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Microlocal parametrices for mixed problems for symmetric hyperbolic systems with diffractive boundary

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§ 1. Introduction

Microlocal parametrices for hyperbolic mixed problems in domains with diffractive boundary have been constructed by Ludwig [9], Melrose [11], Taylor [16], Eskin [3] and others for second-order hyperbolic equations with Dirichlet boundary conditions. Taylor [16] (or [18]) has also obtained such results for Neumann boundary conditions, and Imai and Shirota [4] have obtained such results for certain general boundary conditions which include the Neumann conditions. (See also Shirota [14]). Moreover applying the results in [16] Taylor has obtained in [17] such results for Maxwell's equations in the exterior of a strictly convex perfect conductor.

The purpose of this paper is to give a generalization of the above results.

Let Ω be the open half space $\{x=(x', x_n)=(x_0, x'', x_n)\,;\, x_0\in R^1,\ x''\in R^{n-1}, x_n>0\}$ in R^{n+1} $(n\geqq 2)$ with boundary $\partial\Omega$ and $P(x, D)$ a symmetric system of first order defined on $\bar{\Omega}$ which is hyperbolic with respect to x_0. Consider a mixed problem:

$$\begin{aligned}&P(x, D)\, u=\sum_{j=0}^{n} A_j(x)\, D_j u+C(x)\, u=0 \quad \text{in } \Omega\,,\\ &B(x')\, u=f(x') \quad \text{on } \partial\Omega\,,\\ &u(x)=0 \quad \text{in } \Omega\cap\{x_0<0\}\,,\end{aligned}$$

where $D_j=-i\partial/\partial x_j$, A_j, $j=0, 1, \cdots, n$, are hermitian $m\times m$ matrices, A_0 is positive definite, and A_j, C and B are smooth (i.e., of class C^∞) and are constant for $|x|$ large enough.

Let $f\in\mathscr{E}'(\partial\Omega)$, $f(x')=0$ for $x_0<0$ and the wave front set $WF(f)$ be contained in a conic neighborhood of the diffractive points. We then want to show that there is a parametrix for the mixed problem, i.e., a distribution $u\in\mathscr{D}'(\Omega\cap U)$ with a neighborhood U of sing supp f in R^{n+1} such that $u(x)$ is a C^∞-function of $x_n\geqq 0$ with value in $\mathscr{D}'(R^n_{x'})$ and

(1.1) $\quad P(x, D)\, u\in C^\infty(\bar{\Omega}\cap U)\,,$

(1. 2) $\qquad Bu|_{\partial\Omega}-f\in C^{\infty}(\partial\Omega\cap U)$,

(1. 3) $\qquad u\in C^{\infty}\left(\bar{\Omega}\cap U\cap\{x_0<0\}\right)$,

$WF(u|_{\partial\Omega})\subset WF(f)$ and $WF(u)$ is contained in the set of null bicharacteristics of $P_1(x, \xi)$ passing over $WF(f)$ and going into positive time, where $P_1(x, \xi)$ is the principal symbol of P.

It is assumed that P is of constant multiplicity and hence $\det P_1(x, \xi)$ can be written in the form:

$$\det P_1(x, \xi) = Q_1(x, \xi)^{m_1}\cdots Q_r(x, \xi)^{m_r}\tilde{Q}(x, \xi),$$

where $Q_1, \cdots, Q_r$ and $\tilde{Q}$ are homogeneous polynomials in ξ with $C^{\infty}(\bar{\Omega})$ coefficients which have no common zero in ξ_0, such that $Q_1, \cdots, Q_r$ are strictly hyperbolic with respect to x_0 and $\tilde{Q}(x, \xi)$ is independent of ξ_n for x near $\partial\Omega$. (See Matsuura [10]). Moreover the boundary $\partial\Omega$ is assumed to be noncharacteristic for Q_j, $j=1, \cdots, r$. Then a point $(x', \xi')\in T^*(\partial\Omega)\backslash 0$ is said to be *diffractive* if, for some j and some real ξ_n, $Q_j(x', 0, \xi', \xi_n) = (\partial Q_j/\partial\xi_n)(x', 0, \xi', \xi_n)=0$ and the Poisson bracket $\{Q_j, \partial Q_j/\partial\xi_n\}$ $(x', 0, \xi', \xi_n)$ is positive. We impose furthermore the following condition on Q_j.

(i) *The surface* $Q_j(x', 0, \xi_0, \xi'', \xi_n)=0$ *in the* (ξ'', ξ_n)*-space is bounded and strictly convex for every* $j=1, \cdots, r$, $(x', 0)\in\partial\Omega$ *and* $\xi_0\neq 0$.

It follows then that the real roots of the equation in ξ_n: $Q_j(x, \xi', \xi_n)=0$ are at most double and there is at most such one root for $x_n=0$, $j=1, \cdots, r$. Let $(x^{0\prime}, \xi^{0\prime})\in T^*(\partial\Omega)\backslash 0$ be a diffractive point such that $Q_1(x^{0\prime}, 0, \xi^{0\prime}, \xi_n)=0$ has the real double root ξ_n^0 and let us restrict to a conic neighborhood of $(x^{0\prime}, 0, \xi^{0\prime}, \xi_n^0)$ in $T^*\bar{\Omega}$. Then Q_1 can be writtwn as

(1. 4) $\qquad Q_1(x, \xi) = \left(\left(\xi_n-\lambda(x, \xi')\right)^2-\mu(x, \xi')\right)\cdot(\text{nonzero factor})$,

where $\lambda(x, \xi')$ $(\mu(x, \xi'))$ is a smooth function which is analytic and homogeneous of degree one (two) in ξ' respectively and such that

$$\xi_n^0 = \lambda(x^{0\prime}, 0, \xi^{0\prime}),\ \mu(x^{0\prime}, 0, \xi^{0\prime}) = 0\,.$$

Note that $(\partial\mu/\partial\xi_0)(x^{0\prime}, 0, \xi^{0\prime})\neq 0$ since Q_1 is strictly hyperbolic. For definiteness we assume $(\partial\mu/\partial\xi_0)(x^{0\prime}, 0, \xi^{0\prime})>0$. Then, by the implicit function theorem, μ is factorized as

(1. 5) $\qquad \mu(x, \xi') = \left(\xi_0-\mu_1(x, \xi'')\right)\mu_2(x, \xi')$

with $\xi_0^0=\mu_1(x^{0\prime}, 0, \xi^{0\prime\prime})$ and $\mu_2(x^{0\prime}, 0, \xi^{0\prime})>0$.

