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Parametrices for pseudo-differential equations with double characteristics II.

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1. Introduction.

In our previous paper [11] we constructed parametrices for some double characteristic pseudo-differential operators, whose characteristic sets are closed conic manifolds of codimension 2 in the cotangent space. In this paper we shall show that the condition of Theorem 1. 2 in [11] is necessary and sufficient condition for existence of parametrices.

In order to describe the result more precisely we must recall some notations and hypothese. Let X be a paracompact C^{∞} manifold of dimension n and $T^{*}(X)\backslash 0$ be the cotangential space minus the zero section. $P(x, D)$ is a properly supported classical pseudo-differential operator on X of order m. We denote the principal symbol of P by $p_{m}(x, \xi)\in C^{\infty}(T^{*}(X)\backslash 0)$. For arbitrary $C^{\infty}(T^{*}(X)\backslash 0)$ functions f and g we denote the Poisson bracket of f and g by $\{f, g\}$ and the Hamilton vector field of f by H_{f}. For a nonnegative integer k and a connected closed conic non-involutory submanifold Σ of $T^{*}(X)\backslash 0$ with codim $\Sigma=2$, we use the notations $M^{m,k}(\Sigma, X)$ and $\sigma(P)$ for $P\in M^{m,k}(\Sigma, X)$ when k is odd. These notations are defined in definition 1. 1 of [11].

We consider the following properly supported pseudo-differential operator $L(x, D)$ with double characteristics given by

$$L(x, D)=(P\cdot Q)(x, D)+R(x, D). \tag{1.1}$$

Here $P\in M^{m_{1},k}(\Sigma, X)$, $Q\in M^{m_{2},k}(\Sigma, X)$ and $R\in M^{m_{1}+m_{2}-1,k-1}(\Sigma, X)$.

In the following theorem we write $A\equiv B$ for operators A and B; $\mathscr{D}'(X)\rightarrow\mathscr{D}'(X)$ if $A-B$ is an integral operator with the C^{∞}-kernel. We also write diag $(V)=\{(\rho, \rho)\,;\ \rho\in V\}\subset(T^{*}(X)\backslash 0)\times(T^{*}(X)\backslash 0)$ for any conic subset V of $T^{*}(X)\backslash 0$. Our statement is the following

THEOREM. *Let $L(x, D)$ be a double characteristic pseudo-differential operator defined by* (1. 1), *where k is an odd integer and $\sigma(P)=1$, $\sigma(Q)=-1$. We assume that $(H_{p_{m_{1}}})^{l}q_{m_{2}}(x, \xi)=0$ on Σ for $l=1, \cdots, k-1$ and $(H_{p_{m_{1}}})^{l}r_{m_{1}+m_{2}-1}(x, \xi)=0$ on Σ for $l=1, \cdots, k-2$ where $k>1$. Then the following statements are equivalent.*

i) *For any real number* s *if* $u\in\mathscr{D}'(X)$ *and* $Lu\in H_{\mathrm{loc}}^{s}(X)$ *then* u *belongs to* $H_{\mathrm{loc}}^{s+m_1+m_2-2k/(k+1)}(X)$.

ii) *There exists a properly supported linear operator* F; $\mathscr{D}'(X)\rightarrow\mathscr{D}'(X)$ *which is continuous from* $H_{\mathrm{loc}}^{s}(X)$ *to* $H_{\mathrm{loc}}^{s+m_1+m_2-2k/(k+1)}(X)$ *for all real* s *such that*

$F\cdot L(x, D)\equiv L(x, D)\cdot F\equiv I$ *and* $WF'(F)=diag(T^*(X)\backslash 0)$, *where* I *is the identity in* $\mathscr{D}'(X)$.

iii) *Whatever the positive integer* n, *the function*

$$(H_{p_{m_1}})^{k-1}\left(r_{m_1+m_2-1}+i\lambda\{p_{m_1}, q_{m_2}\}\right)(x, \xi) \tag{1. 2}$$

does not vanishes at any point of Σ, *where* $\lambda=(1-n(k+1))/k$ *or* $-n(k+1)/k$ *and* $H_{p_{m_1}}^{0}$ *is the identity.*

REMARK. 1. *When* $\sigma(P)=-1$ *and* $\sigma(Q)=1$ *then the condition iii) of above Theorem changes the following statement.*

iii)′ *Whatever the positive integer* n, *the function* (1. 2) *does not vanishes on* Σ *when* $\lambda=(1+(n-1)(k+1))/k$ *or* $(n-1)(k+1)/k$.

2. *If* $L(x, D)$ *satisfies the condition iii) of Theorem, then so does the adjoint operator* $L^*(x, D)$.

