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A remark on a closed hypersurface with constant second mean curvature in a Riemann space

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Introduction. Y. Katsurada ([6][1], [5]) proved the following two theorems:

THEOREM A. *Let V^m be a closed orientable hypersurface in an Einstein space which admits a conformal Killing vector field ξ^i. If*

(i) *H_1 is constant,*

(ii) *$N_i\xi^i$ has fixed sign on V^m,*

then every point of V^m is umbilic, where H_1 and N_i denote the first mean curvature of V^m and the covariant component of the unit normal vector to V^m respectively.

THEOREM B. *Let V^m be a closed orientable hypersurface in a Riemann space of constant curvature which admits a conformal Killing vector field ξ^i. If*

(i) *H_ν is constant for a fixed ν $(2\leqq\nu\leqq m-1)$,*

(ii) *$k_1, k_2, \cdots, k_m$ are positive at each point on V^m,*

(iii) *$N_i\xi^i$ has fixed sign on V^m,*

then every point of V^m is umbilic, where k_α $(\alpha=1, 2, \cdots, m)$ and H_ν denote the principal curvature and the ν-th mean curvature of V^m respectively.

The present author [8] proved

THEOREM C. *Let V^m be a closed orientable hypersurface in a Riemann space which admits a conformal Killing vector field ξ^i. If*

(i) *H_2 is constant,*

(ii) *$k_1, k_2, \cdots, k_m$ are positive at each point on V^m,*

(iii) *$C^\alpha{}_{\beta;\alpha}$[2] $=0$ on V^m,*

(iv) *$N_i\xi^i$ has fixed sign on V^m,*

then every point of V^m is umbilic.

It is one of the interesting problems for us to find the conditions that

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1) Numbers in brackets refer to the references at the end of the paper.

2) $C_{\alpha\beta}$ are defined by $b_\gamma^\gamma b_{\alpha\beta}-b_\alpha{}^\gamma b_{\gamma\beta}$, where, $b_{\alpha\beta}$ and $g^{\alpha\beta}$ denoting the covariant component of the second fundamental tensor and the contravariant component of the metric tensor of V^m respectively, $b_\gamma^\gamma = b_{\alpha\beta}g^{\alpha\beta}$ and $b_\alpha{}^\gamma = b_{\alpha\beta}g^{\beta\gamma}$. And $C^\alpha{}_{\beta;\alpha} = C_{\alpha\beta;\gamma}g^{\alpha\gamma}$, where the symbol ";" means the covariant derivative.

an umbilical hypersurface in a Riemann space is isometric to a sphere. On this problem, the following theorems were proved by Y. Katsurada [7]:

THEOREM A–1. *Let* V^m *be a closed orientable hypersurface in an Einstein space* R^{m+1} *which admits a conformal Killing vector field* ξ^i, *i.e.*, $\xi_{i;j}+\xi_{j;i}=2\Phi G_{ij}$[3]. *If*

(i) H_1 *is constant*,

(ii) $N_i\Phi^i$ *has fixed sign on* V^m *and is not constant along* V^m,

then V^m *is isometric to a sphere, where* Φ^i *denote* $G^{ij}\Phi_{;j}$.

THEOREM A–2. *Let* V^m *be a closed orientable hypersurface in an Einstein space which admits a conformal Killing vector field* ξ^i. *If*

(i) H_1 *is constant*,

(ii) $N_i\xi^i$ *has fixed sign on* V^m,

(iii) Φ *is not constant along* V^m,

then V^m *is isometric to a sphere.*

THEOREM B–1. *Let* V^m *be a closed orientable hypersurface in a constant curvature space which admits a conformal Killing vector field* ξ^i. *If*

(i) H_ν *is constant for a fixed* ν $(2\leqq\nu\leqq m-1)$,

(ii) $k_1, k_2, \cdots, k_m$ *are positive at each point on* V^m,

(iii) $N_i\Phi^i$ *has fixed sign on* V^m *and is not constant along* V^m,

then V^m *is isometric to a sphere.*

THEOREM B–2. *Let* V^m *be a closed orientable hypersurface in a constant curvature space which admits a conformal Killing vector field* ξ^i. *If*