Notice that near $\partial\Omega$ the boundary matrix A_n is of constant rank and the number of the positive eigenvalues, say, d is constant. For the boundary operator B we assume

(ii) *$B(x')$ is a $d\times m$ matrix with maximal rank and the kernel of $A_n(x', 0)$ is contained in that of $B(x')$ for each $(x', 0)\in\partial\Omega$.*

Then a Lopatinski determinant $R(x', \xi')$ of the mixed problem may be regarded as an analytic function of $z=\sqrt{\xi_0-\mu_1(x', 0, \xi'')}$ with coefficients smooth in (x', ξ''), where $\sqrt{1}=1$. Set $R(x', \xi')=\tilde{R}(x', \xi'', z)$.

We assume that $\tilde{R}(x^{0\prime}, \xi^{0\prime\prime}, z)$ is simply characteristic at $z=0$, i. e.,

(iii) $(\partial\tilde{R}/\partial z)(x^{0\prime}, \xi^{0\prime\prime}, 0)\neq 0$ *when* $\tilde{R}(x^{0\prime}, \xi^{0\prime\prime}, 0)=0$.

Now let $\tilde{R}(x^{0\prime}, \xi^{0\prime\prime}, 0)=0$. Then $\tilde{R}$ is represented as

$$\tilde{R}(x', \xi'', z)=\left(z-D(x', \xi'')\right)\tilde{R}^{(1)}(x', \xi'', z), \tag{1. 6}$$

where $\tilde{R}^{(1)}(x^{0\prime}, \xi^{0\prime\prime}, 0)\neq 0$ and $D(x', \xi'')$ is smooth in (x', ξ''), homogeneous of degree 1/2 in ξ''.

Finally we impose the following restriction on the range of $D(x', \xi'')$ which was adopted in [4].

(iv) *There are positive constants δ_1, δ_2 with $\delta_1<\pi/2$ such that*

$$\pi/2+\delta_2\leqq\arg\left(e^{i\delta_1}D(x', \xi'')\right)\leqq 3\pi/2-\delta_2\,.$$

Now the main result in this paper is

THEOREM. *Assume conditions (i) to (iv). Let $(x^{0\prime}, \xi^{0\prime})\in T^*(\partial\Omega)\backslash 0$ be an arbitrary diffractive point and let $f\in\mathcal{E}'(\partial\Omega)$, $f(x')=0$ for $x<0$ and $WF(f)$ be contained in a conic neighborhood of $(x^{0\prime}, \xi^{0\prime})$. Then there exists a parametrix for the mixed problem.*

In proving the theorem we will first find an asymptotic solution to $P(x, D)u=0$ in the (microlocal) hyperbolic region by using the phase functions constructed by Eskin [3], and then extend smoothly the solution thus obtained to the elliptic region so that the system of equations is satisfied to infinite order on the boundary as in Taylor [16]. This will enable us to solve (1. 2) in both hyperbolic and elliptic regions by a unified method. For strictly hyperbolic systems the existence of asymptotic solutions is a direct consequence of the solvability of the eikonal and transport equations and Cramer's rule, even in the elliptic region. But in the case of non-strictly hyperbolic systems it seems inadequate to use Cramer's rule in this region and hence we adopt instead an analogue to the method, due to Agranovich [1], which brings a matrix depending smoothly on several parameters to a certain block-diagonal form. (See section 3).

As is well known the transport equation is degenerate on the glancing surface. In the case of strictly hyperbolic equations or systems where the transport equations involve scalar-valued unknowns, it is known that condition (i) guarantees the solvability of the transport equations near the diffractive points. In the present article we will show that condition (i) enables us to reduce the transport equations to symmetric hyperbolic systems which are actually systems of ordinary differential equations with the bicharacteristic curves of $P_1(x,\xi)$ as the directions of differentiation.

It should be pointed out that if $R(x^{0\prime},\xi^{0\prime})=0$ and if the boundary condition is such that an incoming wave creates two or more outgoing waves when the corresponding bicharacteristic curve hits $\partial\Omega$ tangentially at $x^{0\prime}$, then we must in general take the initial data for the transport equation which depend on the boundary operator $B(x')$ so that (1.2) is solvable. (See section 6). Moreover one can construct a parametrix in the special cases where $R(x^0,\xi^{0\prime})\neq 0$ or the function $D(x',\xi'')$ in (1.6) vanishes identcally.

The plan of the paper is as follows. Section 2 contains extensions of phase functions and a construction of a basis of the null space of $P_1(x,\xi)$. In section 3 we look for an asymptotic solution to $P(x,D)u=0$ and in section 4 we solve the transport equations which involve matrix-valued unknowns. In doing so an essential role will be played by Lemma 4.1. In section 5 we solve (1.2) and complete the proof of Theorem. Finally some examples are given in section 6.

§ 2. Preliminaries

NOTATIONS. We often denote a boundary point $(x',0)\in\partial\Omega$ by x' and for instance $\partial\mu(x,\xi)/\partial x$ by $\mu_x(x,\xi)$ or $\partial_x\mu(x,\xi)$.

2.1. Let $(x^{0\prime},\xi^{0\prime})\in T^*(\partial\Omega)$ be an arbitrary fixed diffractive point with $\xi^{0\prime\prime}\neq 0$ and let ξ_n^0 be the real double root of the characteristic equation in ξ_n, say, of $Q_1(x^{0\prime},\xi^{0\prime},\xi_n)=0$. In the present article we adopt the phase functions $\theta(x,\eta')$ and $\rho(x,\eta')$ constructed in [3], where $\eta'=(\eta_0,\eta'')\in R^n$ is a new covariable such that $(\xi^{0\prime},\xi_n^0)=\theta_x(x^{0\prime},\eta^{0\prime})$ with $\eta_0^0=0$ and $\eta^{0\prime\prime}=\xi^{0\prime\prime}$.

LEMMA 2.1. ([3]). *There are real valued functions $\theta(x,\eta')$ and $\rho(x,\eta')$ defined on a conic neighborhood of $(x^{0\prime},\eta^{0\prime})$ in $\bar{\Omega}\times(R^n\backslash 0)$, smooth and homogeneous of degree* 1, 2/3 *in η' respectively, such that*

$$(2.1)\qquad \phi_\pm(x,\eta')=\theta(x,\eta')\pm\frac{2}{3}\rho(x,\eta')^{3/2}$$

are solutions of the eikonal equation

$$(2.2)\qquad \left(\phi_{x_n}-\lambda(x,\phi_{x'})\right)^2-\mu(x,\phi_{x'})=0$$

for $\rho \geqq 0$. *Moreover*

(2.3) $\qquad \det \partial^2\theta/\partial x' \partial \eta' \neq 0 \qquad$ *for* $\quad x_n = 0$,

(2.4) $\qquad \rho = \left(\alpha + O(\alpha^\infty)\right)|\eta'|^{2/3} \qquad$ *for* $\quad x_n = 0$,

where $\alpha = \eta_0/|\eta'|$,

(2.5) $\qquad \partial\rho/\partial x_n > 0 \qquad$ *for* $\quad \rho = 0$.