The present work is closely related to that of Boutet de Monvel and Treves [2], [3], Boutet de Monvel [4] and Sjöstrand [10]. When $k=1$ they showed the hypoellipticity and constructed parametrices of pseudo-differential operators whose characteristic sets are non-involutory manifold. Our theorem is generalization of that in [2] and partial extension of Theorem 8. 6 in [4].

In our previous paper [11] we showed that iii) implies ii). In order to prove the implication i) to iii), Boutet de Monvel-Treves [2], [3] used the Hermite functions and their property essentially. However, when $k>1$ the Hermite functions are not entirely useful. In stead of the Hermite functions we use the functions which are the null solutions of ordinary differential equations with smooth parameters (see Proposition 3. 1.). In the special case these functions are equal to the Hermite functions.

2. Analytic preliminaries.

In this section we shall calculate some special Fourier integral operators, which are pseudo-differential operators with smooth parameters.

We shall define a linear operator A; $C_0^{\infty}(R^{n-1})\rightarrow C^{\infty}(R^n)$ by an integral formula

$$A(x, t, D_x) f = (2\pi)^{-(n-1)} \int e^{i<x,\xi>} \varphi(\xi)^{1/2(k+1)} \times a\left(x, \varphi(\xi)^{1/(k+1)} t, \xi/\varphi(\xi)\right) \hat{f}(\xi)\, d\xi, \tag{2.1}$$

where $\hat{f}$ stands for the Fourier transform of $f \in C_0^\infty(R^{n-1})$ and the element $\varphi(\xi)$ in $C^\infty(R^{n-1})$ is real non-zero function such that $\varphi(\xi) = |\xi|$ if $|\xi| \geq 1$ and $\varphi(\xi) \geq |\xi|$. Moreover $a(x, t, \omega)$ is an element of $C^\infty(R^n \times \Delta)$ and belongs to the space $\mathscr{S}(R)$ of Schwarz as a function of t, where $\Delta = \{\omega \in R^{n-1};\ |\omega| < 2\}$. We remark that $a(x, \varphi(\xi)^{1/(k+1)} t, \xi/\varphi(\xi))$ belongs to $S^0_{1,1/(k+1)}(R^n \times R^{n-1})$ which is the space of the symbols introduced in [9]. We shall investigate the properties of the operators given by (2. 1).

LEMMA 1. 1. *If $a(x, t, \omega)$ has a fixed compact support with respect to x, then for an arbitrary positive integer N there exists a constant C_N such that for all $(\eta, \xi) \in R^{n-1} \times R^{n-1}$*

$$\varphi(\xi)^{1/(k+1)} \int |\hat{a}(\eta, \varphi(\xi)^{1/(k+1)} t, \xi/\varphi(\xi))|^2 dt \leq C_N (1 + |\eta|)^{-2N}, \tag{2.2}$$

where $\hat{a}$ means the Fourier transform with respect to x.

PROOF. Put $s = \varphi(\xi)^{1/(k+1)} t$, then we may show that

$$\int |\eta^\alpha \hat{a}(\eta, s, \xi/\varphi(\xi))|^2 ds \leq C,$$

where $|\alpha| = N$ and $|\eta| \geq 1$. Since a has a fixed compact support with respect to x, by Schwarz's inequality we see that

$$|\eta^\alpha \hat{a}(\eta, s, \xi/\varphi(\xi))|^2 = \left| \int e^{-i<x,\xi>} D_x^\alpha\, a(x, s, \xi/\varphi(\xi))\, dx \right|^2 \leq C \int |D_x^\alpha\, a(x, s, \xi/\varphi(\xi))|^2 dx.$$

Since $D_x^\alpha\, a(x, s, \xi/\varphi(\xi))$ belongs to $C_0^\infty(R^{n-1})$ uniformly with respect to s, ξ, we get the inequality (2. 2). This completes the proof.

By this lemma we can get the following

PROPOSITION 2. 2. *For any real number s the operator $A(x, t, D_x)$ defined in* (2. 1) *is a continuous operator from $H^s_c(R^{n-1})$ to $H^s_{\mathrm{loc}}(R^n)$.*

PROOF. For arbitrary $h(x, t) \in C_0^\infty(R^n)$ and $u \in C_0^\infty(R^{n-1})$ we shall show that

$$\| hAu \|_s \leq C_s \| u \|_s. \tag{2.3}$$

First we shall show (2. 3) when $s = 0$. Let $f(x)$ and $g(t)$ be element of

$C_0^\infty(R^{n-1})$ and $C_0^\infty(R)$ respectively such that $h(x,t)=f(x)g(t)h(x,t)$. Since $\|hAu\|_0=\|fghAu\|_0\le C\|h(fA)u\|_0$, we may assume that $a(x,t,\xi/\varphi(\xi))$ has a fixed compact support with respect to x. For an arbitrary element $v(x,t)$ of $L^2(R^n)$ we get

