Hokkaido Mathematical Journal Vol. 7 (1978) p. 58-73

On the global existence of unique solutions of differential equations in a Banach space

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(Received July 31, 1976: Revised March 22, 1977)

§ 1. Introduction and results.

Let E be a (real or complex) Banach space with the dual space E^*. The norms in E and E^* are denoted by $\| \ \|$. Let D be an open set in E and let F be a closed set in E such that $F \subset D$.

In this paper we consider the Cauchy problem

$$(CP) \qquad x' = f(t, x), \qquad x(t_0) = u_0 \in D, \qquad t_0 \in [0, \infty).$$

Here f is a continuous mapping from $[0, \infty) \times D$ into E. By a solution to (CP) or to $(CP;\ t_0, u_0)$, we mean a continuously differentiable function u from $[t_0, \infty)$into D such that $u(t_0) = u_0$ ond $u'(t) = f(t, u(t))$ for all $t \in [t_0, \infty)$.

As for the existence of a solution of this kind of problem, various results have been established, for example, see F. E. Browder [3], S. Kato [6, 7], N. Kenmochi and T. Takahashi [8], D. L. Lovelady and R. Martin [10], R. Martin [11, 12] and N. Pavel [14].

We say the set F is flow-invariant for f if $u_0 \in F$ implies that $u(t) \in F$ on $[t_0, \infty)$ for the solution to $(CP;\ t_0, u_0)$.

J. Bony [1] and H. Brezis [2] gave sufficient conditions for the set F to be flow-invariant for f in case E is a finite dimensional Euclidean space and f is a locally Lipschitz continuous function of D into E. The sufficient conditions proposed by them were generalized into a class of functions satisfying some dissipative type condition by R. M. Redheffer [15], and moreover some results were extended by R. Martin [12] to the case of general Banach space. Recently, N. Kenmochi and T. Takahashi [8] gave some simplications and improvements of results of [12].

The purpose of this paper is to give a criterion for the set F to be flow-invariant for f under more general dissipative type conditions on f.

If we consider [8, 12] from the view-point of the notion of flow-invariant sets, the condition of the present paper is weaker than those of [8, 12]. In § 5 we shall give some remarks and examples which connect our results with those of others. Our approach is essentially based on the methods in [5, 6, 7, 8].

Let us consider first the following scalar differential equation

$$(1.1) \qquad w'(t) = g\big(t, w(t)\big), \qquad w(t_0) = w_0 .$$

Here $g(t, \tau)$ is a real-valued function defined on $(0, \infty)\times[0, \infty)$ which is measurable in t for each fixed τ, and continuous nondecreasing in τ for each fixed t. We say w is a solution of (1. 1) on an interval $[t_0, t_0+a]$ if w is an absolutely continuous function defined on $[t_0, t_0+a]$ satisfying (1. 1) almost everywhere on $[t_0, t_0+a]$. We assume furthermore that g satisfies the following conditions:

(i) $g(t, 0)=0$ for $a.e.\, t\in(0, \infty)$, and for each bounded subset B of $(0, \infty)\times[0, \infty)$ there exists a function α_B defied on $(0, \infty)$ such that

$$|g(t, \tau)| \leqq \alpha_B(t) \qquad \text{for all } (t, \tau)\in B$$

and α_B is Lebesgue integrable on (t_1, t_2) for each $t_2>t_1>0$.

(ii) For each $T\in[0, \infty)$, $w\equiv 0$ is the only solution of (1. 1) on $[0, T]$ satisfying the condition $w(0)=(D^+w)(0)=0$, where D^+ denotes the right-sided derivative of w.

From the above conditions (i) and (ii) we see that for each $t_1, t_2\in[0, \infty)$ with $t_2<t_1$, $w\equiv 0$ is the only solution of (1. 1) on $[t_1, t_2]$ satisfying $w(t_1)=(D^+w)(t_1)=0$.

We define the functional $[\ ,\ ]$: $E\times E\to R$ by

$$[x, y] = \lim_{h\to -0} (\|x+hy\| - \|x\|)/h .$$

Now, let f be a mapping from $[0, \infty)\times D$ into E and consider the following conditions:

(K_1) f is continuous from $[0, \infty)\times D$ into E.

(K_2) $\big[x-y, f(t, x)-f(t, y)\big] \leqq g(t, \|x-y\|)$

for all x, y in D and for $a.e.\, t\in(0, \infty)$.
Then we have the following main result.

THEOREM. *Suppose that f satisfies the conditions* (K_1) and (K_2). *Then the set F is flow-invariant for f if and only if*

$$(1.2) \qquad \liminf_{h\to +0} p\big(x+hf(t, x), F\big)/h = 0$$

for all $(t, x)\in[0, \infty)\times F$, where $d(z, F)$ denotes the distance from $z\in E$ to F.

The author would like to express his hearty thanks to Professor T. Shirota for the kind criticism. The author thanks also Mr. N. Kenmochi and Mr. T. Takahashi for usefull suggestions.

§ 2. Some lemmas.

In this section we give some lemmas without proof. For proofs of Lemmas 2. 1–2. 3 see [6]. In Lemmas 2. 1–2. 5 we assume that g satisfies the conditions (i) and (ii) stated in § 1.