LEMMA 2.2. *There are smooth extensions of* $\theta|_{\alpha \geqq 0}$ *and* $\rho|_{\alpha \geqq 0}$ *to the region* $\alpha < 0$ *such that*

(2.6) $\qquad \rho(x, \eta') = \alpha|\eta'|^{2/3} \qquad$ *for* $\quad x_n = 0,\ \alpha < 0$,

and $\phi_\pm$ *satisfy* (2.2) *to infinite order on* $x_n = 0$, *i.e.*,

(2.7) $$\left(\theta_{x_n} \pm \sqrt{\rho}\,\rho_{x_n} - \lambda(x, \theta_{x'} \pm \sqrt{\rho}\,\rho_{x'})\right)^2 - \mu(x, \theta_{x'} \pm \sqrt{\rho}\,\rho_{x'}) = 0 \quad \textit{for}$$
$$\rho \geqq 0, \quad = O(x_n^\infty) \quad \textit{as } x_n \to +0 \textit{ for } \alpha < 0.$$

Such an extension has given in [16] when $Q_1(x, \xi)$ is of the second order.

PROOF OF LEMMA 2.2. Since $\mu(x, \xi')$ is analytic in ξ', $\mu(x, \theta_{x'} \pm \sqrt{\rho}\,\rho_{x'})$ may be written as

$$\mu(x, \theta_{x'} \pm \sqrt{\rho}\,\rho_{x'}) = \mu^{(1)} \pm \sqrt{\rho}\,\mu^{(2)},$$

where

$$\mu^{(1)} = \sum_{j=0}^{\infty} \sum_{|\beta|=2j} \rho^j (\rho_{x'})^\beta (\partial_{\xi'}{}^\beta \mu)(x, \theta_{x'})/\beta!,$$

$$\mu^{(2)} = \sum_{j=0}^{\infty} \sum_{|\beta|=2j+1} \rho^j (\rho_{x'})^\beta (\partial_{\beta'}{}^\beta \mu)(x, \theta_{x'})/\beta!.$$

Analogously

$$\lambda(x, \theta_{x'} \pm \sqrt{\rho}\,\rho_{x'}) = \lambda^{(1)} \pm \sqrt{\rho}\,\lambda^{(2)}.$$

Hence the left side of (2.7) is written as

$$\left(\theta_{x_n} - \lambda^{(1)} \pm \sqrt{\rho}\,(\rho_{x_n} - \lambda^{(2)})\right)^2 - (\mu^{(1)} \pm \sqrt{\rho}\,\mu^{(2)}).$$

Thus for $\rho \geqq 0$ (2.7) is equivalent to the pair:

(2.8) $\qquad (\theta_{x_n} - \lambda^{(1)})^2 + \rho(\rho_{x_n} - \lambda^{(2)})^2 = \mu^{(1)}$,

(2.9) $\qquad 2(\theta_{x_n} - \lambda^{(1)}) \cdot (\rho_{x_n} - \lambda^{(2)}) = \mu^{(2)}$.

We shall show, following [16], that θ and ρ restricted to $\alpha \geqq 0$ can be extended to $\alpha < 0$ so that (2.8) and (2.9) as well as (2.6) are satisfied to infinite order on $\partial\Omega$. To this end it suffices to specify θ, ρ and all their normal derivatives on $\partial\Omega$, because Whitney's extension theorem then allows us to extend these quantities smoothly to the whole region $\bar{\Omega}$ intersected with a small open set α close to zero. In what follows we restrict ourselves to $\partial\Omega$ and $|\eta'|=1$.

Define ρ by (2.6) and take θ an arbitrary extension. Then ρ is smooth by virtue of (2.4), and for $\alpha<0$ we have $\mu^{(1)}=\mu(x', \theta_{x'})$, $\mu^{(2)}=0$ and $\lambda^{(1)}=\lambda(x', \theta_{x'})$, $\lambda^{(2)}=0$. Therefore, if we define

$$\theta_{x_n}(x', \eta')=\lambda^{(1)}=\lambda(x', \theta_{x'}) \qquad \text{for} \quad \alpha<0, \tag{2.10}$$

then (2.9) holds. Note that

$$\theta_{x_n}=\lambda^{(1)}+O(\alpha^{\infty})=\lambda(x', \theta_{x'})+O(\alpha^{\infty}) \qquad \text{for} \quad \alpha\geqq 0 \tag{2.11}$$

since $\rho_{x_n}>0$ and $\rho_{x'}=O(\alpha^{\infty})$, so θ_{x_n} is smooth near $\alpha=0$. Next we shall specify ρ_{x_n} for $\alpha<0$. From (2.8), (2.11) and (2.4) we have

$$\mu(x', \theta_{x'})=\alpha(\rho_{x_n})^2+O(\alpha^{\infty}) \qquad \text{for} \quad \alpha\geqq 0. \tag{2.12}$$

Therefore, setting

$$\mu(x', \theta_{x'})=\alpha\mu_3(x', \eta'), \tag{2.13}$$

we see that μ_3 is smooth and positive near $\alpha=0$. We now define

$$\rho_{x_n}(x', \eta')=\sqrt{\mu_3(x', \eta')} \qquad \text{for} \quad \alpha<0.$$

Then ρ_{x_n} is smooth near $\alpha=0$, and (2.8) holds according to (2.6), (2.10) and (2.13).

In order to specify the normal derivatives of higher order we assume inductively that $(\partial/\partial x_n)^{j+1}\theta$, $(\partial/\partial x_n)^{j+1}\rho$, $j=0, 1, \cdots, q-1$, are given to be smooth so that on $\partial\Omega$ the j-th normal derivatives of the left sides of (2.8) and (2.9) equal those of the right sides respectively. Differentiating both sides of (2.9) q times with respect to x_n we have

$$(\rho_{x_n}-\lambda^{(2)})\left(\frac{\partial}{\partial x_n}\right)^q(\theta_{x_n}-\lambda^{(1)})+(\theta_{x_n}-\lambda^{(1)})\left(\frac{\partial}{\partial x_n}\right)^q(\rho_{x_n}-\lambda^{(2)})=\Phi,$$

where Φ does not contain normal derivatives of θ, ρ of order $q+1$. Since $\rho_{x_n}-\lambda^{(2)}\neq 0$ and the second term on the left side vanishes by (2.10), we find that $(\partial^{q+1}\theta/\partial x_n^{q+1})|_{\alpha<0}$ is represented in terms of the normal derivatives of θ and ρ of order up to q. Similarly from (2.8) we have

$$\rho(\rho_{x_n}-\lambda^{(2)})(\partial/\partial x_n)^{q+1}\rho=\Psi,$$

where Ψ is of $O(\alpha)$ and involves the normal derivatives of θ, ρ of order up to $q+1$, q respectively. Since $\rho_{x_n}-\lambda^{(2)}\neq 0$, $(\partial^{q+1}\rho/\partial x_n^{q+1})|_{\alpha<0}$ is represented in terms of θ, ρ of order up to $q+1$, q respectively. Thus the lemma is proved.