$$\begin{aligned}|\langle Au, v\rangle| &= \Big|(2\pi)^{-(n-1)}\iiint v(x,t)\,\varphi(\xi)^{1/2(k+1)}\times \\ &\qquad a(x,\varphi(\xi)^{1/(k+1)}t,\xi/\varphi(\xi))\,\hat{u}(\xi)\,e^{i\langle x,\xi\rangle}\,dx\,dt\,d\xi\Big| \\ &= \Big|(2\pi)^{-2(n-1)}\iiint \hat{v}(-\eta,t)\,\varphi(\xi)^{1/2(k+1)}\times \\ &\qquad \hat{a}(\eta-\xi,\varphi(\xi)^{1/(k+1)}t,\xi/\varphi(\xi))\,\hat{u}(\xi)d\eta\,d\xi\,dt\Big|.\end{aligned}$$

From Schwarz's inequality we have

$$\begin{aligned}|\langle Au, v\rangle|^2 &\le (2\pi)^{-4(n-1)}\Big(\iiint|\hat{v}(-\eta,t)|^2(1+|\eta-\xi|)^{-n} \\ &\quad d\eta\,d\xi\,dt\Big)\Big(\iiint(1+|\eta-\xi|)^n\varphi(\xi)^{1/(k+1)}|\hat{a}(\eta-\xi, \\ &\quad \varphi(\xi)^{1/(k+1)}t,\xi/\varphi(\xi))|^2\,|\hat{u}(\xi)|^2\,d\eta\,d\xi\,dt\Big).\end{aligned}\tag{2.4}$$

By Lemma 1.1 the second term in the right hand side of (2.4) is estimated by

$$C\iint(1+|\eta-\xi|)^{-n}|\hat{u}(\xi)|^2 d\xi\,d\eta\,.$$

Since $\int(1+|\eta-\xi|)^{-n}d\eta$ and $\int(1+|\eta-\xi|)^{-n}d\xi$ is constant, we have (2.3) when $s=0$. If we use (2.3) when $s=0$, we easily show that $\|D_x^\alpha D_t^\beta(hAu)\|_0\le C\|u\|_{|\alpha+\beta|}$, which implies that (2.3) holds for all non-negative integer s. We remark that since by iii) of Theorem 1.4.1 in [9] A is a continuous map from $\mathcal{E}'(R^{n-1})$ to $\mathcal{D}'(R^n)$, Au is well-defined for $u\in H_c^s(R^{n-1})$. Since $C_0^\infty(R^{n-1})$ is a dense set of $H^s(R^{n-1})$, we can regard the operator hA as a continuous operator from $H^s(R^{n-1})$ to $H^s(R^n)$ when s non-negative integer. Therefore by standard interpolation between Sobolev spaces we get (2.3) for all $u\in H^s(R^{n-1})$ and any $s\ge 0$. Let A^* be the dual operator of A via the sesquilinear from $\int f\bar{g}\,dx\,dt$. By same argument we see that A^* is continuous operator from $H_c^s(R^n)$ to $H_{\mathrm{loc}}^s(R^{n-1})$ when $s\ge 0$. From the duality between $H_c^s(R^n)$ and $H_{\mathrm{loc}}^{-s}(R^n)$ we can show that A is continuous from $H_c^s(R^{n-1})$ to $H_{\mathrm{loc}}^s(R^n)$ for any real s. This completes the proof.

Using the above proposition, we can show the following

LEMMA 1.3. *Let $p(x,\xi)$ be an element of $S_{1,0}^m(R^{n-1}\times R^{n-1})$ with compact* *support with respect to* x. *If* $a(x, t, \omega)$ *has a fixed compact support with respect to* x, *then we can write*

$$(2.5) \qquad P(x, D_x) \cdot A(x, t, D_x) u = \tilde{A}(x, t, D_x) u + R_1 u,$$

$$(2.6) \qquad A(x, t, D_x) \cdot P(x, D_x) u = \tilde{A}(x, t, D_x) u + R_2 u.$$

Here R_j $(j=1, 2)$ *is a continuous linear operator from* $H_c^s(R^{n-1})$ *to* $H_{\mathrm{loc}}^{s-m+1}(R^n)$ *and*

$$\tilde{A}(x, t, D_x) u = (2\pi)^{-(n-1)} \int e^{i<x,\xi>} \varphi(\xi)^{1/2(k+1)} \times$$
$$a\Big(x, \varphi(\xi)^{1/(k+1)} t, \xi/\varphi(\xi)\Big) p(x, \xi) \hat{u}(\xi) d\xi .$$