LEMMA 2. 1. *Let $t_1, t_2 \in [0, \infty)$ be such that $t_1 < t_2$ and let $\{w_n\}$ be a sequence of functions from $[t_1, t_2]$ to $[0, \infty)$ converging uniformly on $[t_1, t_2]$ to a function w_0. Let $M > 0$ be such that*

$$|w_n(t) - w_n(s)| \leqq M|t-s| \qquad \text{for all } s, t \in [t_1, t_2] \text{ and } n \geqq 1 .$$

Suppose furthermore that for each $n \geqq 1$ and $\sigma_n \geqq 0$ with $\sigma_n \downarrow 0$

$$w_n'(t) \leqq g\big(t, w_n(t)\big) + \sigma_n$$

for $t \in (t_1, t_2)$ such that $w_n'(t)$ exists. Then

$$w_0'(t) \leqq g\big(t, w_0(t)\big) \qquad \text{for a.e. } t \in (t_1, t_2) .$$

LEMMA 2. 2. *Let $t_1, t_2 \in [0, \infty)$ be such that $t_1 < t_2$ and let Φ be a uniformly bounded family of functions from $[t_1, t_2]$ into $[0, \infty)$ with the property that, for each $s, t \in [t_1, t_2]$ and $w \in \Phi$, $|w(t) - w(s)| \leqq M|t-s|$ for some constant $M > 0$.*
Let $w_0 = \sup\{w ; w \in \Phi\}$ and let $\sigma \geqq 0$ be a constant. Suppose furthermore that for each $w \in \Phi$

$$w'(t) \leqq g\big(t, w(t)\big) + \sigma$$

for $t \in (t_1, t_2)$ such that $w'(t)$ exists. Then

$$w_0'(t) \leqq g\big(t, w_0(t)\big) + \sigma \qquad \text{for a.e. } t \in (t_1, t_2) .$$

LEMMA 2. 3. *Let w be an absolutely continuous function from $[t_1, t_2]$ $(0 \leqq t_1 < t_2 < \infty)$ to $[0, \infty)$ such that $w(t_1) = (D^+ w)(t_1) = 0$ and*

$$w'(t) \leqq g\big(t, w(t)\big) \qquad \text{for a.e. } t \in (t_1, t_2) .$$

Then $w \equiv 0$ on $[t_1, t_2]$.

Let $t_0 > 0$. We define a functian g_{t_0} by

$$g_{t_0}(\mathrm{t}, \tau) = \begin{cases} g(t, \tau) & (t \geqq t_0, \ \tau \geqq 0) \\ 0 & (\text{otherwise}) . \end{cases}$$

For each $t_0 > 0$ we consider the following scalar differential equation

$$(2.1) \qquad w'(t) = g_{t_0}\big(t, w(t)\big) , \qquad w(t_0) = w_0 .$$

Concerning this equation we give the following two lemmas which are used in the proof of the Theorem.

LEMMA 2. 4. *Let* $t_0>0$ *and suppose that the maximal solution* $m_{t_0}(\cdot, w_0)$ *of* (2. 1) *through* (t_0, w_0) *exists over an interval* $[t_0, t_0+a]$. *Then there exists a* $\delta>0$ *such that* (2. 1) *has a maximal solution* $m_{t_0}(\cdot, \sigma)$ *for each* σ, $w_0\leqq\sigma<w_0+\delta$ *on* $[t_0, t_0+a]$ *with* $m_{t_0}(t_0, \sigma)=\sigma$. *Moreover,* $m_{t_0}(\cdot, \sigma)\rightarrow m_{t_0}(\cdot, w_0)$ *as* $\sigma\rightarrow w_0+0$, *uniformly on* $[t_0, t_0+a]$.

For a proof see [4, Theorem 2. 4, p. 47].

LEMMA 2. 5. *Suppose that the hypothesis of Lemma* 2. 4 *are satisfied, and let* w *be an absolutely continuous function on* $[t_0, t_0+a]$. *Suppose furthermore that*

$$w'(t)\leqq g_{t_0}\big(t, w(t)\big) \qquad \textit{for a.e.}\ t\in[t_0, t_0+a]\,.$$

Then $w(t_0)\leqq w_0$ *implies that* $w(t)\leqq m_{t_0}(t, w_0)$ *on* $[t_0, t_0+a]$.

For a proof of the above lemma see [9, Theorem 1. 10. 4, p. 43].

The following lemma on the functional $[\ ,\ ]: E\times E\rightarrow R$ is well-known.

LEMMA 2. 6. *Let* x, y *and* z *be in* E. *Then the functional* $[\ ,\ ]$ *has the following properties*:

(1) $|[x, y]|\leqq\|y\|$.

(2) $[x, y+z]\leqq[x, y]+\|z\|$.

(3) $[x, y]\leqq[x, y-z]+\|z\|$.

(4) *Let* u *be a function from a real interval* I *into* E *such that* $u'(t)$ *and* $\frac{d}{dt}\|u(t)\|$ *exist for a.e.* $t\in I$. *Then*

$$\frac{d}{dt}\big\|u(t)\big\|=\big[u(t), u'(t)\big] \qquad \textit{for a.e.}\ t\in I\,.$$

§3. Local existence.

Assume that conditions (K_1), (K_2) and (1. 2) are satisfied. Then we have the following important

PROPOSITION 3. 1. *Let* $(t_0, u_0)\in[0, \infty)\times F$ *and let* M, r_0 *and* T_1 *be positive numbers such that* $S(u_0, 2r_0)\subset D$ *and*

$$\big\|f(t, x)\big\|\leqq M \qquad \textit{for all}\ (t, x)\in[t_0, t_0+2T_1]\times S(u_0, 2r_0)\,.$$

Then $(CP;\ t_0, u_0)$ *has a unique solution* u *on* $[t_0, t_0+T_0]$ *such that* $u(t)\in F\cap S(u_0, r_0)$ *for all* $t\in[t_0, t_0+T_0]$, *where* $T_0=\mathrm{Min}\,\{r_0/(2M), T_1/2\}$ *and* $S(u_0, r_0)=\{v\,;\ \|v-u_0\|\leqq r_0\}$.

In order to prove this proposition, under the same assumptions and notations as in the proposition for each $\varepsilon>0$ sufficiently small we consider the set H_ε of all pairs (z, a) such that $t_0<a\leqq t_0+T_0$ and $z=z(t)$ is a function from $[t_0, a]$ into $S(u_0, 2r_0)$ satisfying the following conditions:

(i) $z(t_0)=u_0$ and $z(a)\in F$;

(ii) $\|z(t)-z(s)\|\leqq 2M|t-s|$ for all $s, t\in[t_0, a]$;

(iii) $\|z'(t)-f(t, z(t))\|\leqq\varepsilon$ for $a.e.\, t\in[t_0, a]$;

(iv) every subinterval of $[t_0, a]$, with length being $\geqq\varepsilon$, contains at least one point τ such that $z(\tau)\in F$.

