* is p-absolutely summing, therefore it is a nuclear operator (c.f. [11]). This shows that E be a finite dimensional space.

If a Banach space E satisfies the condition (2), then by Lemma 3.1.3., the identity operator from E into E is p-absolutely summing, therefore, it is nuclear. Thus we have the assertion.

If a Banach space E satisfies the condition (3), then by Lemma 3.1.4., the identity operator from E into E is absolutely snmming, therefore, it is nuclear. Thus we have the assertion.

From classical Bochner's Theorem, sufficiency is obvious.

Using Lemma 3.1.4., Theorem 2.3.1. can be generalized for σ-normed spaces, that is the following.

THEOREM 3.1.1. *Let E be a σ-normed space, which satisfies the condition $(*)$ in Lemma* 3.1.4.. *Then, in order that every continuous cylinder measure in E^* be extendable to a σ-additive one, it is necessary and sufficient that E be a nuclear space.*

PROOF. Using Lemma 3. 1. 4. and Pietsch's Theorem (c. f. ([11]), it is easy.

Next, we shall establish theorems analogous to Theorem 2. 3. 2. for σ-normed spaces. From now, if E is a σ-normed space, we shall denote $E=\cap E_n$, where E_n denote the completion of E with respet to the n-th norm.

THEOREM 3. 1. 2. *Let H be a Hilbert space, and $F=\cap F_n$ be a σ-normed space, which satisfies that for each n, a Banach space F_n^* has the $(*)_p$-conditions $(1 \leqq p \leqq 2)$. Also, suppose that T is a continuous linear operator from H into F. Then the following conditions are equivalent.*

(1) *For each n, T is a Hilbert-Schmidt operator from H into F_n.*

(2) *For any continuous cylinder measure μ in F^*, the measure $T^*\mu$ in H^* is σ-additive.*

PROOF.

(1)$\Rightarrow$(2): By the similar arguments for Theorem 2. 3. 2., we have easily the assertion.

(2)$\Rightarrow$(1): By the assumption of F and Lemma 3. 1. 3., for each n, the operator T from H into F_n is p-absolutely summing. Since $1 \leqq p \leqq 2$, using Theorem 2. 1. 1. and Theorem 2. 1. 3., T is a Hilbert-Schmidt operator from H into F_n.

THEOREM 3. 1. 3. *Let $\Phi=\cap \Phi_n$ be a σ-Hilbert space, and $F=\cap F_n$ be a σ-normed space, which satisfies the condition $(*)$ in Lemma 3. 1. 4.. Also, suppose that T is a continuous linear operator from Φ into F. Then the following conditions are equivalent.*

(1) *For any m, there exists n such that the operator T can be extended to a Hilbert-Schmidt operator from Φ_n into F_m.*

(2) *For any continuous cylinder measure μ in F^*, the measure $T^*\mu$ in Φ^* is σ-additive.*

PROOF.

(1)$\Rightarrow$(2): By the similar arguments for the proof of Theorem 2. 3. 2., we have easily the assertion.

(2)$\Rightarrow$(1): By the assumption of F and Lemma 3. 1. 4., for any m, there exists n such that the operator T can be extended to an absolutely summing operator from Φ_n into F_m. Thus, by Theorem 2. 1. 1. and Theorem 2. 1. 3., we have the assertion.

THEOREM 3. 1. 4. *Let $E=\cap E_n$ be a σ-normed space, which satisfies that for each n, a Banach space E_n has the $(*)_p$-conditions $(1 \leqq p \leqq 2)$, and let $\Phi=\cap \Phi_n$ be a σ-Hilbert space, and also suppose that T be a contin-*

uous linear operator from E into Φ. Then the following conditions are equivalent.

(1) *For any m, there exists n such that the operator T can be extended to a Hilbert-Schmidt operator from E_n into Φ_m.*

(2) *For any continuous cylinder measure μ in Φ^*, the measure $T^*\mu$ in E^* is σ-additive.*

(3) *Let μ_m be the Gaussian measure, defined in Φ^* by $(\varphi, \psi)_m^{\Phi}$, then for any m, the measure $T^*\mu_m$ in E^* is σ-additive.*

PROOF.