PROOF. Since $P(x, D_x)$ is a continuous operator from $C_0^\infty(R^{n-1})$ to $C_0^\infty(R^{n-1})$, we have

$$P(x, D_x) A(x, t, D_x) u(x, t) = (2\pi)^{-(n-1)} \int e^{i<x,\xi>} \tilde{a}(x, t, \xi)$$
$$\varphi(\xi)^{1/2(k+1)} \hat{u}(\xi) d\xi ,$$

where $\tilde{a}(x, t, \xi) = e^{-i<x,\xi>} P(x, D_x)(e^{i<x,\xi>} a(x, \varphi(\xi)^{1/(k+1)} t, \xi/\varphi(\xi))$. From Theorem 2.6 in [8], we see that

$$b(x, t, \xi) = \tilde{a}(x, t, \xi) - \sum_{|\alpha|<N} p^{(\alpha)}(x, \xi) D_x^\alpha a\Big(x, \varphi(\xi)^{1/(k+1)} t, \xi/\varphi(\xi)\Big)/\alpha!$$

belongs to $S_{1,1/(k+1)}^{m+n-N}(R^n \times R^{n-1})$. By the proof of Proposition 2. 2 the operator defined by the symbol $\varphi(\xi)^{1/2(k+1)} p^{(\alpha)}(x, \xi) D_x^\alpha a(x, \varphi(\xi)^{1/(k+1)} t, \xi/\varphi(\xi))/\alpha!$ is a continuous operator from $H_c^s(R^{n-1})$ to $H^{s+|\alpha|-m}(R^n)$. For any s and any positive integer M if we take a sufficiently large N, then the operator defined by $\varphi(\xi)^{1/2(k+1)} b(x, t, \xi)$ is continuous from $H_c^s(R^{n-1})$ to $C^M(R^n)$. This shows that R_1 has a desired property. By a similar way we can prove the equality (2. 6). This completes the proof.

PROPOSITION 2. 4. *Let* $A(x, t, D_x)$ *be an operator defined by* (2. 1). *Then the operator* A^*A *is a pseudo-differential operator of order* 0 *and type* 1, 0. *The principal symbol of* $(A^*A)(x, D_x)$ *is given by*

$$(2.7) \qquad \int |a(x, t, \xi/\varphi(\xi))|^2 dt .$$

PROOF. To prove this porposition we shall use the argument of vector valued pseudo-differential operators (see Section 3 in [11]). The operator

$$\alpha(x, \xi) z = \varphi(\xi)^{1/2(k-1)} a\Big(x, \varphi(\xi)^{1/(k+1)} t, \xi/\varphi(\xi)\Big) z$$

belongs to $S^0(R^{n-1}; C, L^2(R))$. Therefore the adjoint operator A^* belongs to $L^0(R^{n-1};\ L^2(R), C)$ and its principal symbol is the operator

$$a_0^*(x,\xi)u(t)=\varphi(\xi)^{1/2(k+1)}\int \bar{a}\left(x,\varphi(\xi)^{1/(k+1)}t,\xi/\varphi(\xi)\right)u(t)\,dt\,.$$

This implies that the operator A^*A belongs to $L^0(R^{n-1};\ C, C)$ and its principal symbol is equal to (2.7). This completes the proof.

3. The proof of Theorem

Since the implication of iii) to ii) was proved in [11], in this section we shall prove the implication ii) to i) and i) to iii).

It is easy to prove the implication ii) to i). Let $k(x,y)$ be the distribution kernel of $F\cdot L$. Since F and L are properly supported operators, both projections π_x, π_y; $suppk\to X$ are proper. We may show that if $u\in\mathscr{D}'(X)$ and $Lu\in H^s_{loc}(X)$ then $hu\in H^{s+m_1+m_2-2k/(k-1)}(X)$ for all $h\in C_0^\infty(X)$. Let $\tilde{h}$ be an element of $C_0^\infty(X)$ such that $\tilde{h}=1$ on $supph\cup\pi_y\cdot\pi_x^{-1}(supph)$. Then we have $h\tilde{h}=h$ and $hFLu=hFL\tilde{h}u$. Since $FL=I+K$ where K is an integral operator with C^∞-kernel, we see that $hFLu=hu+hK\tilde{h}u$. The left hand side belongs to $H^{s+m_1+m_2-2k/(k+1)}(X)$. It implies that $hu\in H^{s+m_1+m_2-2k/(k+1)}(X)$.

In order to prove the implication i) to ii) we shall prepare the following two statements.