2.2. In proving Theorem we may assume without loss of generality that A_0 is the identity matrix and that A_n is of the form

$$(2.14)\qquad A_n(x)=\begin{bmatrix} A(x) & 0 \\ 0 & 0 \end{bmatrix} \quad \text{near} \quad \partial\Omega,$$

where $A(x)$ is a nonsingular $2d\times 2d$ matrix which has d positive and d negative eigenvalues respectively according to condition (i). Hence we write

$$P_1(x,\xi)=\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\xi_n+\begin{bmatrix} A_{\mathrm{I\,I}}(x,\xi') & A_{\mathrm{I\,II}}(x,\xi'') \\ A_{\mathrm{II\,I}}(x,\xi'') & A_{\mathrm{II\,II}}(x,\xi') \end{bmatrix}$$

where $A_{\mathrm{I\,I}}(A_{\mathrm{II\,II}})$ is a square matrix of order $2d(m-2d)$ respectively. Note that $\det A_{\mathrm{II\,II}}(x^{0\prime},\xi^{0\prime})\neq 0$. Set

$$M(x,\xi')=-A^{-1}(A_{\mathrm{I\,I}}-A_{\mathrm{I\,II}}A_{\mathrm{II\,II}}^{-1}A_{\mathrm{II\,I}})(x,\xi').$$

Then we have

$$(2.15)\qquad P_1(x,\xi)=\begin{bmatrix} A(\xi_n I_{2d}-M) & A_{\mathrm{I\,II}}A_{\mathrm{II\,II}}^{-1} \\ 0 & I_{m-2d} \end{bmatrix}\begin{bmatrix} I_{2d} & 0 \\ A_{\mathrm{II\,I}} & A_{\mathrm{II\,II}} \end{bmatrix},$$

where I_k denotes the identity matrix of order k. Therefore it follows that $\det(\xi_n I_{2d}-M(x,\xi'))=Q_1(x,\xi)^{m_1}\cdot$(nonzero factor) for $(x,\xi)\in T^*\bar{\Omega}$ near $(x^{0\prime},\xi^{0\prime})$. Moreover it is convenient to bring the matrix $M(x,\xi')$ to a normal block-diagonal form.

LEMMA 2.3. *There exists a nonsingular smooth matrix $S(x,\xi')$ defined on a conic neighborhood of $(x^{0\prime},\xi^{0\prime})$ in $\bar{\Omega}\times R^n$, analytic and homogeneous of degree zero in ξ', such that*

$$(2.16)\qquad MS=S\tilde{M},\ \tilde{M}=\begin{bmatrix} \tilde{M}_d & & 0 \\ & \tilde{M}_h & \\ 0 & & \tilde{M}_e \end{bmatrix},\ \tilde{M}_d=\begin{bmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_{m_1} \end{bmatrix},$$

where

$$(2.17)\qquad M_j(x,\xi')=\begin{bmatrix} \lambda(x,\xi') & 1 \\ \mu(x,\xi') & \lambda(x,\xi') \end{bmatrix}$$

$$\textit{for}\quad |\xi'|=1 \ \textit{and}\ j=1,\cdots,m_1,$$

the eigenvalues of $\tilde{M}_h$ or $\tilde{M}_e$ are real semisimple or nonreal respectively.

This is a variant of [1], Theorem 1.2 when $A_n(x)$ is non-singular, and can be proved in a way similar to [1] except that A_n may be singular in our case.

PROOF OF LEMMA 2.3. Set $S=[\tilde{S}_d, \tilde{S}_h, \tilde{S}_e]$ and $\tilde{S}_d=[s_1, \cdots, s_{2m_1}]$. We shall first construct $\tilde{S}_d$. There is a smooth $m\times 2d$ matrix $V(x, \xi'', \xi_n)$ with maximal rank, analytic in (ξ'', ξ_n), such that

$$P_1(x, \xi)\, V = VD(x, \xi)\,, \tag{2.18}$$

where

$$D(x, \xi) = \xi_0 I_{2d} - \begin{bmatrix} \overbrace{\tau_1 \cdots \tau_1}^{m_1} & & & & \\ & \tau_2 & & & \\ & & \ddots & & \\ & & & & \tau_q \end{bmatrix},$$

and $\tau_j(x, \xi'', \xi_n)$, $j=1, \cdots, q$, are the mutually distinct roots of the equation in ξ_0: $Q_1(x, \xi)\cdots Q_r(x, \xi)=0$ with $\tau_1(x^{0\prime}, \xi^{0\prime\prime}, \xi_n^0)=\xi_0^0$. Set $V=\begin{bmatrix} V_{\mathrm{I}} \\ V_{\mathrm{II}} \end{bmatrix}$ with V_{I} the $2d\times 2d$ block. Then (2.18) is written, by (2.15), as

$$(\xi_n I_{2d} - M)\, V_{\mathrm{I}} + A^{-1} A_{\mathrm{I\,II}} A_{\mathrm{II\,II}}^{-1}(A_{\mathrm{II\,I}} V_{\mathrm{I}} + A_{\mathrm{II\,II}} V_{\mathrm{II}}) = A^{-1} V_{\mathrm{I}} D$$

with

$$A_{\mathrm{II\,I}} V_{\mathrm{I}} + A_{\mathrm{II\,II}} V_{\mathrm{II}} = V_{\mathrm{II}} D\,. \tag{2.19}$$

Note that $A_{\mathrm{II\,II}}(x, \xi') - (\xi_0 - \tau_j(x, \xi'', \xi_n))\, I_{m-2d}$ is nonsignular for $j=1, \cdots, q$. Therefore V_{II} is linearly dependent on V_{I} and hence V_{I} is nonsingular. Thus, with V_{II} defined by (2.19), (2.18) is equivalent to

$$\left(\xi_n I_{2d} - M(x, \xi')\right) V_{\mathrm{I}} = A^{-1}(V_{\mathrm{I}} - A_{\mathrm{I\,II}} A_{\mathrm{II\,II}}^{-1} V_{\mathrm{II}})\, D\,. \tag{2.20}$$