(3)$\Rightarrow$(1): For any m, a positive definite continuous function $\hat{\mu}_m(\varphi)$ on Φ_m satisfies the condition (*) in Lemma 3. 1. 4., and therefore, by the similar arguments for the proof of Lemma 3. 1. 4., there exists n such that the operator T can be extended to an absolutely summing operator from E_n into Φ_m. Also, by the assumption, a Banach space E_n has the $(*)_p$-conditions, therefore T^* (conjugate of T) is a p-absotlutely summing operator from Φ_m^* into E_n^* (c. f. Theorem 2. 1. 1. and 2. 1. 4.). Since $1 \leqq p \leqq 2$, by Theorem 2. 1. 1. and Theorem 2. 1. 3., T^* is a Hilbert-Schmidt operator, and therefore, T is a Hilbert-Schmidt operator from E_n into Φ_m.

For the part of (1)$\Rightarrow$(2) and (2)$\Rightarrow$(3), it is easy.

Next, we shall show the Sazonov's Theorem concerning Gaussian measures for σ-normed spaces (c. f. Theorem 2. 3. 3).

DEFINITION 3. 1. 1. (c. f. [16])

Let E be a locally convex Hausdorff space and H a Hilbert space. We shall call a continuous linear map T: $E \to H$ a Hilbert-Schmidt map if it can be factored into

$$E \xrightarrow[U]{} H_1 \xrightarrow[V]{} H,$$

where H_1 is a Hilbert space, U is a continuous linear map and V is a Hilbert-Schmidt map.

The Hilbert-Schmidt topology τ_{HS} on E will be the coarsest topology on E for which all Hilbert-Schmidt maps are continuous.

THEOREM 3. 1. 5. *Let $E = \cap E_n$ be a σ-normed space, which satisfies that for each n, a Banach space E_n has the $(*)_p$-conditions $(1 \leqq p \leqq 2)$, and let $\|x\|_H$ be a continuous Hilbertian norm on E.*

Then, in order that a Gaussian measure μ_H, defined in E^* by $\|x\|_H$, be σ-additive, it is necessary and sufficient that $\hat{\mu}_H(x)$ be continuous relative to the Hilbert-Schmidt topology.

PROOF. First we prove the necessity of the condition. Let H be a completion of E with respect to the Hilbertian norm $\|x\|_H$.

Since $\hat{\mu}_H(x)$ (Fourier transform of the measure μ_H) is defined by

$$\hat{\mu}_H(x)=\exp(-\|x\|_H^2/2), \qquad \text{for} \quad x\in E,$$

in order to prove that $\hat{\mu}_H(x)$ is continuous relative to the Hilbert-Schmidt topology, it is sufficient to show that the natural map from E into H is a Hilbert-Schmidt map. Thus, by the assumption of E and Theorem 3.1.4., we have easily the assertion.

The sufficiency of the condition is obvious.

REMARK 3.1.3. In Theorem 3.1.5., if E is a σ-Hilbert space, then the Hilbert-Schmidt topology on E coincide with the nuclear topology. (For the nuclear topology on σ-Hilbert spaces, c.f. [2], [19])

Next, by Theorem 2.1.4. and Theorm 2.4.2., we obtain that the following theorem for the existence of quasi-invariant measures. That is the generalization of the author's result (c.f. Theorem B in [20]).

THEOREM 3.1.6. *Let E be a Banach space, and let $\mathfrak{F}$ be the totality of weak Borel sets in E. Let Φ be a linear subspace of E, and suppose that $\Phi=\cap\Phi_n$ itself is a complete σ-Hilbert space.*

Also, suppose that the inclusion mapping T from Φ into E is continuous. Then, if E^* (dual of E) has the $(*)_p$-conditions $(1\leqq p\leqq 2)$, the following conditions are equivalent.