PROPOSITION 3.1. *Let k be an odd integer and $L_\lambda(x,t,\omega,D_t)$ be an ordinary differential operator with parameter (x,ω) given by*

$$L_\lambda(x,t,\omega,D_t)=(D_t-ia(x,\omega)t^k)(D_t-ib(x,\omega)t^k)\\ +\lambda(a-b)(x,\omega)t^{k-1}\,.$$

Here $a(x,\omega)$ and $b(x,\omega)$ are elements of $C^\infty(R^{n-1}\times\Delta)$ and $Rea(x,\omega)>0$, $Reb(x,\omega)<0$. If $\lambda=-1-(k+1)n$ for a positive integer n, then there exists a non-trivial solution $\mathscr{H}_n^+(x,t,\omega)$ such that

i) $L_\lambda(x,t,\omega,D_t)\mathscr{H}_n^+(x,t,\omega)=0$.

ii) *$\mathscr{H}_n^+(x,t,\omega)$ is an element of $C^\infty(R^n\times\Delta)$ and belongs to $\mathscr{S}(R)$ as a function of t.*

iii) *$\mathscr{H}_n^+(x,t,\omega)$ is even and real analytic as a function of t.*

If $\lambda=-(k+1)n$ for some positive integer n, then there exists a non-trivial solution $\mathscr{H}_n^-(x,t,\omega)$ such that i) and ii) holds and $\mathscr{H}_n^-(x,t,\omega)$ is odd and real analytic as a function of t.

PROOF. First we shall grantee the existence of the nontrivial null solution of L_λ when $\lambda=-1-(k+1)n$ or $-(k+1)n$ for any positive integer n.

We shall change variable; $s=t^{k+1}/(k+1)=\varphi(t)$. Note that s varies only in the non-negative half-line $\bar{R}_+=\{s\in R;\ s\geq 0\}$. We seek functions $w_n^+(s)$ and $w_n^-(s)$ such that $L_\lambda(w_n^+\cdot\varphi)=0$ and $L_\mu(t(w_n^-\cdot\varphi))=0$ in the whole real line when $\lambda=1-(k+1)n$ and $\mu=-(k+1)n$. By the easy computation (see [5]), $L_\lambda(w_n^+\cdot\varphi)=0$ and $L_\mu(t(w_n^-\cdot\varphi))=0$ are equivalent to $(L_\lambda^+ w_n^+)(s)=0$ and $(L_\mu^- w_n^-)(s)=0$ respectively, where

$$L_\lambda^+(x, s, \omega, D_s) = \big(D_s - ia(x,\omega)\big)\big(D_s - ib(x,\omega)\big)s + i\frac{k+2}{k+1}D_s + \left(a + \frac{b+\lambda(a-b)}{k+1}\right)(x,\omega),$$

$$L_\mu^-(x, s, \omega, D_s) = \big(D_s - ia(x,\omega)\big)\big(D_s - ib(x,\omega)\big)s + i\frac{k}{k+1}D_s + \left(\frac{ka+\mu(a-b)}{k+1}\right)(x,\omega).$$

We shall apply the following theorem to L_λ^+ and L_μ^-.

THEOREM (*Theorem* 2.3 *in Chapter* 3 *of* [1]). *Let* $L(s, D_s)$ *be an ordinary differential operator given by*

$$L(s, D_s)u(s) = P^2(D_s)(su) + P^1(D_s)u.$$

Here $P^1(D_s)=p_1^1 D_s + p_0^1$, $p_j^1\in C(j=1,2)$ *and* $P^2(D_s)=D_s^2+p_1^2 D_s+p_0^2$, $p_j^2\in C(j=0,1)$ *such that the polynomial* $P^2(\tau)$ *has the roots* τ_+ *and* τ_- *with* $Im\ \tau_+>0$ *and* $Im\ \tau_-<0$. *If there exists a positive integer* n *such that* $P^1(\tau_+)=in(\tau_+-\tau_-)$, *then the dimension of KerL and CokerL are* 1 *as an operator from* $W_1^{p+2}(R_+)$ *to* $H^p(R_+)$. *Here* $p>Im p_1^1-3/2$ *and* $W_1^{p+2}(R_+)=\{u\in H^{p+1}(R_+);\ su\in H^{p+2}(R_+)\}$.

Using this Theorem, we continue the proof of Proposition 3.1. By above Theorem it implies that there exists a non-trivial solution $w_n^+(s)$ of $L_\lambda^+ u=0$ when $\lambda=1-(k+1)n$ for all positive integer n and there also exists a non-zero null solution w_n^- of L_μ^- when $\mu=-(k+1)n$ for any positive integer n. By Sobolev's lemma and trace theorem it implies that $w_n^+(s)$ and $w_n^-(s)$ belong to $C^2(\bar{R}_+)$ when $p>3$. If we put $W_n^+(t)=(w_n^+\cdot\varphi)(t)$ and $W_n^-(t)=t(w_n^-\cdot\varphi)(t)$, then they belong to $C^2(R)$ and $L_{1-(k+1)n}W_n^+=L_{-(k+1)n}W_n^-=0$ in the whole real line.