Suppose for instance that the uppermost left $m_1\times m_1$ block of V_{I} is nonsingular. We shall then define $s_1, \cdots, s_{2m_1}$ by

$$v_{2(j-1)+k}(x, \xi', y) = \int_\Gamma e^{iyz}\, V_{\mathrm{I}}(x, \xi'', z)\, e_j \frac{(z-\lambda(x, \xi'))^{2-k}}{(z-\lambda(x, \xi'))^2 - \mu(x, \xi')}\, dz \tag{2.21}$$

and

$$s_{2(j-1)+k}(x, \xi') = v_{2(j-1)+k}(x, \xi', 0), \quad j=1, \cdots, m_1, \ k=1, 2,$$

where $\{e_1, \cdots, e_{2d}\}$ is the canonical basis of R_{2d} and Γ is a closed Jorden curve enclosing ξ_n^0 only of the roots of $\det P_1(x^{0\prime}, \xi^{0\prime}, \xi_n)=0$. Notice that

$$\xi_0 - \tau_1(x, \xi'', \xi_n) = (\xi_n - \lambda(x, \xi')^2 - \mu(x, \xi'))\cdot(\text{nonzero factor})\,,$$

for (x, ξ) near $(x^{0\prime}, \xi^0)$ according to (1.4), since $\xi_0=\tau_1(x, \xi'', \xi_n)$ is a root of $Q_1(x, \xi)=0$. Hence from (2.20) and (2.21) we have

$$\left(-i\partial/\partial y - M(x,\xi')\right) v_l(x,\xi',y) = 0 \qquad \text{for} \quad y \geqq 0,\ l = 1, \cdots, 2m_1 .$$

Setting $y=0$ we obtain

$$Ms_{2(j-1)+1} = \lambda s_{2(j-1)+1} + \mu s_{2(j-1)+2}\,,$$

$$Ms_{2(j-1)+2} = s_{2(j-1)+1} + \lambda s_{2(j-1)+2}, \quad j = 1, \cdots, m_1$$

which means $M\tilde{S}_d = \tilde{S}_d \tilde{M}_d$.

Next we shall show that $v_1(y), \cdots, v_{2m_1}(y)$ are linearly independent. Suppose that this is not the case at fixed point (x, ξ'). Then there are polynomials $R_j(z)$, $j=1, \cdots, m_1$, of degree one such that

$$\text{(2. 22)} \qquad \sum_{j=1}^{m_1} \oint_\Gamma e^{iyz} V_0(z)\, e_j \frac{R_j(z)}{(z-\lambda)^2 - \mu} dz = 0 \qquad \text{for} \quad y \geqq 0\,,$$

where V_0 is the uppermost left $m_1 \times m_1$ block of V_{I} and e_j are regarded as m_1-vectors. Let z_1 and z_2 be the roots of $(z-\lambda)^2 - \mu = 0$. Then, without destroying (2. 22), we can replace $V_0(z)$ by a corresponding Lagrange interporation polynomial

$$P_0(z) = V_0(z_1) + \frac{z-z_1}{z_2-z_1}\left(V_0(z_2) - V_0(z_1)\right).$$

Applying the differential operator $(\operatorname{cof} P_0)(-i\partial/\partial y)$ to the left side of (2. 22) we have

$$\sum_{j=1}^{m_1} \oint_\Gamma \left(\det P_0(z)\right) e_j \frac{R_j(z)}{(z-\lambda)^2 - \mu} dz = 0\,,$$

which leads to a contradiction, since

$$\det P_0(z_l) = \det V_0(z_l) \neq 0 \qquad \text{for} \quad l = 1, 2\,.$$

Now $\tilde{S}_h$ and $\tilde{S}_e$ with the required property can be constructed as usual. This proves the lemma.

By means of (2. 15) and Lemma 2. 3 we obtain a basis of the null space of $P_1(x, \xi)$ which is very convenient. In fact, define an $m \times m_1$ matrix $W(x, \xi)$ by

$$\text{(2. 23)} \qquad W(x,\xi) = \begin{bmatrix} I_{2d} \\ -A_{\mathrm{II\,II}}^{-1} A_{\mathrm{II\,I}} \end{bmatrix} (x,\xi')\, S(x,\xi')\, \tilde{W}_{\mathrm{I}}(x,\xi)$$

with

$$\tilde{W}_{\mathrm{I}}(x,\xi) = [e_1, e_3, \cdots, e_{2m_1-1}] + \left(\xi_n - \lambda(x,\xi')\right) |\xi'|^{-1} [e_2, e_4, \cdots, e_{2m_1}]\,,$$

where $\{e_1, e_2, \cdots, e_{2d}\}$ is the canonical basis of R^{2d}. Then W is of maximal rank and satisfies

(2.24)
$$
\begin{aligned}
&P_1(x,\xi)\,W(x,\xi)\\
&=\begin{bmatrix} I_{2d}\\ 0\end{bmatrix} A(x)\,S(x,\xi')\left(\xi_n I_{2d}-\widetilde{M}(x,\xi')\,\widetilde{W}_{\mathrm{I}}(x,\xi)\right)\\
&=\begin{bmatrix} I_{2d}\\ 0\end{bmatrix} A(x)\,S(x,\xi')\,[e_2,e_4,\cdots,e_{2m_1}]\cdot\left(\left(\xi_n-\lambda(x,\xi')\right)^2-\mu(x,\xi')\right)/|\xi'|\,.
\end{aligned}
$$

2.3. In the present article we use one of Airy functions defined by

(2.25)
$$
\begin{aligned}
A(s)&=2\pi e^{\pi i/3}A_i(se^{\pi i/3})\\
&=\int_{\Gamma} e^{i(k^3/3-sk)}\,dk\,,
\end{aligned}
$$

where L is a path running from $\infty\cdot e^{-\pi i/2}$ to $\infty\cdot e^{\pi i/6}$. The following asymptotic formula which is valid uniformly in a sector $-\pi+\delta\leqq\arg z\leqq\pi-\delta$ with arbitrary $\delta>0$, is given in [2].

(2.26)
$$A_i(z)=\Phi(z)\,e^{-2z^{3/2}/3}$$

with

$$\Phi(z)\sim\frac{1}{2\sqrt{\pi}}z^{-1/4}\left(1+O(z^{-3/2})\right)\qquad\text{as}\quad |z|\to\infty\,,$$

which implies

$$\Phi'(z)=O(z^{-1-1/4})\qquad\text{as}\quad |z|\to\infty\,.$$

Therefore we have

(2.27)
$$\frac{A'(s)}{A(s)}=\begin{cases}-is^{1/2}(1+O(s^{-3/2})) & \text{as}\quad s\to\infty\,,\\ -|s|^{1/2}(1+O(|s|^{-3/2})) & \text{as}\quad s\to-\infty\,.\end{cases}$$

Notice that $A(s)$ and $A'(s)$ do not vanish for s real.

Define a symbol K by

(2.28)
$$K(\eta')=\left(\frac{A'}{A}\right)(\alpha|\eta'|^{2/3})=\left(\frac{A'}{A}\right)(\eta_0|\eta'|^{-1/3})\,.$$

It follows (2.25) and (2.26) that $(A'/A)(s)\in S^{1/2}_{1,0}(R^1)$, as pointed out in [16]. Therefore $K(\eta')\in S^{1/3}_{1/3,0}$. More precisely we obtain the following estimates which will be used in § 5.