(1) *There exists a Φ-quasi-invariant finite measure (non-trivial) on $(E, \mathfrak{F})$.*

(2) *There exists n such that the conjugate operator T^* from E^* into Φ_n^* is absolutely summing.*

(3) *There exists Hilbert spaces H_1 and H_2 such that*

$$\Phi \underset{I}{\subset} H_1 \underset{J}{\subset} H_2 \underset{K}{\subset} E$$

$T=K\circ J\circ I$ where injection map I and K are continuous, J is a Hilbert-Schmidt operator, respectively.

PROOF is easy.

Now, we shall establish main theorems analogous to Theorem A when G and H belongs to some suitable class of complete separable σ-normed spaces. In the ensuing discussions of this subsection, the totality of weak Borel sets is denoted by $\mathfrak{B}$ and $\mathfrak{F}$.

THEOREM 3.1.7. *Let $F=\cap F_n$ be a σ-normed space, which satisfies that for each n, a Banach space F_n^* (dual of F_n) has the $(*)_p$-conditions $(1\leqq p\leqq 2)$. Let H be a subspace of F, and suppose that H itself is a Hilbert* *space. Also, suppose that the inclusion mapping T from H into F is continuous. Then, the following conditions are equivalent.*

(1) *For each n, T is a Hilbert-Schmidt operator from H into F_n.*

(2) *For each n, there exists H-quasi-invariant finite measure (non-trivial) on $(F_n, \mathfrak{B})$.*

(3) *For any positive definite continuous function L on F with $L(0)=1$, there exits a unique probability measure μ on $(H^*, \mathfrak{F})$ such that*

$$L(x)=\int_{H^*} e^{ix^*(x)}\,d\mu(x^*), \qquad \text{for} \quad x\in H.$$

Namely, in our sense, Bochner-Minlos' Theorem is valid for (H, F).

PROOF.

(1)$\Rightarrow$(2): By the similar arguments for the proof of Theorem 2.4.3., it is obvious (c.f. [20]).

(2)$\Rightarrow$(1): By the assumption of F, and by Theorem 2.1.4. and 2.4.2., it is obvious.

(1)$\Rightarrow$(3): By the similar arguments for the proof of Theorem 2.3.2., and by Lemma 2.2.4., it is easily seen that we have the assertion.

(3)$\Rightarrow$(1): By the assumption of F, and Lemma 2.2.4., Theorem 3.1.2., it is easily seen that we have the assertion.

That completes the proof.

REMARK 3.1.4. In the above theorem, we can not apply Theorem 2.4.4. for the proof of (2)$\Rightarrow$(3). However, if we consider the following condition (2)$'$ instead of the condition (2), then, by Theorem 2.4.4., we can prove that the condition (2)$'$ implies the condition (2).

(2)$'$ *There exists a H-quasi-invariant regular finite measure (non-trivial) on $(F, \mathfrak{B})$.*

THEOREM 3.1.8. *Let $F=\cap F_n$ be a σ-normed spac, which satisfies the condition $(*)$ in Lemma 3.1.4., and let Φ be a linear subspace of F, and suppose that $\Phi=\cap\Phi_n$ itself is a complete σ-Hilbert space. Also, suppose that the inclusion mapping T from Φ into F. Then, the following conditions are equivalent.*

(1) *For any m, there exists n such that the operator T can be extended to a Hilbert-Schmidt operator from Φ_n into F_m.*

(2) *For each m, there exists a Φ-quasi-invariant regular finite measure (non-trivial) on $(F_m, \mathfrak{B})$.*

(3) *For any positive definite continuous function L on F with $L(0)=1$, there exists a unique probability measure μ on $(\Phi^*, \mathfrak{F})$ such that*

$$L(x)=\int_{\Phi^*} e^{ix^*(x)}\, d\mu(x^*), \qquad for \quad x\in\Phi.$$

Namely, in our sense, Bochner-Minlos' Theorem is valid for (Φ, F).

PROOF.

(1)$\Rightarrow$(2): By the similar arguments for the proof of Theorem 2.4.3., it is obvious.

(2)$\Rightarrow$(1): Let assume the condition (2), then by Theorem 2.4.4., we have that the Bochner-Minlos' Theorem is valid for (Φ, F_m). Hence, by the assumption of F_m and Theorem 3.1.3., we have the assertion.