(i) H_ν *is constant for a fixed* ν $(2\leqq\nu\leqq m-1)$,

(ii) $k_1, k_2, \cdots, k_m$ *are positive at each point on* V^m,

(iii) $N_i\xi^i$ *has fixed sign on* V^m,

(iv) Φ *is not constant along* V^m,

then V^m *is isometric to a sphere.*

To prove Theorem B, C, B–1 and B–2, the restriction that at each point on V^m, the principal curvature $k_1, k_2, \cdots, k_m$ of V^m are postive plays a very important role. But, for Theorem A, A–1 and A–2, this restriction is not necessary. The purpose of the present paper is to prove some theorems except its restriction for closed orientable hypersurfaces with positive constant second mean curvature. § 1 is devoted to give notations and fundamental formulas in the theory of hypersurfaces in a general Riemann space R^{m+1}. In § 2 we derive the integral formulas which are valid for a closed orientable hypersurface in R^{m+1}. In § 3 we apply the integral formulas obtained in § 2 to a closed orientable hypersurface whose

3) G_{ij} denote the covariant component of the metric tensor of R^{m+1}.

second mean curvature H_2 is positive constant, and give some theorems. In the last section 4, making use of results obtained in § 3, we give characteristic properties of a hypersurface which is isometric to a sphere.

The present author wishes to express his very sincere thanks to Professor Y. Katsurada for her valuable advices and kind guidances.

§ 1. Notations and fundamental formulas.

We consider an $(m+1)$-dimensional Riemann space R^{m+1} of class C^r $(r \geqq 3)$ with the positive definite metric tensor G_{ij}, which admits a continuous one-parameter transformation group G of R^{m+1} generated by an infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i(x)\,\delta\tau\,^{4)},$$

where x^i are local coordinates in R^{m+1}. If the generating vector field ξ^i satisfies the equation

$$\underset{\xi}{\pounds} G_{ij} \equiv \xi_{i;j} + \xi_{j;i} = 2\Phi G_{ij} \qquad (\xi_i = G_{ij}\xi^j)$$

for a scalar field Φ in R^{m+1}, ξ^i is called a conformal Killing vector field and G a conformal transformation group, where $\underset{\xi}{\pounds} G_{ij}$ denotes the Lie derivative of the metric tensor G_{ij} with respect to ξ^i.

We now consider a closed orientable hypersurface $V^m (m \geqq 3)$ imbedded in R^{m+1} whose local expression is

$$x^i = x^i(u^\alpha),$$

where u^α are local coordinates on V^m. If we put

$$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha},$$

then $B^i_1, B^i_2, \cdots, B^i_m$ are m linearly independent vectors tangent to V^m, and the covariant component $g_{\alpha\beta}$ of the metric tensor of V^m are given by

$$g_{\alpha\beta} = G_{ij} B^i_\alpha B^j_\beta .$$

And we choose the unit vector N^i normal to V^m in such a way that

$$B^i_1, B^i_2, \cdots, B^i_m, N^i$$

give the positive orientation in V^m.

Denoting by " ; " the operation of covariant differentiation due to van

4) Throughout this paper Latin indices take the values 1 to $m+1$ and Greek indices the values 1 to m.

der Waerden-Bortolotti, we have the following Gauss's formula and Weingarten's formula:

$$B^i_{\alpha;\beta} = b_{\alpha\beta} N^i, \tag{1.1}$$

$$N^i_{;\alpha} = -b_\alpha{}^\beta B^i_\beta, \tag{1.2}$$

where $b_{\alpha\beta}$ is the second fundamental tensor of V^m and $b_\alpha{}^\beta = b_{\alpha\gamma} g^{\gamma\beta}$. We also obtain the equations of Gauss

$$R_{\alpha\beta\gamma\delta} = b_{\alpha\delta} b_{\beta\gamma} - b_{\alpha\gamma} b_{\beta\delta} + K_{ijkl} B^i_\alpha B^j_\beta B^k_\gamma B^l_\delta \tag{1.3}$$

for hypersurface V^m, where $R_{\alpha\beta\gamma\delta}$, K_{ijkl} is the covariant component of the curvature tensor of V^m and of R^{m+1} respectively.