Also, define an order "$\leqq$" in H_ε by the following manner: $(z_1, a_1)\leqq(z_2, a_2)$ if and only if $a_1\leqq a_2$ and $z_1(t)=z_2(t)$ for all $t\in[t_0, a_1]$. Then H_ε becomes a partially ordered set and we have

LEMMA 3.1. *H_ε is non-empty and inductive with respect to the order "$\leqq$".*

PROOF. For simplicity we may assume that $t_0=0$. Let $(t^0, v_0)\in[0, 2T_0]\times(F\cap S(u_0, r_0))$. Now, take a number δ so that

$$0<\delta<\mathrm{Min}\,\{r, \varepsilon_0, M\}$$

and

$$\|f(t, x)-f(t^0, v_0)\|\leqq\varepsilon/2 \tag{3.1}$$

whenever $t^0\leqq t\leqq t^0+\delta$ and $\|x-v_0\|\leqq\delta$, and by using (1.2), take a number h_1 with $0<h_1<\mathrm{Min}\,\{\delta/(\delta+2M), \delta\}$ having the property: for each $h\in(0, h_1]$ there is $v_h\in F$ such that

$$\|(v_h-v_0)/h-f(t^0, v_0)\|\leqq\delta/2\,. \tag{3.2}$$

Then it follows from (3.2) that

$$\begin{aligned}\|v_h-v_0\|/h &\leqq\delta/2+\|f(t^0, v_0)\|\\ &\leqq\delta/2+M\leqq\delta/2h\end{aligned} \tag{3.3}$$

for all $h\in(0, h_1]$. Therefore, defining

$$Q(t)=Q(t;\, v_0, t^0, h)=v_0+(t-t^0)(v_h-v_0)/h \tag{3.4}$$

for $t\in[t^0, t^0+h]$ with $h\in(0, h_1]$, we have by (3.3)

$$\|Q(t)-v_0\|\leqq\|v_h-v_0\|\leqq\delta/2<r_0$$

and hence $Q(t)\in S(u_0, 2r_0)$ for all $t\in[t^0, t^0+h]$. In particular $Q(t^0)=v_0\in F$ and $Q(t^0+h)=v_h\in F$. Besides it follows from (3.2) and (3.3) that

$$\begin{aligned}\|Q(t)-Q(s)\| &= |t-s|\,\|v_h-v_0\|/h \\ &\leqq (\delta/2+M)|t-s| \leqq 2M|t-s|\end{aligned}$$

and

$$\begin{aligned}\|Q'(t)-f(t, Q(t))\| &= \|(v_h-v_0)/h-f(t, Q(t))\| \\ &\leqq \|(v_h-v_0)/h-f(t^0, v_0)\|+\|f(t^0, v_0)-f(t, Q(t))\| \\ &\leqq \delta/2+\varepsilon/2 \leqq \varepsilon\end{aligned}$$

for all $t, s\in[t^0, t^0+h]$. Thus $(Q, h)\in H_\varepsilon$ if we take $t^0=0$ and $v_0=u_0$, so that $H_\varepsilon\neq\phi$.

Next we show that H_ε is inductive. Let $L=\{(z_\lambda, a_\lambda)\,;\ \lambda\in\Lambda\}$ be any totally ordered subset of H_ε, and put

$$\bar{a}=\sup\{a_\lambda\,;\ \lambda\in\Lambda\}\,.$$

If $\bar{a}=a_\lambda$ for some $\lambda\in\Lambda$, then (z_λ, a_λ) is clearly an upper bound for L. In case $a_\lambda<\bar{a}$ for all $\lambda\in\Lambda$, define a function $z:[0, \bar{a})\to S(u_0, 2r_0)$ by putting

$$z(t)=z_\lambda(t) \qquad \text{if } t<a_\lambda\,.$$

Then it is easy to see that z satisfies the properties (ii), (iii) and (iv) on $[0, \bar{a})$. Since $\|z(a_\lambda)-z(a_\gamma)\|\leqq 2M|a_\lambda-a_\gamma|$ for $\lambda, \gamma\in\Lambda$, the limit $z(\bar{a})=\lim_{t\uparrow\bar{a}} z(t)$ exists and $z(\bar{a})\in F$. If we denote again by z the function extended on $[0, \bar{a}]$ by the limit, the pair $(z, \bar{a})$ is clearly an upper bound for L. Thus H_ε is inductive. Q. E. D.

LEMMA 3. 2. *H_ε has a maximal element $(z_\varepsilon, a_\varepsilon)$ such that $a_\varepsilon=t_0+T_0$.*

PROOF. Since H_ε is inductive by Lemma 3. 1, it has at least one maximal element $(z_\varepsilon, a_\varepsilon)$. Moreover $a_\varepsilon=t_0+T_0$. In fact, suppose for contradiction that $a_\varepsilon<t_0+T_0$. Then $z_\varepsilon(a_\varepsilon)\in F\cap S(u_0, r_0)$ by (i) and (ii), and hence we can extend z_ε to the interval $[t_0, a_\varepsilon+h]$ by means of $Q(t)=Q(t\,;\ z_\varepsilon(a_\varepsilon), a_\varepsilon, h)$ on $[a_\varepsilon, a_\varepsilon+h]$, where h is a sufficiently small positive number and $Q(t)$ is the function as constructed in the previous lemma. This contradicts the fact that $(z_\varepsilon, a_\varepsilon)$ is maximal. Q. E. D.

PROOF of PROPOSITION 3. 1. Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\varepsilon_n\downarrow 0$ as $n\to\infty$ and let (z_n, t_0+T_0) be a maximal element in H_{ε_n} for each n.