Since, by Lemma 2.2.4. and Theorem 3.1.3., the codition (1) and the condition (3) are equivalent, thus we complete the proof.

REMARK 3.1.5. Since a Banach space F_m be separable (recall the assumptions of this section), for any σ-additive cylinder measure μ on $(F_m, \mathfrak{B})$, by Lemma 2.2.1. and well known results, the measure μ is a regular Borel measure. Therefore, the regularity of the measure μ is not necessarily a essential condition.

THEOREM 3.1.9. *Let* $\Phi=\cap\Phi_n$ *be a* σ*-Hilbert space,* E *be a linear subspace of* Φ*, and suppose that* $E=\cap E_n$ *itself is a complete* σ*-normed space, which satisfies that for each* n*, a Banach space* E_n *has the* $(*)_p$*-conditions* $(1\leqq p\leqq 2)$. *Also, suppose that the inclusion mapping* T *from* E *into* Φ *is continuous. Then, the following conditions are equivalent.*

(1) *For any* m*, there exists* n *such that the operator* T *can be extended to a Hilbert-Schmidt operator from* E_n *into* Φ_m.

(2) *For each* m*, there exists a* E*-quasi-invariant regular finite measure (non-trivial) on* $(\Phi_m, \mathfrak{B})$.

(3) *For any positive definite continuous function* L *on* Φ *with* $L(0)=1$*, there exists a unique probability measure* μ *on* $(E^*, \mathfrak{F})$ *such that*

$$L(x)=\int_{E^*} e^{ix^*(x)}\, d\mu(x^*), \qquad for \quad x\in E.$$

Namely, in our sense, Bochner-Minlos' Theorem is valid for (E, Φ).

PROOF. The assertion can be proved in a quite similar way as before, so we omit it.

2°. $L_p(X, \mu)$ and $l_p(\mathrm{a}_n)$ cases

In this subsection, we shall consider the special cases of the subsection 1°. Then, we can obtain an interesting result.

Notations. Let X be a set, $\sum$ be a σ-algebra in X, and let μ and

ν_n $(n=1, 2, \cdots)$ be σ-finite measures on $(X, \sum)$. Also, suppose that ν_n satisfies the following conditions;

$$\nu_n(B) \leqq \nu_{n+1}(B), \qquad \text{for any} \quad B \in \sum \quad (n=1, 2, \cdots).$$

Then, $L_p(X, \mu)$ be a Banach space, and $\cap L_p(X, \nu_n)$ be a complete σ-normed space with the norms $\|f\|_n$ defined by

$$\|f\|_n = \left(\int_X \left|f(x)\right|^p d\nu_n(x)\right)^{1/p}, \qquad \text{for} \quad f \in \cap L_p(X, \nu_n).$$

Similarly, the sequence $l_p(a_n)$ be a Banach space, and $\bigcap_n l_p(a_{m,n})$ be a complete σ-normed space (Köthe space).

LEMMA 3. 2. 1. *Let $F = \cap L_p(X, \nu_n)$ be a complete separable σ-normed space, and $L_2(X, \mu)$ be a separable Hilbert space, such that $L_2(X, \mu)$ is a linear subspace of F and the inclusion mapping T from $L_2(X, \mu)$ into F is continuous. Let $\mathfrak{F}$ denote the totality of weak Borel sets in F. Then, the following implications* (1)⇒(2)⇒(3)⇒(4) *holds.*

(1) *There exists a $L_2(X, \mu)$-quasi-invariant regular finite measure (non-trivial) on $(F, \mathfrak{F})$.*

(2) *In our sense, Bochner-Minlos' Theorem is valid for $(L_2(X, \mu), F)$.*

(3) *For any n, the conjugate operator T^* from $L_p(X, \nu_n)^*$ into $L_2(X, \mu)^*$ is absolutely summing.*

(4) *For any n, and for any $\{X_j\} \subset X$ which is measurable and pairwise disjoint with $0 < \mu(X_j) < \infty$, $0 < \nu_n(X_j) < \infty$, we have*

$$\sum_{j=1}^{\infty} \frac{\mu(X_j)}{\nu_n(X_j)^{p/2}} < \infty .$$

PROOF.