It is clear that $W_n^+(x,t,\omega)$ and $W_n^-(x,t,\omega)$ belong to $L^2(R)$ as a function of t. Therefore by Theorem 2.2 in [6] it implies that $W_n^+(x,t,\omega)$ and $W_n^-(x,t,\omega)$ belongs to $\mathscr{S}(R)$ as a function of t. We shall show that $W_n^+(x,0,\omega)$ and $\partial W_n^-/\partial t(x,0,\omega)$ are non-zero for every $(x,\omega)\in R^{n-1}\times\Delta$. Suppose $W_n^+(x,0,\omega)=0$ for some $(x,\omega)\in R^{n-1}\times\Delta$. Since $W_n^+(x,t,\omega)$ is an even function as a function of t, $\partial W_n^+/\partial t(x,t,\omega)=0$. By the uniquness theorem of an ordinary differential operator it implies that $W_n^+(x, t, \omega) \equiv 0$ in some neighbourhood of the original point of R. Since the coefficients of $L_{1-(k+1)n}(x, t, \omega, D_t)$ are real analytic as a function of t, the null solution of $L_{1-(k+1)n}$ is also real analytic (see Theorem 7.5.1 in [7]). By the unique continuation theorem of real analytic function with one variable it implies that $W_n^+(x, t, \omega) \equiv 0$ in the whole real line. This contradicts non-triviality of $W_n^+(x, t, \omega)$. By a similar argument we have $\partial W_n^-/\partial t\,(x, 0, \omega) \neq 0$ for all $(x, \omega) \in R^{n-1} \times \Delta$.

Put

$$\mathscr{H}_n^+(x, t, \omega) = W_n^+(x, t, \omega)/W_n^+(x, 0, \omega)\,,$$

$$\mathscr{H}_n^-(x, t, \omega) = W_n^-(x, t, \omega)/(\partial W_n^-/\partial t)(x, 0, \omega)\,.$$

Then the statements i) and iii) are clear. we shall only prove that $\mathscr{H}_n^+(x, t, \omega)$ and $\mathscr{H}_n^-(x, t, \omega)$ belongs to $C^\infty(R^n \times \Delta)$. Since the initial condition at $t=0$ is independent of a parameter (x, ω), by a well-known theorem of an ordinary differential operator theory it implies that $\mathscr{H}_n^+(x, t, \omega)$ and $\mathscr{H}_n^-(x, t, \omega)$ belong to $C^\infty(R^n \times \Delta)$. This completes the proof.

REMARK. 1. *If $a(x, \omega)=1$, $b(x, \omega)=-1$ and $k=1$, then $\mathscr{H}_n^+(t)$ and $\mathscr{H}_n^-(t)$ are $(2n-2)$-th and $(2n-1)$-th Hermite function respectively, where j-th Hermite function $H_j(t)$ is defined by*

$$H_j(t) = (2^j j!)^{-1/2}(\partial/\partial t - t)^j \exp(-t^2/2)\,.$$

2. *The following two statements are easily verfied by the definition of $\mathscr{H}_n^+(x, t, \omega)$ and $\mathscr{H}_n^-(x, t, \omega)$.*

i) *For any sequences $(n_1, \cdots, n_N)$ and $(m_1, \cdots, m_M)$ such that n_i, m_j are positive integers with $n_i \neq n_j$ and $m_i \neq m_j$ if $i \neq j$, the function $\mathscr{H}_{n_i}^+(x, t, \omega)$, $\mathscr{H}_{m_j}^-(x, t, \omega)$ $(i=1, \cdots, N,\ j=1, \cdots, M)$ are linear independent, i.e., if $\sum_i c_i \mathscr{H}_{n_i}^+ + \sum_j d_j \mathscr{H}_{m_j}^- = 0$ then $c_i = d_j = 0$ $(i=1, \cdots, N,\ j=1, \cdots, M)$.*

ii) *If $b(x, \omega) = -a(x, \omega)$ then we have*

$$\int \mathscr{H}_n^+(x, t, \omega)\,\mathscr{H}_m^-(x, t, \omega)\,t^{k-1}\,dt = 0 \quad \text{for all } n \text{ and } m\,,$$

$$\int \mathscr{H}_n^\pm(x, t, \omega)\,\mathscr{H}_m^\pm(x, t, \omega)\,t^{k-1}\,dt = 0 \quad \text{if } n \neq m\,.$$

LEMMA 3.2. *Let $P(x, D)$ be a classical pseudo-differential operator with the principal symbol $p_m(x, \xi)$. We assume $p_m(x_0, \xi^0)=0$ for some point of $R^n \times R^n \backslash \{0\}$. Let Γ_0 be an arbitrary open conic neighbourhood of (x_0, ξ^0). Then for any $\varepsilon > 0$ and all $s \in R$ there exists $f_s \in H^{s-\varepsilon}(R^n)$ such that $P(x, D) f_s \in H^{s-m}(R^n)$, $f_s \notin H^s(R^n)$ and $Wf(f_s) \subset \Gamma_0$.*