We show that the sequence $\{z_n\}$ converges uniformly on $[t_0, t_0+T_0]$. For simplicity we assume again that $t_0=0$. Let $w_{mn}(t)=\|z_m(t)-z_n(t)\|$ for $t\in[0, T_0]$ and $m>n\geqq 1$, and remark first that $w'_{mn}(t)$ exists for *a.e.* $t\in[0, T_0]$ since

$$\left|w_{mn}(t)-w_{mn}(s)\right| \leqq 4M|t-s| \qquad \text{for all } s, t \in [0, T_0]. \tag{3.5}$$

Thus we have by Lemma 2.6 and the condition (K_2)

$$\begin{aligned} w'_{mn}(t) &= \left[z_m(t)-z_n(t), z'_m(t)-z'_n(t)\right] \\ &\leqq g\left(t, \left\|z_m(t)-z_n(t)\right\|\right) + \left\|z'_m(t)-f\left(t, z_m(t)\right)\right\| \\ &\qquad\qquad + \left\|z'_n(t)-f\left(t, z_n(t)\right)\right\| \\ &\leqq g\left(t, w_{mn}(t)\right) + 2\varepsilon_n \end{aligned} \tag{3.6}$$

for $a.e. t \in (0, T_0]$ and $m > n \geqq 1$.
Let $w_n(t) = \sup_{m>n} \{w_{mn}(t)\}$ for $t \in [0, T_0]$. Then $w_n(0)=0$ for all $n \geqq 1$. It thus follows from (3.5), (3.6) and Lemma 2.2 that

$$\left|w_n(t)-w_n(s)\right| \leqq 4M|t-s| \qquad \text{for all } s, t \in [0, T_0] \tag{3.7}$$

and

$$w'_n(t) \leqq g\left(t, w_n(t)\right) + 2\varepsilon_n \qquad \text{for } a.e. t \in (0, T_0]. \tag{3.8}$$

Since $0 \leqq w_n(t) \leqq w_n(0) + 4Mt \leqq 4MT_0$ for $t \in [0, T_0]$ and $n \geqq 1$, the sequence $\{w_n\}$ is equicontinuous and uniformly bounded, and hence it has a subsequence converging uniformly on $[0, T_0]$ to a function $w = w(t)$, and obviously $w(0)=0$. From (3.8) and Lemma 2.1 we have

$$w'(t) \leqq g\left(t, w(t)\right) \qquad \text{for all } a.e. t \in (0, T_0].$$

We show next that $(D^+ w)(0)=0$. For each $\varepsilon > 0$ we can fined a $\delta > 0$ such that

$$\left\|f(t, x)-f(0, u_0)\right\| < \varepsilon \qquad \text{for all } (t, x) \in [0, \delta] \times S(u_0, \delta).$$

Let $\delta_0 = \mathrm{Min}\,\{\delta, \delta/2M\}$. Since $\|z_n(t)-u_0\| \leqq 2Mt \leqq \delta$ by (ii),

$$\left\|f\left(t, z_m(t)\right)-f\left(t, z_n(t)\right)\right\| < 2\varepsilon$$

whenever $m > n \geqq 1$ and $t \in [0, \delta_0]$. From Lemma 2.6 we have

$$\begin{aligned} w'_{mn}(t) &= \left[z_m(t)-z_n(t), z'_m(t)-z'_n(t)\right] \\ &\leqq \left\|z'_m(t)-f\left(t, z_m(t)\right)\right\| + \left\|z'_n(t)-f\left(t, z_n(t)\right)\right\| \\ &\qquad\qquad + \left\|f\left(t, z_m(t)\right)-f\left(t, z_n(t)\right)\right\| \\ &< 2(\varepsilon + \varepsilon_n) \end{aligned}$$

for $a.e. t \in [0, \delta_0]$, and hence, by integrating the above inequality, $0 \leqq w_{mn}(t) \leqq 2(\varepsilon+\varepsilon_n)t$, whence $(D^+w)(0)=0$. Consequently, from Lemma 2. 3 we deduce now that $w \equiv 0$, and this implies that the sequencs $\{z_n\}$ is uniformly convsrgent on $[0, T_0]$. The limit $z=z(t)$ of of this sequence satisfies

$$z(t) = u_0 + \int_0^t f\big(s, z(s)\big)\, ds \qquad \text{for } t \in [0, T_0]\,.$$

Thus $z=z(t)$ is a solution to $(CP;\ 0, u_0)$ and $z(t) \in F \cap S(u_0, r_0)$ on $[0, T_0]$. Since the uniqueness of a solution to $(CP;\ 0, u_0)$ is well-known (cf. [6, Theorem 1], the proof of Proposition 3. 1 is complete.

§ 4. Proof of Theorem.

Before proving Theorem, we prepare the following two lemmas.

LEMMA 4. 1. *Let b be any positive number and let $u_0 \in F$. Then there exists a $\delta > 0$ for which $(CP;\ s, u_0)$ has a solution u on $[s, s+\delta]$ for each $s \in [0, b]$ such that $u(t) \in F$ for all $t \in [s, s+\delta]$.*

PROOF. We first see from the continuity of f on $[0, \infty) \times D$ that there exist positive constants r_0 and M such that

$$\big\|f(t, x)\big\| \leqq M \qquad \text{for all } (t, x) \in [0, 4b] \times S(u_0, 2r_0)\,.$$

Let $\delta = \mathrm{Min}\,\{3b/4, r_0/2M\}$. Then, by Proposition (3. 1), $(CP;\ s, u_0)$ has a unique solution u on $[s, s+\delta]$ for each $s \in [0, b]$ such that

$$u(t) \in F \text{ for all } t \in [s, s+\delta]\,. \qquad \text{Q.E.D.}$$

LEMMA 4. 2. *Let $t_0 > 0$ and $u_0 \in F$. Suppose that T is a positive number such that $(CP;\ t_0, u_0)$ has a solution u such that $u(t) \in F$ for all $t \in [t_0, t_0+T]$. Then there exists a positive number r having the property: for each $v_0 \in F \cap S(u_0, r)$, $(CP;\ t_0, v_0)$ has a solution v such that $v(t) \in F$ for all $t \in [t_0, t_0+T]$.*