If we denote by $k_1, k_2, \cdots, k_m$ the principal curvatures of V^m, that is, the roots of the characteristic equation

$$\det(b_{\alpha\beta} - k g_{\alpha\beta}) = 0,$$

then the ν-th mean curvature H_ν of V^m is defined to be the ν-th elementary symmetric function of $k_1, k_2, \cdots, k_m$ divided by the number of terms, that is,

$$\binom{m}{\nu} H_\nu = \sum_{\alpha_1 < \alpha_2 < \cdots < \alpha_\nu} k_{\alpha_1} k_{\alpha_2} \cdots k_{\alpha_\nu} \qquad (1 \leqq \nu \leqq m).$$

From the above definition, the first mean curvature H_1, the second mean curvature H_2 and the third mean curvature H_3 of V^m are respectively given by

$$m H_1 = \sum_\alpha k_\alpha = b_\alpha{}^\alpha, \tag{1.4}$$

$$\binom{m}{2} H_2 = \sum_{\alpha<\beta} k_\alpha k_\beta = \frac{1}{2}\left\{(b_\alpha{}^\alpha)^2 - b_\alpha{}^\beta b_\beta{}^\alpha\right\} \tag{1.5}$$

and

$$\binom{m}{3} H_3 = \sum_{\alpha<\beta<\gamma} k_\alpha k_\beta k_\gamma = \frac{1}{3!}\left\{(b_\alpha{}^\alpha)^3 + 2 b_\alpha{}^\beta b_\beta{}^\gamma b_\gamma{}^\alpha - 3 b_\alpha{}^\alpha (b_\beta{}^\gamma b_\gamma{}^\beta)\right\}, \tag{1.6}$$

where $b_\alpha{}^\alpha = b_{\alpha\beta} g^{\alpha\beta}$ and $b_\alpha{}^\beta = b_{\alpha\gamma} g^{\gamma\beta}$.

§ 2. Some integral formulas for a closed orientable hypersurface in R^{m+1}.

At the each point of the hypersurface V^m we can put as follows

$$\xi^i = B^i_\alpha \xi^\alpha + \Theta N^i \tag{2.1}$$

for some vector ξ^α and scalar Θ on V^m. Since $G_{ij} B^i_\alpha N^j = 0$, it follows immediately that

$$\xi_\alpha = B^i_\alpha \xi_i ,$$

where $\xi_\alpha = \xi^\beta g_{\beta\alpha}$ and $\xi_i = \xi^j G_{ji}$. We differentiate covariantly the above equation along V^m, making use of (1.1) and (2.1), we get

$$\xi_{\alpha;\beta} = b_{\alpha\beta}\Theta + B^i_\alpha B^j_\beta \xi_{i;j} .$$

Multiplying both sides by the contravariant metric tensor $g^{\alpha\beta}$ of V^m, contracting and using (1.4), we get

$$\xi_{\alpha;\beta} g^{\alpha\beta} = mH_1\Theta + \frac{1}{2} g^{\alpha\beta} B^i_\alpha B^j_\beta \underset{\xi}{\pounds} G_{ij} . \tag{2.2}$$

If we put

$$\underset{\xi}{\pounds} g_{\alpha\beta} = B^i_\alpha B^j_\beta \underset{\xi}{\pounds} G_{ij} , \tag{2.3}$$

then (2.2) is rewritten as follows:

$$\frac{1}{m}\xi^\alpha_{;\alpha} \equiv \frac{1}{m}\xi_{\alpha;\beta} g^{\alpha\beta} = H_1\Theta + \frac{1}{2m} g^{\alpha\beta} \underset{\xi}{\pounds} g_{\alpha\beta} .$$

Since V^m is orientable and closed, we have

$$\int_{V^m} \xi^\alpha_{;\alpha} dA = 0 ,$$

where dA is the area element of V^m [10]. Hence we obtain the following integral formula:

$$\int_{V^m} H_1 \Theta dA + \frac{1}{2m}\int_{V^m} g^{\alpha\beta} \underset{\xi}{\pounds} g_{\alpha\beta} dA = 0 . \tag{2.4}$$