(1)⇒(2): By Theorem 2. 4. 4., it is obvious.

(2)⇒(3): Since a Hilbert space $L_2(X, \mu)$ has the $(*)_1$-conditions, by Lemma 3. 1. 3., we have the assertion.

(3)⇒(4): The proof of this part can be found in [20].

REMARK 3. 2. 1. In Lemma 3. 2. 1., if $1 \leqq p \leqq 2$, then a Banach space $L_p(X, \nu_n)^*$ has the $(*)_p$-conditions, and therefore, by Theorem 3. 1. 7., the conditions (1)′, (2), (3), and (4)′ are equivalent; where the conditions (1)′ and (4)′ are defined by the following:

(1)′ *For each n, there exists a $L_2(X, \mu)$-quasi-invariant finite measure (non-trivial) on $(L_p(X, \nu_n), \mathfrak{B})$.*

(4)′ *For each n, the operator T from $L_2(X, \mu)$ into $L_p(X, \nu_n)$ is* *a Hilbert-Schmidt operator.*

THEOREM 3.2.1. *Let* $1 \leqq p < \infty$, *let* $\{a_{m,n}\}$ *be a double sequence of positive numbers, which satisfies that* $0 < a_{m,n} \leqq a_{m,n+1} < \infty$ $(m, n = 1, 2, \cdots)$. *Let* $F = \cap l_p(a_{m,n})$ *denote the totality of real number sequences* $\xi = \{\xi_m\}$ *which satisfies the following conditions;*

$$\|\xi\|_n = \left(\sum_{m=1}^{\infty} a_{m,n} |\xi_m|^p\right)^{1/p} < \infty, \qquad (n = 1, 2, \cdots).$$

Then, F forms a complete separable σ-normed space (Köthe space) with respect to the sequence of norms $\|\xi\|_n$ $(n = 1, 2, \cdots)$.

Let $\mathfrak{B}$ *be the σ-algebra in F generated by the totality of Borel cylinders*

$$\left\{\xi \middle| (\xi_1, \xi_2, \cdots, \xi_m) \in B\right\}$$

where B represents an arbitrary Borel sets in m-dimensional space.

Let l_2 *be a subspace of F, and suppose that the natural injection T from* l_2 *into F be continuous. Then, the following conditions are equivalent.*

(1) *There exists a* l_2*-quasi-invariant regular finite measure (non-trivial) on* $(F, \mathfrak{B})$.

(2) *In our sense, Bochner-Minlos' Theorem is valid for* (l_2, F).

(3) *For any n, the conjugate operator* T^* *from* $l_p(a_{m,n})^*$ *into* l_2^* *is absolutely summing.*

(4) *For any n, we have*

$$\sum_{m=1}^{\infty} a_{m,n} < \infty.$$

PROOF.

(4)⇒(1): Let μ_H be a standard Gaussian mersure on l_2, then, by the Kolmogorov's extension theorem, μ_H is a σ-additive measure on R^∞, which is l_2-quasi-invariant (c.f. [2]). Now, if we assume the condition (4), then it is easily seen (c.f. [2]) that for any n, we have

$$\mu_H\Big(l_p(a_{m,n})\Big) = 1.$$

Therefore, we have that $\mu_H(F) = 1$. Thus, restricting μ_H to $(F, \mathfrak{B})$, we obtain a regular finite measure which is l_2-quasi-inveariant, that is, the condition (1) holds.

Since, by Lemma 3.2.1., the implications (1)⇒(2)⇒(3)⇒(4) is valid, thus, we complete the proof.

REMARK 3.2.2. In Theorem 3.2.1., if $1 \leqq p \leqq 2$, the conditions (1), (2), (3), (4) and (5) are equivalent (c.f. [20]); where the condition (5) is the following:

(5) *For each n, the operator T from l_2 into $l_p(a_{m,n})$ is a Hilbert-Schmidt operator.*

However, if $2<p<\infty$, the condition (2) is not necessarily equivalent to the condition (5) (c. f. [20]).

§ 4. Appendix

In this section, we shall consider the rotationally invariant measures on separable Banach spaces. Throughout this section, we assume that Banach spaces are separable with real coefficients.