LEMMA 2.4. *There are positive constants C_1 and C_2 such that*

(2.29)
$$C_1(1+|\alpha|\cdot|\eta'|^{2/3})^{1/2}\leqq|K(\eta')|\leqq C_2(1+|\alpha|\cdot|\eta'|^{2/3})^{1/2}\,.$$

Furthermore $\partial_{\eta'}K\in S^{-1/3}_{1/3,0}$ and for every j, β there are constants $C_{j,\beta}$ such that

(2.30)
$$\left|\partial^j_{\eta_0}\partial^\beta_{\eta''}K(\eta')\right|\leqq C_{j,\beta}|\eta'|^{-|\beta|-j/3}\left|K(\eta')\right|^{1-2j}\,.$$

COROLLARY 2. 5. *For real number q let $|K|^q$ be the pseudo-differential operator with symbol $|K(\eta')|^q$ and let $a(y', \eta') \in S^0_{1/3,0}$. Then the symbol of the commutator $[a, |K|^q]$ is such that*

$$\sigma\left([a, |K|^q]\right) \in S^{-1/3}_{1/3,0} \quad \textit{if} \quad q \leqq 2 . \tag{2. 31}$$

Moreover

$$\sigma\left([a, |K|^q]\right) \in S^{-2/3}_{1/3,0} \quad \textit{if} \quad q \leqq 1 \textit{ and } a = O(\alpha) . \tag{2. 32}$$

PROOF. From (2. 29) and (2. 30) we have

$$\left|\partial^\beta_{\eta'} |K|^q\right| \leqq \text{const.}\ |\eta'|^{-|\beta|/3} |K|^{q-2|\beta|}$$

for each q and β. Hence $|K|^q$ belongs to $S^{q/3}_{1/3,0}$ for $q \geqq 0$, to $S^0_{1/3,0}$ for $q<0$ and $\sigma([a, |K|^q])$ has the asymptotic expansion

$$\sum_{|\beta| \geqq 1} (D^\beta_{y'} a) \left(\partial^\beta_{\eta'} |K|^q\right) / \beta ! .$$

Moreover $\partial^\beta_{\eta'} |K|^q \in S^{-|\beta|/3}_{1/3,0}$ for $|\beta| \geqq 1$ and $q \leqq 2$. Therefore we obtain (2. 31). To derive (2. 32) it suffices to note that for some constant $C>0$

$$|\alpha| \cdot \left|K(\eta')\right|^{-1} \leqq C |\alpha|^{1/2} |\eta'|^{-1/3} . \tag{2. 33}$$

§ 3. Asymptotic solutions

Let $(x^{0'}, \xi^{0'}) \in T^*(\partial\Omega)$ and ξ^0_n be the same point and root as in the preceding section, and let θ, ρ be so extended phase functions as in Lemma 2. 2. We shall look for a solution u to (1. 1), (1. 2) and (1. 3) in the form $u = Gv \in \mathscr{D}'(\Omega)$ with $v = {}^t(v_1, \cdots, v_d) \in \mathscr{E}'(R^n)$ and

$$(Gv)(x) = \sum_{j=1}^{3} G^{(j)} v^{(j)} , \tag{3. 1}$$

where $G^{(2)}$ is a Fourier integral operator with classical symbol which corresponds to the (microlocal) hyperbolic part of P_1, and $G^{(3)}$ is a classical pseudodifferential operator corresponding to the elliptic part. $G^{(1)}$ is of the form

$$(G^{(1)} v^{(1)})(x) = \int_{R^n} \left\{ \int_L e^{i\phi(x,\eta',k)} a(x, \eta', k)\, dk \right\} \frac{\chi(\eta')}{A(\alpha|\eta'|^{2/3})} \hat{v}^{(1)}(\eta')\, d\eta . \tag{3. 2}$$

Here $\phi(x, \eta', k) = \theta(x, \eta') - k\rho(x, \eta') + k^3/3$, L is the path in (2. 25), a has asymptotic expension $a \sim \sum_{j=-\infty}^{0} a_j$, $a_j(x, \eta', k)$ are polynomials in k whose coefficients are $m \times m_1$ matrices smooth in (x, η') and which are homogeneous of degree j in the sense: $a_j(x, t\eta', t^{1/3}k)=t^j a_j(x, \eta', k)$ for $t>0$, $v^{(1)}={}^t(v_1, \cdots, v_{m_1})$, $\hat{v}^{(1)}$ is the Fourier transform of $v^{(1)}$, $A(s)$ is the function defined by (2. 25), and χ is a cutoff function such that $\chi(\eta')=\chi^{(1)}(|\eta'/|\eta'|-\eta^{0\prime}/|\eta^{0\prime}||)\cdot\chi^{(2)}(|\eta'|)$, where $\chi^{(j)}\in C^\infty(R^1)$, $\chi^{(1)}(t)$ is equal to one for $|t|<\delta$ and to zero for $|t|>2\delta$ with small $\delta>0$, $\chi^{(2)}(t)$ is equal to one for $|t|>2\delta^{-1}$ and to zero for $|t|<\delta^{-1}$.

The present and following sections will be devoted to construct the amplitude $a(x, \eta', k)$ so that

$$\begin{aligned}(3.3)\qquad &\int_L P(x, D)\left(e^{i\phi(x,\eta',k)}\, a(x, \eta', k)\right) dk \\ &\quad=\int_L e^{i\phi(x,\eta',k)}\left\{\sum_{j=-\infty}^{1}\left(b_{1j}(x, \eta')+kb_{2j}(x, \eta')\right)\right\} dk\,,\end{aligned}$$

where, for every $j=1, 0, -1, -2, \cdots$, $b_{1j}(x, \eta')$ and $b_{2j}(x, \eta')$ are smooth in (x, η'), homogeneous of degree j, $j-1/3$ in η' respectively and satisfy

$$(3.4)_j \qquad b_{lj}(x, \eta')=0 \qquad \text{for} \quad \rho\geqq 0,\ l=1, 2\,,$$

$$(3.5)_j \qquad b_{lj}(x, \eta')=O(x_n^\infty) \text{ as } x_n\to+0 \qquad \text{for} \quad \alpha<0,\ l=1, 2\,,$$

which yield (1. 1) for $u=G^{(1)}v^{(1)}$. (See [16]).