Next, if we put

$$\eta_\beta = 2C_\beta{}^\alpha B^i_\alpha \xi_i ,$$

where the symbols $C_{\beta\gamma}$ are the component of the symmetric tensor of V^m defined by

$$C_{\beta\gamma} = b_\alpha{}^\alpha b_{\beta\gamma} - b_\beta{}^\alpha b_{\alpha\gamma} , \tag{2.5}$$

and $C_\beta{}^\alpha = C_{\beta\gamma} g^{\gamma\alpha}$, then we have, by covariant differentiation along V^m and using (1.1),

$$\eta_{\beta;\gamma} = 2C_\beta{}^\alpha{}_{;\gamma} B_\alpha{}^i \xi_i + 2C_\beta{}^\alpha b_{\alpha\gamma}\Theta + 2C_\beta{}^\alpha B^i_\alpha B^k_\gamma \xi_{i;k} .$$

Multiplying both sides by $g^{\beta\gamma}$ and summing for β and γ, we get

$$\eta_{\beta;\gamma} g^{\beta\gamma} = 2C_\beta{}^\alpha{}_{;\gamma} g^{\beta\gamma} B^i_\alpha \xi_i + 2C_\beta{}^\alpha b_\alpha{}^\beta \Theta + C^{\gamma\alpha} B^i_\alpha B^k_\gamma \underset{\xi}{\pounds} G_{ik} . \tag{2.6}$$

On the other hand, from (1.4) and (1.5), the equation (1.6) is also rewritten as follows:

(2. 7) $$2C_\beta^{\ \gamma} b_\gamma^{\ \beta} = m(m-1)\left\{mH_1H_2-(m-2)H_3\right\}.$$

Consequently, by substituting (2. 7) and (2. 3) into (2. 6), we find that

$$\eta^\beta{}_{;\beta} = 2C^{\beta\alpha}{}_{;\beta} B^i_\alpha \xi_i + m(m-1)\left\{mH_1H_2-(m-2)H_3\right\}\Theta + C^{\beta\alpha} \underset{\xi}{\pounds} g_{\alpha\beta},$$

where $C^{\beta\alpha}{}_{;\beta} = C_\beta{}^\alpha{}_{;\gamma} g^{\beta\gamma}$, and further, since $m \geqq 3$, we may write it

$$\frac{1}{m(m-1)}\eta^\beta{}_{;\beta} = \frac{2}{m(m-1)} C^{\beta\alpha}{}_{;\beta} B^i_\alpha \xi_i + \left\{mH_1H_2-(m-2)H_3\right\}\Theta + \frac{1}{m(m-1)} C^{\beta\alpha} \underset{\xi}{\pounds} g_{\alpha\beta}.$$

Therefore, since V^m is orientable and closed, we get the required integral formula

(2. 8) $$\frac{2}{m(m-1)}\int_{V^m} C^{\beta\alpha}{}_{;\beta} B^i_\alpha \xi_i dA + \int_{V^m}\left\{mH_1H_2-(m-2)H_3\right\}\Theta dA + \frac{1}{m(m-1)}\int_{V^m} C^{\beta\alpha} \underset{\xi}{\pounds} g_{\alpha\beta} dA = 0.$$

We now assume that the vector field ξ^i is conformal, that is, $\underset{\xi}{\pounds} G_{ij} = 2\Phi G_{ij}$, then (2. 4) becomes

(2. 9) $$\int_{V^m} H_1 \Theta dA + \int_{V^m} \Phi dA = 0$$

and, since it follows, from (1. 5) and (2. 5), that $C_{\beta\gamma} g^{\beta\gamma} (= C_\beta^{\ \beta}) = m(m-1)H_2$, (2. 8) becomes

(2. 10) $$\frac{2}{m(m-1)}\int_{V^m} C^{\beta\alpha}{}_{;\beta} B^i_\alpha \xi_i dA + \int_{V^m}\left\{mH_1H_2-(m-2)H_3\right\}\Theta dA + 2\int_{V^m} \Phi H_2 dA = 0,$$

where the integral formula (2. 9) is due to Y. Katsurada [5].