PROOF. Let F be the set of all $u\in\mathscr{D}'(R^n)$ such that $u\in H^{s-\varepsilon}(R^n)$, $Pu\in H^{s-m}(R^n)$ and $WF(u)\subset\Gamma_0$. We introduce in F the weakest topology making the following map continuous;

$$F\hookrightarrow H^{s-\varepsilon}(R^n)\,,\quad F\ni u\to Pu\in H^{s-m}(R^n)\ \text{and}$$
$$F\ni u\to Au\in C^\infty(R^n)\,,$$

where A is a properly supported pseudo-differential operator with $WF(A)\cap\Gamma_0=\phi$. If is clear F is a Frèchet space for we need only consider countable many choices of A and the completeness is obvious. Suppose that $F\subset H^s(R^n)$. Then by the closed graph theorem the inclusion map $F\hookrightarrow H^s(R^n)$ is continuous. There exists properly supported pseudo-differential operators $A_1,\cdots,A_N$ with $WF(A_j)\cap\Gamma_0=\phi$ $(j=1,\cdots,N)$ such that for all $u\in F$

$$||u||_s^2\leq C\Big(||Pu||_{s-m}^2+||u||_{s-\varepsilon}^2+\sum_j\Big(\sup_{K_j}|\sum_{|\alpha|\leq N_j}D^\alpha(A_ju)|\Big)^2\Big)\,. \tag{3.1}$$

Here K_j is a compact set of R^n and N_j is a positive integer. Let K be a compact set of R^n such that X_0 is an interior point of K and $K\times\{\xi^0\}\subset\Gamma_0$. By the Sobolev's lemma and (3.1) it implies for all $u\in C_0^\infty(K)$

$$||u||_s^2\leq C\Big(||Pu||_{s-m}^2+||u||_{s-\varepsilon}^2+\sum_j||A_ju||_{N_j+[n/2]+1}^2\Big)\,. \tag{3.2}$$

Set $u_t=e^{i<x,t\xi^0>}u$. Then by the definition of pseudo-differential operator (see Theorem 2.6 in [8]) we have

$$t^{-l}e^{-i<x,t\xi^0>}Qu_t\to q_l(x,\xi^0)\,u\,,$$

when $t\to\infty$. Here Q is a pseudo-differential operator of order l with the principal symbol $q_l(x,\xi)$. Since $WF(A_j)\cap(K\times\{\xi^0\})=\phi$,

$$t^N||e^{-i<x,t\xi^0>}\Lambda^{N_j+[n/2]+1}A_ju_t||_0\to 0\qquad\text{as } t\to\infty\,,$$

where Λ is the pseudo-differential operator with the symbol $(1+|\xi|^2)^{1/2}$ and N is an arbitrary integer. By this fact and (3.2) it implies that for all $u\in C_0^\infty(K)$

$$|\xi^0|^{2m}\int|u|^2dx\leq C\int|p_m(x,\xi^0)\,u|^2\,dx\,.$$

This implies that $|\xi^0|^m\leq C'|p_m(x,\xi^0)|$ for an interior point of K, since u is arbitrary. This contradicts to non-ellipticity of (x_0,ξ^0). This completes the proof.

We shall start the proof of the implication i) to iii). We shall consider the following statement instead of i).

i)′ Whatever $s\in R$ and the point $\rho\in T^*(X)\backslash 0$, there is a conic open neighbourhood Γ of ρ such that for all $u \in \mathscr{D}'(X)$ if $Pu \in H^s_{\mathrm{loc}}(X)$ and $WF(u) \subset \Gamma$ then $u \in H^{s+m_1+m_2-2k/(k+1)}_{\mathrm{loc}}(X)$.

Since it is clear i)$\to$i)$'$, we shall prove the implication i)$'$ to iii). Before embarking on the proof, let us observe that all the statements are microlocal. We shall therefore be reasoning in a conic open subset Γ of $T^*(X)\backslash 0$ which intersects $\sum$ (in the complement of $\sum$ L is elliptic, and there the various statements are well known). Microlocalizing the pseudo-differential operator L (see Section 2 in [11]), we may consider the second order pseudo-differential operator $M(x, t, D_x, D_t)$ given by

$$\begin{aligned} M(x, t, D_x, D_t) = & \left(D_t - ia(x, t, D_x)\, t^k\right)\left(D_t - ib(x, t, D_x)\, t^k\right) \\ & + c(x, t, D_x, D_t)\, t^{k-1} + t^k A(x, t, D_x, D_t) \\ & + B(x, t, D_x, D_t)\, D_t + C(x, t, D_x, D_t)\,. \end{aligned}$$

Here $A \in L^1(R^n)$ and $B,\ C \in L^0(R^n)$. Moreover $a(x, t, D_x)$, $b(x, t, D_x)$ are pseudo-differential operators defined by the symbol $a(x, t, \xi)$, $b(x, t, \xi)$ respectively, where a, b are elements of $S^1_{1,0}(R^n \times R^{n-1})$ and positively homogeneous of degree 1 when $|\xi| \geq 1$ and $Rea > 0$, $Reb < 0$. $c(x, t, D_x, D_t)$ is a pseudo-differential opetator with symbol $c(x, t, \xi, \tau)$ which belongs to $S^1_{1,0}(R^n \times R^n)$ and is positively homogeneous of degree 1 when $|(\xi, \tau)| \geq 1$. Futhermore we may assume that $a(x, t, \xi)$, $b(x, t, \xi)$ and $c(x, t, \xi, \tau)$ have a compact support with respect to x.