PROOF. By the condition (ii) in § 1, $w \equiv 0$ is a maximal solution on $[t_0, t_0+T]$ of (2. 1) with $w(t_0)=(D^+w)(t_0)=0$. It thus follows from Lemma 2. 4 that there exists a positive number δ such that (2. 1) has a maximal solution $m_{t_0}(\cdot, \sigma)$ for each σ, $0 \leqq \sigma < \delta$ on $[t_0, t_0+T]$ with $m_{t_0}(t_0, \sigma)=\sigma$. Moreover, $m_{t_0}(\cdot, \sigma)$ converges to 0 uniformly on $[t_0, t_0+T]$ as $\sigma \to +0$. Since the set $\{(t, u(t));\ t \in [t_0, t_0+T]\}$ is compact in $[t_0, t_0+T] \times D$, there exist positive constants ρ and M such that

$$\big\|f(t, x)\big\| \leqq M \qquad \text{for all } t \in [t_0, t_0+T] \text{ and } x \in S\big(u(t), \rho\big)\,. \tag{4. 1}$$

Here we may choose ρ such that $S(u(t), \rho) \subset D$ for all $t \in [t_0, t_0+T]$. Consequently, we can choose a positive number r such that $0 < r < \mathrm{Min}\,\{\delta, \rho\}$ and

$$(4.2) \qquad \left|m_{t_0}\left(t, \|v_0-u_0\|\right)\right|<\rho$$

for all $(t, v_0)\in[t_0, t_0+T]\times(F\cap S(u_0, r))$.
By virtue of Proposition 3. 1, $(CP;\ t_0, v_0)$ has a unique local solution v with $v(t)\in F$ on some interval $[t_0, t_0+T(v_0))$ for each $v_0\in F\cap S(u_0, r)$. Assume that $T(v_0)\leqq T$ and $[t_0, t_0+T(v_0))$ is a maximal interval of existence of v with the property that $v(t)\in F$ on $[t_0, t_0+T(v_0))$.
Since $\|v(t)-u(t)\|$ is absolutely continuous on each closed interval $[t_0, t_0+T(v_0))$ we have

$$\begin{aligned}\frac{d}{dt}\left\|v(t)-u(t)\right\| &= \left[v(t)-u(t), f\left(t, v(t)\right)-f\left(t, u(t)\right)\right]\\ &\leqq g\left(t, \left\|v(t)-u(t)\right\|\right)\end{aligned}$$

for $a.e.\, t\in[t_0, t_0+T(v_0))$. Hence we have by Lemma 2. 5

$$\left\|v(t)-u(t)\right\|\leqq m_{t_0}\left(t, \left\|v_0-u_0\right\|\right) \qquad \text{for all } t\in\left[t_0, t_0+T(v_0)\right).$$

It thus follows from (4. 1) and (4. 2) that

$$\left\|f\left(t, v(t)\right)\right\|\leqq M \qquad \text{for all } t\in\left[t_0, t_0+T(v_0)\right),$$

and this implies that $\lim_{t\to T(v_0)} v(t)$ exists in F. Applying Proposition 3. 1 once again we have a contradiction. Thus $T<T(v_0)$ and the proof is complete.

PROOF of the THEOREM. The method of the following proof is essentially based on that of [8].

Let $(t_0, u_0)\in[0, \infty)\times F$. Then, by Proposition 3. 1, $(CP;\ t_0, u_0)$ has a unique local solution u on some interval $[t_0, t_1]$ such that $u(t)\in F$ for all $t\in[t_0, t_1]$. We note that $t_1>0$ and $u(t_1)\in F$. Let b be any positive number such that $b>t_1$. Then, by Lemma 4. 1, there exists a positive constant δ such that $(CP;\ s, u(t_1))$ has a solution v with $v(t)\in F$ on $[s, s+\delta]$ for each $s\in(0, b]$. We note here that if $s=0$, then we can not apply Lemmas 2. 4, 2. 5 and 4. 2 in the following discussion. Therefore, we omit the case $s=0$.

Now, let C be a connected component in F containing $u(t_1)$ and let

$$G_s=\left\{x\in C;\ (CP;\ s, x) \text{ has a solution } v \text{ such that } v(t)\in F \text{ for } t\in[s, s+\delta]\right\} \qquad \text{for each } s\in(0, b].$$

Then G_s is not empty since $u(t_1)\in G_s$ for each $s\in(0, b]$ by Lemma 4. 1. Moreover, G_s is relatively open in C for each fixed $s\in(0, b]$ by Lemma 4. 2. We show that G_s is also relatively closed in C. For this, let $\{x_n\}$ be any sequence in G_s which converges to $x \in C$ and let v_n be a solution to $(CP;\, s, x_n)$ on $[s, s+\delta]$. Then

$$\frac{d}{dt}\left\|v_n(t)-v_m(t)\right\| = \left[v_n(t)-v_m(t), f\left(t, v_n(t)\right)-f\left(t, v_m(t)\right)\right]$$
$$\leqq g\left(t, \left\|v_n(t)-v_m(t)\right\|\right)$$