§ 3. Closed orientable hypersurfaces with H_2=positive constant.

From (1. 5) and (1. 4), it follows that

(3. 1) $$m^2 H_1^2 = m(m-1)H_2 + b_\alpha^{\ \beta} b_\beta^{\ \alpha},$$

and then the second term of the right hand member has non-negative sign, because of $b_\alpha^{\ \beta} b_\beta^{\ \alpha} = b_{\alpha\beta} b^{\alpha\beta}$. If we assume that the second mean curvature H_2 is positive constant, then the left hand member of (3. 1) is positive, that is, there exists not any point P on V^m satisfying $H_1(P)=0$. Accordingly, since the differentiability of H_1 on the closed hypersurface V^m is assumed, H_1 must have fixed sign on V^m. Therefore we have

LEMMA 3.1 *If the second mean curvature H_2 is positive constant, then the first mean curvature H_1 has fixed sign on V^m.*

Now we shall prove the following theorem:

THEOREM 3.2 *Let V^m be a closed orientable hypersurface in a Riemann space R^{m+1} which admits a conformal Killing vector field ξ^i. If*
(i) *H_2 is positive constant,*
(ii) *$C^\alpha{}_{\beta;\alpha}=0$ on V^m,*
(iii) *$N_i\xi^i(=\Theta)$ has fixed sign on V^m,*
then every point of V^m is umbilic.

Proof. Multiplying the formula (2.9) in §2 by $2H_2$(=positive constant), we obtain

$$\int_{V^m} 2H_1H_2\Theta dA + 2\int_{V^m}\Phi H_2 dA = 0,$$

and subtracting the above formula from (2.10), we find

$$\frac{2}{m(m-1)}\int_{V^m} C^{\beta\alpha}{}_{;\beta}B^i_\alpha\xi_i dA + (m-2)\int_{V^m}\{H_1H_2-H_3\}\Theta dA = 0.$$

And, from the assumption $C^{\beta\alpha}{}_{;\beta}=0$ and $m\geqq 3$, we have

$$\int_{V^m}\{H_1H_2-H_3\}\Theta dA = 0. \tag{3.2}$$

Moreover, since $H_1\neq 0$ for any point on V^m, the scalar field on V^m defined by $H_1H_2-H_3$ is rewritten as follows:

$$H_1H_2-H_3 = \frac{1}{H_1}\{H_2(H_1^2-H_2)+(H_2^2-H_1H_3)\}. \tag{3.3}$$

On the other hand, we know the fact that

$$H_\nu^2 - H_{\nu-1}H_{\nu+1} \geqq 0 \qquad (\nu=1, 2, \cdots, m-1) \text{ ([1], [3])},$$

where $H_0=1$. As a special case, we see that

$$H_1^2-H_2\geqq 0 \quad \text{and} \quad H_2^2-H_1H_3\geqq 0.$$

Accordingly, making use of the assumption H_2 = positive constant and Lemma 3.1, from (3.3) we have that $H_1H_2-H_3\geqq 0$ (or $\leqq 0$) on V^m. Hence, from (3.2), we find that

$$H_1H_2 - H_3 = 0 \qquad \text{on } V^m,$$

from which, by virtue of (3. 3), we get

$$H_2(H_1^2 - H_2) + (H_2^2 - H_1H_3) = 0 \qquad \text{on } V^m.$$

On making use of $H_2(H_1^2 - H_2) \geqq 0$ and $H_2^2 - H_1H_3 \geqq 0$ on V^m, we obtain $H_2(H_1^2 - H_2) = 0$, from which

$$H_1^2 - H_2 = 0.$$

Therefore, from $H_1^2 - H_2 = \dfrac{1}{m^2(m-1)} \sum (k_\alpha - k_\beta)^2$, we find that

$$k_1 = k_2 = \cdots = k_m$$

at each point on V^m. (Then we have $H_1 = k_1$, $H_2 = k_1^2$ and $H_3 = k_1^3$, from which we get $H_2^2 - H_1H_3 = 0$.) This is the required result.