DEFINITION 4. 1. (c. f. [22])

Let E be a Banach space, $\|x\|_H$ be a continuous Hilbertian norm on E, and let H be the completion of E with respect to the norm $\|x\|_H$. An unitary operator u on H is called a rotation of E, if it satisfies;

(1) *u maps E onto E.*

(2) *u is homeomorphic on E.*

Whole of rotations of E forms a group, which we call the rotation group of E and denote it with $O_H(E)$.

For any $u \in O_H(E)$, its conjugate operator u^* becomes a homeomorphic transformation on E^*. Thus, identifying u^* with u^{-1}, (i. e. identifying u with u^{-1*}), $O_H(E)$ can be regarded as a transformation group on E^*.

Next, let $\mathfrak{F}$ be the totality of weak Borel sets in E^*, and let μ be a measure on $(E^*, \mathfrak{F})$. Then, we say that μ is $O_H(E)$-invariant, if $u^*\mu=\mu$ for all $u \in O_H(E)$.

REMARK 4. 1. Let E be a Banach space, $\|x\|_F$ be a continuous norm on E, and let F be the completion of E with respect to the norm $\|x\|_F$. Then, similarly in the case of Hilbertian norm, whole of rotations of E forms a group, which we call the rotaion group of E and denote it with $O_F(E)$.

DEFINITION 4. 2. (c. f. [15])

Let E be a Banach space, and $1 \leqq p < \infty$.

(1) *We say that a cylinder measure μ on E is of type p, if there exists a positive constant C such that the following inequality holds;*

$$\int_E \left| x^*(x) \right|^p d\mu(x) \leqq C\|x^*\|^p, \qquad \textit{for all} \quad x^* \in E^*.$$

(2) *We say that a probability Radon measure μ on E is of order p, if the following inequality holds;*

$$\int_E \|x\|^p \, d\mu(x) < \infty .$$

EXAMPLE 4.1. (c.f. [15])

Let H be a Hilbert space, then, a standard Gaussian measure μ on H is of type p, for all $1 \leqq p < \infty$.

LEMMA 4.1. (c.f. [15])

Let H be a Hilbert space, E be a Banach space, and suppose that μ_H be a standard Gaussian measure on H. Also, suppose that the operator T from H into E be a Hilbert-Schmidt operator. Then, we have that the measure $T\mu_H$ on E is a Radon measure of order p, $(1 \leqq p < \infty)$.

LEMMA 4.2. *Let E be a Banach space, $\|x\|_H$ be a continuous Hilbertian norm on E, and let H be the completion of E with respect to the norm $\|x\|_H$. Then, if there exists a $O_H(E)$-invariant Radon measure of order p on E^* (except the Dirac measure, $1 \leqq p < \infty$), we have that the natural injection from E into H is p-absolutely summing.*

PROOF. Suppose that μ is a $O_H(E)$-invariant Radon measure (except the Dirac measure) of order p on E^*.

CLAIM (a): *For any $x, y \in E$, if $\|x\|_H = \|y\|_H$, then we have*

$$\int_{E^*} \left| \langle x^*, x \rangle \right|^p d\mu(x^*) = \int_{E^*} \left| \langle x^*, y \rangle \right|^p d\mu(x^*) < \infty .$$

REASON: For any $x, y \in E$, let R be the two-dimensional subspace of E which is generated by $\{x, y\}$. If $\|x\|_H = \|y\|_H$, x is mapped to y by a suitable rotation u_R of R. However, since R is finite dimensional, u_R can be extended to a rotation u of E, i.e. to an unitary operator on H which is homeomorphic on E. Since μ is $O_H(E)$-invariant, we have easily the equality, and also, since μ is of order p, therefore we have the inequality.

CLAIM (b): *There exists a positive constant C_1 such that*

$$C_1 = \int_{E^*} \left| \langle x^*, x \rangle \right|^p d\mu(x^*), \qquad \textit{for all} \quad x \in E \quad \textit{with} \quad \|x\|_H = 1 .$$

REASON: To prove this, by claim (a), it is sufficient to show that C_1 be positive.