for $a.e.\, t \in [s, s+\delta]$. Thus we have by Lemma 2. 5

$$\left\|v_n(t)-v_m(t)\right\| \leqq m_s\left(t, \left\|x_n-x_m\right\|\right)$$

for all $t \in [s, s+\delta]$ and for sufficiently large positive integers n and m. Since $\lim_{n,m\to\infty} \|x_n - x_m\| = 0$, the sequence $\{v_n\}$ converges uniformly on $[s, s+\delta]$ to a function v by Lemma 2. 4, and clearly v is a solulion to $(CP;\, s, x)$ on $[s, s+\delta]$ and hence $x \in G_s$. Consequently, $G_s = C$ for all $s \in (0, b]$. In particular, $u(t_1) \in G_{t_1} = C$ and hence $(CP;\, t_1, u(t_1))$ has a solution v on $[t_1, t_1+\delta]$ such that $v(t) \in F$ for $t \in [t_1, t_1+\delta]$. If $t_1+\delta < b$, then $(CP;\, t_1+\delta, v(t_1+\delta))$ has a solution w on $[t_1+\delta, t_1+2\delta]$ such that $w(t) \in F$ for $t \in [t_1+\delta, t_1+2\delta]$, because $v(t_1+\delta) \in G_{t_1+\delta} = C$. Obviously

$$\tilde{v}(t) = \begin{cases} u(t) & (t_0 \leqq t \leqq t_1) \\ v(t) & (t_1 \leqq t \leqq t_1+\delta) \\ w(t) & (t_1+\delta \leqq t \leqq t_1+2\delta) \end{cases}$$

is a solution to $(CP;\, t_0, u_0)$ on $[t_0, t_1+2\delta]$. Repeating this argument we see that $(CP;\, t_0, u_0)$ has a solution on $[t_0, b]$. Since b was arbitrary number such that $b > t_1$, it is proved that $(CP;\, t_0, u_0)$ has a solution u^* on $[t_0, \infty)$ such that $u^*(t) \in F$ for all $t \in [t_0, \infty)$. Thus the sufficiency is proved.

Conversely, suppose that the set F is flow-invariant for f and let u be a solution to $(CP;\, t_0, u_0)$ on $[t_0, \infty)$ such that $u(t) \in F$ for all $t \in [t_0, \infty)$. Then

$$d\left(u_0 + hf(t_0, u_0), F\right)/h \leqq \left\|\left(u(t_0+h)-u(t_0)\right)/h - f(t_0, u_0)\right\|$$

and

$$\left\|\left(u(t_0+h)-u(t_0)\right)/h - f(t_0, u_0)\right\| \to 0 \text{ as } h \to +0\,.$$

Hence the necessity follows. Q. E. D.

§ 5. Remarks and examples.

In this section we give some remarks and examples which connect our results with those of others.

REMARK 1. In the previous paper [6] we used the functional

$$\langle x, y\rangle = \big([x, y]-[x, -y]\big)/2\,.$$

But it can be easily seen that $[x, y]\leqq\langle x, y\rangle$ for each x, y in E. Hence the Theorem of the present paper gives an improvement of Theorem 2 in [6].

Let J be the duality mapping from E into 2^{E^*} (i. e., for each x in E, $J(x)=\{x^*\in E^*\,;\ x^*(x)=\|x\|^2=\|x^*\|^2\}$.
For each x, y in E, define

$$\langle x, y\rangle_i = \inf\left\{\mathrm{Re}\big(x^*(y)\big)\,;\ x^*\in J(x)\right\}.$$

Then for each $x\neq 0$ and y in E, $[x, y]=\langle x, y\rangle_i/\|x\|$ (see [11]). Thus the condition (K_2) is equivalent to the following:

$$(5.\ 1)\qquad \big\langle x-y,\ f(t, x)-f(t, y)\big\rangle_i \leqq \|x-y\|\, g(t, \|x-y\|)$$

for all $x, y\in D$ and for $a.e.\,t\in(0, \infty.)$
We note also that Proposition 3. 1 remains valid even if F is a relatively closed subset of D. Hence, this fact and (5. 1) imply that our Theorem gives a generalization of Theorems 3 and 4 in R. M. Redheffer [15] into a general Banach space.

REMARK 2. Let β be a real-valued function defined on $(0, \infty)$ satisfying the following conditions:

(β_1) For each $t_1, t_2\in(0, \infty)$ with $t_1<t_2$, β is Lebesgue integrable on (t_1, t_2).

(β_2) For each $t>0$, $\limsup_{\varepsilon\to+0}\left[\varepsilon \exp\left(\int_\varepsilon^t\beta(\tau)\,d\tau\right)\right]<+\infty$.

The condition (β_2) was considered by C. V. Pao [13] to prove the uniqueness of solutions to $(CP\,;\ 0, u_0)$.
If $g(t, \tau)=\beta(t)\,\tau$, then the conclusion of our Theorem remains valid. In fact, it is obvious that this function $\beta(t)\,\tau$ satisfies the condition (i) in § 1. To prove that $\beta(t)\,\tau$ satisfies also the condition (ii) in § 1, let w be a solution of the equation $w'(t)=\beta(t)\,w(t)$ on $[0, T]$ satisfying $w(0)=(D^+w)(0)=0$. Then for each $\varepsilon>0$, we have

$$\begin{aligned}0\leqq w(t) &= w(\varepsilon)\exp\left(\int_\varepsilon^t\beta(\tau)\,d\tau\right)\\ &= \varepsilon\exp\left(\int_\varepsilon^t\beta(\tau)\,d\tau\right)\big(w(\varepsilon)-w(0)\big)/\varepsilon\end{aligned}$$

for $t\in[\varepsilon, T]$. This implies that $w\equiv 0$ on $[0, T]$. Thus $\beta(t)\,\tau$ satisfies (i) and (ii) in § 1. However, the function $\beta(t)\,\tau$ need not be nondecreasing in τ for fixed t. The nondecreasing nature is used only in establishing

Lemma 2. 3 (see [6]) which is valid for $g(t, \tau)=\beta(t)\tau$. Thus our result extends those of [10, 11, 14] when $g(t, \tau)=\beta(t)\tau$.

REMARK 3. Recently, N. Kenmochi and T. Takahashi [8] proved the following theorem which gives an improvement of [12].