We now assume that the Riemann space R^{m+1} is an Einstein space: $K_{jk} = \dfrac{K}{m+1} G_{jk}$, where $K_{jk}(=K_{ijkl}G^{il})$ and $K(=K_{jk}G^{jk})$ are the Ricci tensor and the scalar curvature of R^{m+1} respectively. Multiplying (1. 3) by $g^{\alpha\delta}$ and summing for α and δ, we have

$$R_{\beta\gamma} = b_\alpha{}^\alpha b_{\beta\gamma} - b_\beta{}^\alpha b_{\alpha\gamma} + K_{ijkl} B_\alpha^i B_\beta^j B_\gamma^k B_\delta^l g^{\alpha\delta},$$

where $R_{\beta\gamma}$ is the Ricci tensor of V^m. Remembering $C_{\beta\gamma} = b_\alpha{}^\alpha b_{\beta\gamma} - b_\beta{}^\alpha b_{\alpha\gamma}$ and $g^{\alpha\delta} B_\alpha^i B_\delta^l = G^{il} - N^i N^l$, we can write in the form

$$R_{\beta\gamma} = C_{\beta\gamma} + K_{jk} B_\beta^j B_\gamma^k - K_{ijkl} N^i B_\beta^j B_\gamma^k N^l.$$

Accordingly, because of an Einstein space, we obtain

$$R_{\beta\gamma} = C_{\beta\gamma} + \frac{K}{m+1} g_{\beta\gamma} - K_{ijkl} N^i B_\beta^j B_\gamma^k N^l.$$

Moreover, multiplying by $g^{\beta\gamma}$ and summing for β and γ, we have

$$R = C_{\beta\gamma} g^{\beta\gamma} + \frac{mK}{m+1} - K_{il} N^i N^l,$$

where R is the scalar curvature of V^m. Since R^{m+1} is also an Einstein space and $C_{\beta\gamma} g^{\beta\gamma}(=C_\beta^\beta) = m(m-1) H_2$, we obtain

$$R = m(m-1) H_2 + \frac{m-1}{m+1} K.$$

Therefore, remembering the fact that the scalar curvature K in an Einstein space is constant, we finally reach the following

LEMMA 3. 3 *Let V^m be a hypersurface in an Einstein space R^{m+1}.*

Then, necessary and sufficient condition that the second mean curvature H_2 be constant is that the scalar curvature R of V^m be constant.

Next, as a special case of it, let R^{m+1} be a Riemann space of constant curvature κ : $K_{ijkl}=\kappa(G_{il}G_{jk}-G_{ik}G_{jl})$, where $\kappa=\frac{K}{m(m+1)}$. Then, the equation of Gauss (1. 3) is written in the form

$$R_{\alpha\beta\gamma\delta}=b_{\alpha\delta}b_{\beta\gamma}-b_{\alpha\gamma}b_{\beta\delta}+\kappa(g_{\alpha\delta}g_{\beta\gamma}-g_{\alpha\gamma}g_{\beta\delta}).$$

Similarly, multiplying by $g^{\alpha\delta}$ and summing for α and δ, we have

$$R_{\beta\gamma}=C_{\beta\gamma}+(m-1)\,\kappa g_{\beta\gamma}.$$

By covariant differentiation along V^m, we get

$$R_{\beta\gamma;\alpha}=C_{\beta\gamma;\alpha}+(m-1)\,\kappa_{;\alpha}g_{\beta\gamma}.$$

Since κ is constant, we obtain

$$R_{\beta\gamma;\alpha}=C_{\beta\gamma;\alpha},$$

and, moreover, multiplying by $g^{\beta\alpha}$ and summing for β and α, we have $R_{\beta\gamma;\alpha}g^{\beta\alpha}=C_{\beta\gamma;\alpha}g^{\beta\alpha}$, that is,

(3. 4) $$R^{\beta}{}_{\gamma;\beta}=C^{\beta}{}_{\gamma;\beta}.$$

On the other hand, as well-known, the following equation is valid for any Riemann space :

(3. 5) $$R^{\beta}{}_{\gamma;\beta}=\frac{1}{2}R_{;\gamma}\left(=\frac{1}{2}\frac{\partial R}{\partial u^{\gamma}}\right).$$