If we assume that C_1 is equal to zero. Then, we have

$$(*) \qquad \int_{E^*} \left| \langle x^*, x \rangle \right|^p d\mu(x^*) = 0, \qquad \text{for all} \quad x \in E.$$

Since a Banach space E be separable (recall that the assumption of this section), it is easily seen that there exists a weakly p-summable sequence $\{x_i\}$ in E such that the totality of its linear combinations is dense in E. Let $f(x^*)$ be a real valued function on E^* defined by

$$f(x^*)=\sum_{i=1}^{\infty}\left|\langle x^*, x_i\rangle\right|^p, \qquad \text{for} \quad x^*\in E^*,$$

then, by the assumptions of $\{x_i\}$, it is easily seen that we have

$$(**) \qquad f(x^*)>0 \qquad \text{for all} \quad x^*\in E^*-\{0\}.$$

Thus, by the condition $(*)$, we have

$$\int_{E^*} f(x^*)\, d\mu(x^*)=0.$$

From this and condition $(**)$, it can be shown that μ be the Dirac measure concentrated to $\{0\}$. That is a contradiction.

Now, we shall prove that the natural injection from E into H is p-absolutely summing.

Let $\{x_i\}\subset E$ be weakly p-summable, namely, the following inquality is satisfied ;

$$\sum_{i=1}^{\infty}\left|\langle x^*, x_i\rangle\right|^p<\infty, \qquad \text{for all} \quad x^*\in E^*.$$

Putting

$$p(x^*)=\left(\sum_{i=1}^{\infty}\left|\langle x^*, x_i\rangle\right|^p\right)^{1/p}, \qquad \text{for} \quad x^*\in E^*,$$

obviously $p(x^*)$ is a lower semicontinuous seminorm on E^*.

Since a Banach space E^* be second category, using Gelfand's theorem, $p(x^*)$ is continuous. Therefore, there exists a positive number C_2 such that

$$p(x^*)\leqq C_2\|x^*\|, \qquad \text{for} \quad x^*\in E^*.$$

Next, from claim (b), we have that

$$C_1\|x_i\|_H^p=\int_{E^*}\left|\langle x^*, x_i\rangle\right|^p d\mu(x^*), \qquad (i=1, 2, \cdots).$$

Hence, we have

$$\begin{aligned} C_1\sum_{i=1}^{\infty}\|x_i\|_H^p &= \sum_{i=1}^{\infty}\int_{E^*}\left|\langle x^*, x_i\rangle\right|^p d\mu(x^*) \\ &\leqq C_2^p\int_{E^*}\|x^*\|^p\, d\mu(x^*). \end{aligned}$$

Since the measure μ be of order p, and C_1 be positive, therofore, we have

$$\sum_{i=1}^{\infty}\|x_i\|_H^p<\infty.$$

This shows that the natural injection from E into H is p-absolutely smming.

REMARK 4. 2. Throughout this section, we assume that a Banach space E be separable. However, if E is not necessarily separable, under the assumption that E^* (dual of E) be separable, we can prove Lemma 4. 2. in a similar way.

THEOREM 4. 1. *Let G be a Hilbert space, $\|x\|_H$ be a continuous Hilbertian norm on G, and let H be the completion of G with respect to the norm $\|x\|_H$. Then, the following conditions are equivalent.*

(1) *The natural injection from G into H is a Hilbert-Schmidt operator.*

(2) *There exists a $O_H(G)$-invariant Radon measure of order p on G^* (except the Dirac measure, $1\leqq p<\infty$).*

PROOF.

(1)$\Rightarrow$(2): Let assume the condition (1), then the natural injection from H^* into G^* is a Hilbert-Schmidt operator. Let μ_H be a standard Gaussian measure on H^*, then, by Lemma 4. 1., the measure μ_H on G^* is a Radon measure of order p. Since μ_H is $O_H(G)$-invariant (c. f. [22]), thus we have the assertion.

(2)$\Rightarrow$(1): Let assume the condition (2), then, by Lemma 4. 2., the natural injection from G into H is a p-absolutely summing operator, and therefore, by Theorem 2. 1. 2., it is a Hilbert-Schmidt operator.