THEOREM A. *Let F be a closed subset of E. Suppose that f satisfies the following conditions*:

$$f \text{ is continuous from } [0, \infty)\times F \text{ into } E. \tag{5.2}$$

$$\langle x-y, f(t, x)-f(t, y)\rangle_i \leqq \omega(t)\|x-y\|^2 \tag{5.3}$$

for all (t, x), $(t, y)\in[0, \infty)\times F$, where ω is a real-valued continuous function defined on $[0, \infty)$. Suppose furthermore that

$$\liminf_{h\to+0} d(x+hf(t, x), F)/h=0 \tag{5.4}$$

for all $(t, x)\in[0, \infty)\times F$. Then $(CP;\ 0, u_0)$ has a unique global solution u defined on $[0, \infty)$ for each $u_0\in F$.

This result is intimately related to the notion of flow-invariant sets. If we consider this theorem from the view-point of the notion of flow-invariant sets we have the following

THEOREM B. *Let D be an open set in E and let F be a closed set in E such that $F\subset D$. Suppose that f satisfies* (5. 4) *and the following conditions*:

$$f \text{ is continuous from } [0, \infty)\times D \text{ into } E. \tag{5.5}$$

$$\langle x-y, f(t, x)-f(t, y)\rangle_i \leqq \omega(t)\|x-y\|^2 \tag{5.6}$$

for all (t, x), $(t, y)\in[0, \infty)\times D$. Then the set F is flow-invariant for f.

Since (5. 1) implies (5. 6), our Theorem contains Theorem B.

The following examples show that the condition (K_2) is strictly more general than (5. 6).

EXAMPLE 1. Let $a(t)$ be the function defined by

$$a(t)=\begin{cases} t^{3/2} & (0\leqq t\leqq \rho) \\ \rho^{3/2} & (t\geqq \rho), \end{cases}$$

where ρ is a constant such that $\rho>1$. Consider the function G defined by

$$G(t, u)=\begin{cases} \dfrac{\sqrt[3]{u}}{1+\sqrt[3]{a(t)}}+b(t)u^3 & \big(t\geqq 0,\ u\geqq a(t)\big) \\ \dfrac{\sqrt[3]{a(t)}}{1+\sqrt[3]{a(t)}}+b(t)u^3 & \big(t\geqq 0,\ u< a(t)\big), \end{cases}$$

where b is a real-valued continuous function from $[0, \infty)$ into $(-\infty, 0]$. It is easily verified that the function G satisfies the following inequality:

$$
\begin{aligned}
(5.7) \qquad & \left|u-v+h\left(G(t, u)-G(t, v)\right)\right| \\
& \geqq \left(1+h/3\sqrt[3]{a(t)^2}\left(1+\sqrt[3]{a(t)}\right)\right)|u-v|
\end{aligned}
$$

for all $h \leqq 0$, $t>0$ and $u, v \in (-\infty, \infty)$.

Let us take as E the Banach space ℓ^∞ of bounded sequences of real numbers. For eabh $x=(x_n)$ and $t \geqq 0$, define $f(t, x)=(G(t, x_n))$. Then f is continuous from $[0, \infty)\times E$ into E. For each $x=(x_n)$, $y=(y_n)$ in E, $h<0$, we have by (5.7)

$$
\begin{aligned}
& \sup_n \left|x_n-y_n+h\left(G(t, x_n)-G(t, y_n)\right)\right| - \sup_n |x_n-y_n| \\
& \geqq \frac{h}{3\sqrt[3]{a(t)^2}\left(1+\sqrt[3]{a(t)}\right)} \sup_n |x_n-y_n| .
\end{aligned}
$$

This implies that

$$
\left[x-y, f(t, x)-f(t, y)\right] \leqq \|x-y\|/3\sqrt[3]{a(t)^2}\left(1+\sqrt[3]{a(t)}\right)
$$

for all x, y in E and $t>0$. Let $\beta(t)=1/3\sqrt[3]{a(t)^2}\left(1+\sqrt[3]{a(t)}\right)$. Then $\int_0^\rho \beta(\tau)\,d\tau = \int_0^\rho dt/3t(1+\sqrt{t}) = +\infty$. However, it is easy to see that $\beta(t)$ satisfies the condition (β_1) in Remark 2. Moreover, by a simple calculation, we have

$$
\begin{aligned}
& \varepsilon \exp\left(\int_\varepsilon^t \beta(\tau)\,d\tau\right) \\
& \leqq \begin{cases} (\varepsilon^2 t)^{1/3} & (0<\varepsilon<t\leqq \rho) \\ (\varepsilon^2 t)^{1/3} \exp\left((t-\rho)/3\rho(1+\sqrt{\rho})\right) & (0<\varepsilon<\rho<t) . \end{cases}
\end{aligned}
$$

Thus, $\beta(t)$ satisfies also the condition (β_2).
Consequently, for each $(t_0, u_0) \in [0, \infty)\times E$, $(CP;\ t_0, u_0)$ has a unique global solution for the above defined f.

On the other hand, for each $x=(x_n)$ and $y=(y_n)$ in E such that $x_1 > y_1 > 0$ and $x_n=y_n=0$ for $n \geqq 2$,

$$
\begin{aligned}
& \left[x-y, f(0, x)-f(0, y)\right] \\
& = \left\{\frac{1}{\sqrt[3]{x_1^2}-\sqrt[3]{x_1 y_1}+\sqrt[3]{y_1^2}} - b(0)\left(x_1^2-x_1y_1+y_1^2\right)\right\} \|x-y\| .
\end{aligned}
$$

Hence we can not apply [8, 10, 11, 12, 13] to this example for the Cauchy problem $(CP;\ 0, u_0)$.