Accordingly, from (3. 4) and (3. 5), we have

$$C^{\beta}{}_{\gamma;\beta}=\frac{1}{2}R_{;\gamma}.$$

Consequently, it follows from Lemma 3. 3 that, if the second mean curvature H_2 is constant, then $C^{\beta}{}_{\gamma;\beta}=0$. Therefore, as a special case of the Theorem 3. 2, we conclude the following

COROLLARY 3. 4 *Let V^m be a closed orientable hypersurface in a Riemann space of constant curvature which admits a conformal Killing vector field ξ^i. If*

(i) *H_2 is positive constant,*

(ii) *$N_i\xi^i(=\Theta)$ has fixed sign on V^m,*

then every point of V^m is umbilic.

§ 4. Characteristic properties of a hypersurface which is isometric to a sphere.

To prove that the hypersurface is isometric to a sphere, we use the following theorem due to M. Obata [9].

THEOREM D. *Let V^m $(m \geqq 2)$ be a complete Riemannian manifold which admits a non-null function φ such that*

$$\varphi_{;\alpha;\beta} = -c^2 \varphi g_{\alpha\beta} \qquad (c = constant).$$

Then V^m is isometric to a sphere of radius $1/c$.

Now we consider an Einstein space R^{m+1} which has the scalar curvature $K \neq 0$ and admits a proper conformal Killing vector field ξ^i, that is, ξ^i satisfies an equation:

$$\underset{\xi}{\pounds} G_{ij} = \xi_{i;j} + \xi_{j;i} = 2\Phi G_{ij}.$$

Then the Lie derivative of the curvature tensor $K^h_{\ ijk}$ with respect to ξ^i is given by

$$\underset{\xi}{\pounds} K^h_{\ ijk} = \delta^h_j \Phi_{i;k} - \delta^h_k \Phi_{i;j} + G_{ik}\Phi^h_{;j} - G_{ij}\Phi^h_{;k} \qquad ([11]), \tag{4.1}$$

where $\Phi_i = \Phi_{;i}$, $\Phi^i = G^{ij}\Phi_j$ and δ^i_j is the Kronecker delta. Since R^{m+1} is an Einstein space, we have

$$K_{ij} = \frac{K}{m+1} G_{ij} \qquad (K = \text{constant}). \tag{4.2}$$

Making use of (4.1) and (4.2), after some calculations we obtain the following result:

$$\Phi_{i;j} = \rho\Phi G_{ij} \qquad \left(\rho = -\frac{K}{m(m+1)}\right), \tag{4.3}$$

and then, by virtue of (4.2) and the assumption $K \neq 0$, we see that $\rho =$ non-zero constant. Consequently, we find that the Einstein space R^{m+1} admitting the proper conformal Killing vector field ξ^i must always admit the vector field Φ^i satisfying (4.3), which is the special conformal Killing vector field.

THEOREM 4.1 *Let R^{m+1} be an Einstein space with $K \neq 0$ which admits a conformal Killing vector field ξ^i and V^m a closed orientable hypersurface such that*

(i) *H_2 is positive constant,*

(ii) *$C^\alpha_{\ \beta;\alpha} = 0$ on V^m,*

(iii) *$N_i\Phi^i$ has fixed sign on V^m and is not constant along V^m.*

Then V^m is isometric to a sphere.

Proof. (This method is due to Y. Katsurada [7].) By virtue of Theorem 3. 2, every point of V^m is umbilic, that is,

$$k_1 = k_2 = \cdots = k_m ,$$

from which we have

$$H_2 = k_1^2 = H_1^2 . \tag{4. 4}$$

Accordingly, from the assumption (i) we obtain $H_1 =$ non-zero constant. Consequently, since V^m is the umbilical hypersurface, we find that

$$b_{\alpha\beta} = H_1 g_{\alpha\beta} \qquad (H_1 = \text{constant}) . \tag{4. 5}$$