THEOREM 4. 2. *Let $1\leqq p\leqq 2$. Let E be a Banach space, which has the $(*)_p$-conditions, $\|x\|_H$ be a continuous Hilbertian norm on E, and let H be the completion of E with respect to the norm $\|x\|_H$. Then, the following conditions are equivalent.*

(1) *The natural injection from E into H is a Hilbert-Schmidt operator.*

(2) *Let μ_H be a standard Gaussian measure on H^*, then the measure μ_H on E^* is a Radon measure of order p.*

(3) *There exists a $O_H(E)$-invariant Radon measure of order p on E^* (execpt the Dirac measure).*

PROOF.

(3)$\Rightarrow$(1): Let assume the condition (3), then, by Lemma 4. 2., the natural injection from E into H is p-absolutely summing, and therefore, by the assumption of E, the conjugate operator from H^* into E^* is p-absolutely summing (c. f. Theorem 2. 1. 4.). Since $1\leqq p\leqq 2$, then, by Theorem 2. 1. 1. and 2. 1. 3., the operator from H^* into E^* is a Hilbert-Schmidt operator, and therefore, the natural injection from E into H is a Hilbert-Schmidt operator.

Next, by Lemma 4. 1., the implications (1)⇒(2)⇒(3) holds. That completes the proof.

REMARK 4. 3. In Theorem 4. 2., if $2<p<\infty$, and E has the $(*)_p$-conditions, then, the implications (1)⇒(2)⇒(3) is valid, however, (3)⇒(1) is not valid. The following example shows this.

EXAMPLE 4. 2. *Let $l_p(a_n)$ be a Banach space with the norm $\|\xi\|$ defined by*

$$\|\xi\| = \left(\sum_{n=1}^{\infty} a_n |\xi_n|^p\right)^{1/p}, \qquad \textit{for} \quad \xi = \{\xi_n\} \in l_p(a_n),$$

where $1 \leqq p < \infty$, and $0 < a_n < \infty$ $(n=1, 2, \cdots)$. Suppose that l_2 be a subspace of $l_p(a_n)$, and the natural injection from l_2 into $l_p(a_n)$ be continuous. Then, the following conditions are equivalent.

(1) *There exists a $O_{l_2}(l_p(a_n))$-invariant Radon measure of order p on $l_p(a_n)$ (except the Dirac measure).*

(2) *The conjugate operator from $l_p(a_n)^*$ into l_2^* is p-absolutely summing.*

(3) $\sum_{n=1}^{\infty} a_n < \infty$.

PROOF.

(1)⇒(2): By the similar arguments for the proof of Lemma 4. 2., it is obvious.

(2)⇒(3): The proof of this part can be found in [20].

(3)⇒(1): Let μ_H be a standard Gaussian measure on l_2. If we assume that the condition (3) holds, then, it is easily seen that the measure μ_H can be extended to a Radon measure on $l_p(a_n)$ (c. f. [2]). Then, by easy calculations, we can show that a Radon measure μ_H on $l_p(a_n)$ is $O_{l_2}(l_p(a_n))$-invariant and of order p. That is the assertion.

REMARK 4. 4.

(1) In the above example, $O_{l_2}(l_p(a_n))$ means that the following: An linear operator u on $l_p(a_n)$ belongs to $O_{l_2}(l_p(a_n))$, if it satisfies;

(a) u is homeomorphic on $l_p(a_n)$.

(b) u is a unitary operator on l_2.

Then, a Radon measure μ on $l_p(a_n)$ is called $O_{l_2}(l_p(a_n))$-invariant, if it satisfies that

$$u\mu = \mu, \qquad \text{for all} \quad u \in O_{l_2}\big(l_p(a_n)\big).$$

(2) In the above example, if $1 \leqq p \leqq 2$, then, these conditions are equivalent to the following condition $(*)$;

$(*)$ *The natural injection from l_2 into $l_p(a_n)$ is a Hilbert-Schmidt* *operator.*

However, if $2<p<\infty$, then, these conditions are not necessarily equivalent to the condition $(*)$ (c. f. [20]).

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