EXAMPLE 2. Next, let us take as E the Banach space ℓ^p $(1<p<\infty)$ of sequences of real numbers. Let $a(t)$ be as in Example 1 and let $M=(\sum_{n=1}^{\infty} 1/n^p)^{1/p}$. For each $x=(x_n)\in E$, define

$$f_n(t, x)=\begin{cases} \dfrac{\sqrt[3]{x_n}}{n(1+\sqrt[3]{a(t)})}-b(t)\,x_n & \big(t\geqq 0,\ x_n\geqq a(t)\big) \\ \dfrac{\sqrt[3]{a(t)}}{n(1+\sqrt[3]{a(t)})}-b(t)\,x_n & \big(t\geqq 0,\ x_n< a(t)\big). \end{cases}$$

Here $b(t)$ is a real-valued continuous function defined on $[0, \infty)$ satisfying $b(t)>M/\sqrt{\rho}$ for all $t\geqq 0$.

Define $f(t, x)=(f_n(t, x))$ for $(t, x)\in[0, \infty)\times E$. Then f is continuous from $[0, \infty)\times E$ into E. Let

$$F=\Big\{x\,;\ E\ni x=(x_n) \text{ such that } x_n\geqq 0 \text{ for } n\geqq 1 \text{ and } \|x\|\leqq\rho\Big\}.$$

Then F is closed in E. We shall show that the mapping f does not satisfy (5. 3) but does satisfy all the conditions of our Theorem. For this note that

$$[x, y]=\textstyle\sum_{n=1}^{\infty}\operatorname{sgn}(x_n)|x_n|^{p-1}y_n/\|x\|^{p-1} \tag{5. 8}$$

for all $x\neq 0$ and y in E.

Using (5. 8) we can verify easily that

$$\begin{aligned}&\big[x-y, f(t, x)-f(t, y)\big] \\ &\qquad\leqq\Big(b(t)+1/3\,\sqrt[3]{a(t)^2}\,\big(1+\sqrt[3]{a(t)}\big)\Big)\,\|x-y\|\end{aligned}$$

for all x, y in E and $t>0$. Let $\beta(t)=1/3\,\sqrt[3]{a(t)^2}\,\big(1+\sqrt[3]{a(t)}\big)$. Then $\int_0^{\rho}\beta(t)\,dt=+\infty$. But $\beta(t)$ satisfies the conditions (β_1) and (β_2) in Remark 2 by the same argument as in Example 1. Thus the above defined f satisfies (K_1) and (K_2) in § 1.

To show that f satisfies (5. 4) we note that

$$\begin{aligned}&x+hf(t, x) \\ &\qquad=\Big(\big(1-hb(t)\big)\,x_n+h\big(\sqrt[3]{x_n}\ \text{or}\ \sqrt[3]{a(t)}\big)/n\big(1+\sqrt[3]{a(t)}\big)\Big)\end{aligned}$$

for each $x=(x_n)\in F$ and $t\geqq 0$. Thus it follows that

$$\begin{aligned}\big\|x+hf(t, x)\big\|&\leqq\big(1-hb(t)\big)\,\|x\|+h\sqrt{\rho}\,M \\ &\leqq\rho+\big(\sqrt{\rho}\,M-\rho b(t)\big)\,h\,.\end{aligned}$$

By the assumption on b we have for each $x\in F$ and $t\geqq 0$

$$x+hf(t, x)\in F \qquad \text{for } 0<h<\mathrm{Min}\left\{1/b(t), \sqrt{\rho}/(\sqrt{\rho}\, b(t)-M)\right\}.$$

Consequently, the set F is flow-invariant for f by our Theorem.

On the other hand, for each $x=(x_n)$ and $y=(y_n)$ in F such that $\rho\geqq x_1>y_1>0$ and $x_n=y_n=0$ for $n\geqq 2$,

$$\begin{aligned}&\left[x-y, f(0, x)-f(0, y)\right]\\ &\qquad =\left(1/(\sqrt[3]{x_1^2}+\sqrt[3]{x_1 y_1}+\sqrt[3]{y_1^2})-b(0)\right)\|x-y\|,\end{aligned}$$

so that we can not apply [8] to this example.

References

[1] J. Bony: Principe du maximum, inégalité de Harnack et unicité des problèmes de Cauchy pour les opérateurs elliptiques dégénérés, Ann. Inst. Fourier, Grenoble, 19 (1969), 277–304.

[2] H. Brezis: On a characterization of flow-invariant sets, Comm. Pure Appl. Math. 23 (1970), 261–263.

[3] F. E. Browder: Nonlinear equations of evolution, Ann. of Math. 8 (1964), 485–523.

[4] E. Coddington and N. Levinson: Theory of ordinary differential equations, Academic Press, 1955.

[5] P. Hartman: On invariant sets and on a Theorem of Wazewski, Proc. Amer. Math. Soc., 32 (1972), 511–520.

[6] S. Kato: Some remarks on nonlinear differential equations in Banach spaces, Hokkaido Math. Jour. 4 (1975), 205–226.

[7] S. Kato: On flow-invariant sets in a Banach space, unpublished.

[8] N. Kenmochi and T. Takahashi: On the global existence of solutions of differential equations on closed subsets of a Banach space, Proc. Japan Acad., 51 (1975), 520–525.

[9] V. Lakshimikantham and S. Leela: Differential and integral inequalities, Academic Press, 1969.

[10] D. L. Lovelady and R. Martin: A gloval existence theorem for a nonautonomous differential equation in a Banach space, Proc. Amer. Math. Soc., 35 (1972), 445–449.

[11] R. Martin: A global existence theorem for an autonomous differential equation in a Banach space, Proc. Amer. Math. Soc., 26 (1970), 307–314.

[12] R. Martin: Differential equations on closed subsets of a Banach space, Trans. Amer. Math. Soc., 179 (1973), 399–414.

[13] C. V. Pao: On the uniqueness problem of differential equations in normed spaces, Applicable Analysis, 4 (1975), 269–282.

[14] N. PAVEL: Équations nonlinéaires d'évolution, Matematica., 14 (1972), 289–300.
[15] R. M. REDHEFFER: The theorems of Bony and Brezis on flow-invariant sets, Amer. Math. Monthly., 79 (1972), 740–747.

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