Now we put $\Psi = \Phi_i N^i$. Then, by covariant differentiation along V^m, we have

$$\Psi_{;\alpha} = \Phi_{i;j} B_\alpha^j N^i + \Phi_i N^i_{;\alpha} ,$$

from which, by means of (4. 3), (1. 2) and (4. 5), it follows that

$$\Psi_{;\alpha} = -H_1 \Phi_i B_\alpha^i \qquad (H_1 = \text{constant}), \tag{4. 6}$$

that is to say, $\Psi_{;\alpha} + H_1 \Phi_{;\alpha} = 0$. Accordingly, since H_1 is constant, we get

$$\Psi = -H_1 \Phi + C \qquad (C = \text{constant}) . \tag{4. 7}$$

Moreover, by covariant differentiation of (4. 6) along V^m, we get

$$\Psi_{;\alpha;\beta} = -H_1 (\Phi_{i;j} B_\alpha^i B_\beta^j + \Phi_i B^i_{\alpha;\beta}) .$$

Substituting (4. 3) and (1. 1) into the right hand member of the last equation and remembering $\Psi = \Phi_i N^i$, we obtain

$$\Psi_{;\alpha;\beta} = -H_1 (\rho \Phi g_{\alpha\beta} + \Psi b_{\alpha\beta}) .$$

Making use of (4. 4), (4. 5) and (4. 7), the last equation is written as follows:

$$\Psi_{;\alpha;\beta} = -\left\{(H_2 - \rho)\Psi + \rho C\right\} g_{\alpha\beta} \qquad \left(\rho = -\frac{K}{m(m+1)}\right) . \tag{4. 8}$$

If $H_2 - \rho = 0$, then (4. 8) becomes $\Psi_{;\alpha;\beta} = -\rho C g_{\alpha\beta}$, from which $\Delta\Psi = -m\rho C$, that is, $\Delta\Psi =$ constant, where $\Delta\Psi = \Psi_{;\alpha;\beta} g^{\alpha\beta}$. However this is impossible, unless $\Psi =$ constant along V^m ([4], [2]). Therefore, $H_2 - \rho$ being different from zero, we have, from (4. 8),

$$\left(\Psi + \frac{\rho C}{H_2 - \rho}\right)_{;\alpha;\beta} = -(H_2 - \rho)\left(\Psi + \frac{\rho C}{H_2 - \rho}\right) g_{\alpha\beta} , \tag{4. 9}$$

from which we get

$$\Delta\left(\Psi + \frac{\rho C}{H_2 - \rho}\right) = -m(H_2 - \rho)\left(\Psi + \frac{\rho C}{H_2 - \rho}\right) .$$

Consequently it follows that $H_2-\rho>0$ [12]. Thus, according to Theorem D, the equation (4.9) shows that the hypersurface V^m is isometric to a sphere.

Especially, restricting R^{m+1} to a Riemann space of constant curvature, from Corollary 3.4 we obtain the following

COROLLARY 4.2 *Let R^{m+1} be a Riemann space of constant curvature which admits a conformal Killing vector field ξ^i and V^m a closed orientable hypersurface such that*

(i) *H_2 is positive constant,*

(ii) *$N_i\Phi^i$ has fixed sign on V^m and is not constant along V^m.*

Then V^m is isometric to a sphere.

Similarly, making use of the condition that Φ is not constant along V^m instead of that $N_i\Phi^i$ is not constant along V^m, we can prove the following theorem and corollary in a similar way to the proof of Theorem 4.1.

THEOREM 4.3 *Let R^{m+1} be an Einstein space with $K\neq 0$ which admits a conformal Killing vector field ξ^i and V^m a closed orientable hypersurface such that*

(i) *H_2 is positive constant,*

(ii) *$C^{\alpha}_{\beta;\alpha}=0$ on V^m,*

(iii) *$N_i\xi^i$ has fixed sign on V^m,*

(iv) *Φ is not constant along V^m.*

Then V^m is isometric to a sphere.

COROLLARY 4.4 *Let R^{m+1} be a Riemann space of constant curvature which admits a conformal Killing vector field ξ^i and V^m a closed orientable hypersurface such that*

(i) *H_2 is positive constant,*

(ii) *$N_i\xi^i$ has fixed sign on V^m,*

(iii) *Φ is not constant along V^m.*

Then V^m is isometric to a sphere.

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