To accomplish the purpose above we often use the following well known device. Consider the integral

$$\int_L e^{i\phi(x,\eta',k)}\, b(x, \eta', k)\, dk\,,$$

where $b(x, \eta', k)$ is a polynomial of k with coefficients smooth in (x, η') which is homogeneous of degree q in the sense above. Then b is represented as

$$(3.6)\qquad b(x, \eta', k)=b_1(x, \eta', \rho)+kb_2(x, \eta', \rho)+(k^2-\rho)\, b_3(x, \eta', k)\,,$$

where

$$(3.7)\qquad \begin{aligned} b_1(x, \eta', \rho)&=\frac{b(x, \eta', \sqrt{\rho})+b(x, \eta', -\sqrt{\rho})}{2}\,.\\ b_2(x, \eta', \rho)&=\frac{b(x, \eta', \sqrt{\rho})-b(x, \eta', -\sqrt{\rho})}{2\sqrt{\rho}}\,,\end{aligned}$$

Note that $b_3(x, \eta', k)$ is homogeneous of degree $q-2/3$ and $b_l(x, \eta', \rho(x, \eta'))$ is smooth in (x, η'), homogeneous in η' of degree q for $l=1$, $q-1/3$ for $l=2$. Moreover, if $b(x, \eta', \pm\sqrt{\rho})=0$ for $\rho\geqq 0$ then $b_l(x, \eta', \rho)=0$ for $\rho\geqq 0$, $l=1, 2$, and if $b(x, \eta', \pm\sqrt{\rho})=O(x_n^\infty)$ for $\alpha<0$ then $b_l(x, \eta', \rho)=O(x_n^\infty)$ for $\alpha<0$, $l=1, 2$. Furthermore $(k^2-\rho)\, b_3(x, \eta', k)$ may be regarded as a term homogeneous of degree $q-1$, since $(k^2-\rho)\, e^{i\phi}=-i\partial e^{i\phi}/\partial k$ so that

$$\int_L e^{i\phi}(k^2-\rho)\, b_3\, dk=i\int_L e^{i\phi}(\partial b_3/\partial k)\, dk\,,$$

where $\partial b_3/\partial k$ is homogeneous of degree $q-1$. (See [3]).

We shall first derive $(3.4)_1$ and $(3.5)_1$. Since the amplitude on the left side in (3.3) is

$$e^{-i\phi}P(e^{i\phi}a)=P_1(x,\theta_x-k\rho_x)\,a+P(x,D)\,a\,, \tag{3.8}$$

we take a_0 in the form

$$a_0(x,\eta',k)=\left(W_1(x,\eta')-kW_2(x,\eta')\right)\left(g_0(x,\eta')-kh_0(x,\eta')\right) \tag{3.9}$$

where with the matrix $W(x,\xi)$ defined by (2.23)

$$W_1(x,\eta')=\frac{W(x,\theta_x+\sqrt{\rho}\,\rho_x)+W(x,\theta_x-\sqrt{\rho}\,\rho_x)}{2}\,,$$

$$W_2(x,\eta')=\frac{W(x,\theta_x+\sqrt{\rho}\,\rho_x)-W(x,\theta_x-\sqrt{\rho}\,\rho_x)}{2\sqrt{\rho}}$$

so that

$$W(x,\theta_x\pm\sqrt{\rho}\,\rho_x)=W_1(x,\eta')\pm\sqrt{\rho}\,W_2(x,\eta')\,, \tag{3.10}$$

$g_0(x,\eta')$ and $h_0(x,\eta')$ are smooth $m_1\times m_1$ matrices homogeneous in η' of degree 0 and $-1/3$ respectively with g_0 nonsingular. Then it follows from (2.7) and (2.24) that

$$P_1(x,\theta_x\pm\sqrt{\rho}\,\rho_x)\left(W_1(x,\eta')\pm\sqrt{\rho}\,W_2(x,\eta')\right)=0 \tag{3.11}$$
$$\text{for } \rho\geqq 0,\ =O(x_n^\infty) \text{ for } \alpha<0\,,$$

which gives $(3.4)_1$ and $(3.5)_1$.

We shall next establish $(3.4)_0$ and $(3.5)_0$. By (3.8) the relevant terms are

$$\int_L e^{i\phi}\left\{P_1(x,\theta_x-k\rho_x)\,a_{-1}+P(x,D)\,a_0\right\}dk+\int_L e^{i\phi}P_1(x,\theta_k-k\rho_x)\,a_0\,dk\,.$$

To the amplitude in the last integral we apply (3.6) with $b(x,\eta',k)=P_1(x,\theta_x-k\rho_x)(W_1-kW_2)$. Then

$$P_1(x,\theta_x-k\rho_x)\left(W_1(x,\eta')-kW_2(x,\eta')\right) \tag{3.12}$$
$$=b_1(x,\eta')+kb_2(x,\eta')+(k^2-\rho)\,b_3(x,\eta')\,,$$

where $b_l(x,\eta')$, $l=1,2$, satisfy $(3.4)_1$ and $(3.5)_1$ by virtue of (3.11), and b_3 is homogeneous of degree 1/3. Thus we need only to establish

$$P_1(x,\theta_x\pm\sqrt{\rho}\,\rho_x)\,a_{-1}(x,\eta',\mp\sqrt{\rho})+F_0(x,\eta',\mp\sqrt{\rho})=0 \tag{3.13}$$
$$\text{for } \rho\geqq 0,\ =O(x_n^\infty) \text{ as } x_n\to+0 \text{ for } \alpha<0\,,$$

where

$$(3.14) \qquad F_0(x, \eta', k) = P(x, D)\, a_0(x, \eta', k) - ib_3(x, \eta')\, h_0(x, \eta')$$

and $b_3(x, \eta') = P_1(x, \rho_x)\, W_2$.

We shall now look for a special solution $a^0_{-1}(x, \eta', \mp\sqrt{\rho})$ of (3.13) with $F_0(x, \eta', \mp\sqrt{\rho})$ regarded as given. For convenience set $\xi = \theta_x \pm \sqrt{\rho}\, \rho_x$ and

$$(3.15) \qquad a^0_{-1}(x, \eta', \mp\sqrt{\rho}) = \begin{bmatrix} a_{\mathrm{I}} \\ a_{\mathrm{II}} \end{bmatrix}, \quad F_0(x, \eta', \mp\sqrt{\rho}) = -\begin{bmatrix} F_{\mathrm{I}} \\ F_{\mathrm{II}} \end{bmatrix},$$

where a_{I} and F_{I} are the $2d \times m_1$ blocks. Then by (2.15) we have

$$P_1(x, \xi)\, a^0_{-1}(x, \eta', \mp\sqrt{\rho}) = \begin{bmatrix} A(\xi_n I_{2d} - M)\, a_{\mathrm{I}} + A_{\mathrm{I}\,\mathrm{II}} A^{-1}_{\mathrm{II}\,\mathrm{II}}(A_{\mathrm{II}\,\mathrm{I}} a_{\mathrm{I}} + A_{\mathrm{II}\,\mathrm{II}} a_{\mathrm{II}}) \\ A_{\mathrm{II}\,\mathrm{I}} a_{\mathrm{I}} + A_{\mathrm{II}\,\mathrm{II}} a_{\mathrm{II}} \end{bmatrix}.$$