Suppose that there exist $\rho \in \sum$ and the positive integer n such that

$$(H_{p_{m_1}})^{k-1}\left(r_{m_1+m_2-1} + i(1-(k+1)\, n)/k\, \{p_{m_1}, q_{m_2}\}\right)(\rho) = 0\,.$$

Then by the proof of Proposition 3.4 in [11] there exists $(x_0, \xi^0) \in T^*(R^{n-1})$ such that

$$c(x_0, 0, \xi^0, 0) - \left(1-(k+1)\, n\right)(a-b)(x_0, 0, \xi^0) = 0\,. \tag{3.3}$$

We may assume $|\xi^0| \geq 1$. Let $\mathscr{H}^+_n(x, t, \omega)$ be the function given by Proposition 3.1 and $H^+_n(x, t, D_x)$ be the operator defined by (1.1) with the symbol $\varphi(\xi)^{1/2(k+1)}\, h(x)\ \mathscr{H}^+_n(x, \varphi(\xi)^{1/(k+1)}\, t, \xi/\varphi(\xi))$, where $h(x) \in C^\infty_0(R^{n-1})$ and $h(x) = 1$ in the neghbourhood of x_0. Then by Proposition 2.2 we have

$$\begin{aligned} M(x, t, D_x, D_t)\, H^+_n(x, t, D_x) = & \left(D_t - ia(x, 0, D_x)\, t^k\right) \\ & \times\left(D_t - ib(x, 0, D_x)\, t^k\right) H^+_n \\ & + c(x, 0, D_x, 0)\, \tilde{H}^+_n\, \varphi(D_x)^{2/(k+1)} + R_1\,, \end{aligned} \tag{3.4}$$

where $\tilde{H}^+_n(x, t, D_x)$ is the operator defined by a integral from (1.1) with the symbol $h(x)\varphi(\xi)^{1/2(k+1)}(\varphi(\xi)^{1/(k+1)}t)^{k-1}\mathscr{H}_n^+(x, \varphi(\xi)^{1/(k+1)}t, \xi/\varphi(\xi))$ and R_1 is a continuous operator from $H_c^s(R^{n-1})$ to $H_{\mathrm{loc}}^{s-1/(k+1)}(R^n)$ for all $s\in R$. By Lemma 2.3 and (3.4) it implies that

$$M(x, t, D_x, D_t)\, H_n^+ = \tilde{H}_n^+ \Lambda(x, D_x) + R_2, \tag{3.5}$$

where Λ is a pseudo-differential operator with the symbol $(c(x, 0, \xi/\varphi(\xi), 0) - (1-(k+1)n)(a-b)(x, 0, \xi/\varphi(\xi)))\varphi(\xi)^{2/(k+1)}$ and R_2 is a continuous operator from $H_c^s(R^{n-1})$ to $H_{\mathrm{loc}}^{s-1/(k+1)}(R^n)$. From (3.3) $\Lambda(x, D_x)$ is not elliptic at (x_0, ξ^0). Therefore we can apply Lemma 1.2 to Λ as $m=2/(k+1)$, $\varepsilon=1/(k+1)$ and $\Gamma_0=\pi(\Gamma)$ where π is the projection $T^*(R^n)\to T^*(R^{n-1})$ along (t, τ). Let f_s be the distribution which satisfies the conditions of Lemma 3.2 and *supp* $(f_s)\subset supp\,(h)$. Since $WF(H_n^+ f_s)\subset\{(x, 0, \xi, 0); (x, \xi)\in WF(f_s)\}$, we have $WF(H_n^+ f_s)\subset\Gamma$. From (3.5) we see that $M(x, t, D_x, D_t)\, H_n^+ f_s\in H_{\mathrm{loc}}^{s-2/(k+1)}(R^n)$. Finally we observe that $H_n^+ f_s \not\ni H_{\mathrm{loc}}^s(R^n)$, otherwise since by Proposition 2.4 $(H_n^+)^* H_n^+$ is elliptic in *supp* (h), we should have $f_s\in H_{\mathrm{loc}}^s(R^{n-1})$. This contradicts the statement i)'. We complete the proof of Theorem.

SUPPLEMENT. In the previous paper [11], we announced that the proof of Proposition 2.1 of [11] is verified in this paper. However, that is proved in [12].

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