Hence, if we define a_{II} by

$$(3.16) \qquad a_{\mathrm{II}} = A^{-1}_{\mathrm{II}\,\mathrm{II}}(F_{\mathrm{II}} - A_{\mathrm{II}\,\mathrm{I}} a_{\mathrm{I}}),$$

then the equation $P_1(x, \xi)\, a^0_{-1} + F_0 = 0$ is equivalent to

$$(3.17) \qquad A(\xi_n I_{2d} - M)\, a_{\mathrm{I}} = F_{\mathrm{I}} - A_{\mathrm{I}\,\mathrm{II}} A^{-1}_{\mathrm{II}\,\mathrm{II}} F_{\mathrm{II}},$$

which becomes, by (2.16),

$$(\xi_n I_{2d} - \widetilde{M})\, S^{-1} a_{\mathrm{I}} = S^{-1} A^{-1}(F_{\mathrm{I}} - A_{\mathrm{I}\,\mathrm{II}} A^{-1}_{\mathrm{II}\,\mathrm{II}} F_{\mathrm{II}}).$$

Moreover set

$$(3.18) \qquad a_{\mathrm{I}} = S \begin{bmatrix} \tilde{a}_d \\ \tilde{a}_h \\ \tilde{a}_e \end{bmatrix}, \quad S^{-1} A^{-1}(F_{\mathrm{I}} - A_{\mathrm{I}\,\mathrm{II}} A^{-1}_{\mathrm{II}\,\mathrm{II}} F_{\mathrm{II}}) = \begin{bmatrix} \tilde{F}_d \\ \tilde{F}_h \\ \tilde{F}_e \end{bmatrix},$$

where $\tilde{a}_d$ and $\tilde{F}_d$ are $2m_1 \times m_1$ matrices. Then (3.17) is written as

$$(3.19) \qquad (\xi_n I_{2m_1} - \widetilde{M}_d)\, \tilde{a}_d = \tilde{F}_d,$$

$$(3.20) \qquad \left(\xi_n I_{2d-2m_1} - \begin{bmatrix} \widetilde{M}_h & 0 \\ 0 & \widetilde{M}_e \end{bmatrix}\right) \begin{bmatrix} \tilde{a}_h \\ \tilde{a}_e \end{bmatrix} = \begin{bmatrix} \tilde{F}_h \\ \tilde{F}_e \end{bmatrix}.$$

Since $\xi_n I_{2d-2m_1} - \begin{bmatrix} \widetilde{M}_h & 0 \\ 0 & \widetilde{M}_e \end{bmatrix}(x, \xi')$ is nonsingular, (3.20) is uniquely solvable. On the other hand the rank of $\xi_n I_{2m_1} - \widetilde{M}_d(x, \xi')$ is equal to m_1 when $\rho \geqq 0$. Set now

$$(3.21) \qquad \tilde{L}_d(x, \xi) = \big(\xi_n - \lambda(x, \xi')\big)\, {}^t[e_1, e_3, \cdots, e_{2m_1-1}] + {}^t[e_2, e_4, \cdots, e_{2m_1}],$$

where $\{e_1, e_2, \cdots, e_{2m_1}\}$ is the canonical basis of R^{2m_1}. Then $\tilde{L}_d(x, \xi)$ is of rank m_1 and its rows are, by (2.17), left null vectors of $\xi_n I_{2m_1} - \widetilde{M}_d(x, \xi')$ for $\rho \geqq 0$.

We shall define a solution $\tilde{a}_d$ of (3.19) by

(3.22) $$\tilde{a}_d = -{}^t[0, F_1, 0, F_3, \cdots, 0, F_{2m_1-1}]\,,$$

where $\tilde{F}_d = {}^t[F_1, F_2, \cdots, F_{2m_1}]$. Then by (2.17) and (3.21) we obtain

$$\left(\xi_n I_{2m_1} - \widetilde{M}_d(x, \xi')\right)\tilde{a}_d - \tilde{F}_d = -{}^t[0, H_1, 0, H_2, \cdots, 0, H_{m_1}]\,,$$

where $\tilde{L}_d \tilde{F}_d = {}^t[H_1, H_2, \cdots, H_{m_1}]$. With a_{-1}^0 thus defined, the left side of (3.13) is dominated by $\tilde{L}_d \tilde{F}_d$. Moreover we shall show that the latter is estimated by $W^*(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\, F_0(x, \eta', \pm\sqrt{\rho})$.

Let $\rho \geqq 0$ and set $W_{\mathrm{I}}(x, \xi) = S(x, \xi')\,\widetilde{W}_{\mathrm{I}}(x, \xi)$, where $\widetilde{W}_{\mathrm{I}}$ is the matrix in (2.23). Then from (2.15) and (3.11) we have

$$\begin{bmatrix} A(\xi_n I_{2d} - M)\, W_{\mathrm{I}} \\ 0 \end{bmatrix} = P_1(x, \xi)\, W(x, \xi) = 0\,.$$

Since $A(\xi_n I_{2d} - M)$ is hermitian and (2.23) yields $W^* = W_{\mathrm{I}}^* [I_{2d}, -A_{\mathrm{I\,II}} A_{\mathrm{II\,II}}^{-1}]$, it follows that $W_{\mathrm{I}}^* A(\xi_n I_{2d} - M) = 0$ and hence the rows of $W_{\mathrm{I}}^* AS$ are left null vectors of $\xi_n I_{2d} - \widetilde{M}$ according to (2.16). On the other hand, the rows of the $m_1 \times 2d$ matrix $[\tilde{L}_d(x, \xi), 0]$ is also left null vectors of $\xi_n I_{2d} - \widetilde{M}$. Hence there is a (nonsingular) $m_1 \times m_1$ matrix $T(x, \eta')$ such that $[\tilde{L}_d, 0] = TW_{\mathrm{I}}^* AS$. Furthermore it follows from (2.23) and (3.15) that $W^* F_0 = -W_{\mathrm{I}}^*(F_{\mathrm{I}} - A_{\mathrm{I\,II}} A_{\mathrm{II\,II}}^{-1} F_{\mathrm{II}})$. Therefore by (3.18) we obtain $\tilde{L}_d \tilde{F}_d = -TW^* F_0$ as desired. Thus the left side of (3.13) is estimated by $W^*(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\, F_0(x, \eta', \mp\sqrt{\rho})$ not only for $\rho \geqq 0$ but also for $\rho < 0$ by continuity. Summing up we have proved

PROPOSITION 3.1. *Define $a_{-1}^0(x, \eta', \mp\sqrt{\rho})$ by* (3.15), (3.16), (3.18), (3.20) *and* (3.22). *Then the left hand side of* (3.13) *with a_{-1} replaced by a_{-1}^0 is estimated by $W^*(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\, F_0(x, \eta', \mp\sqrt{\rho})$.*

Thus, if we define a_{-1} by

$$\begin{aligned} a_{-1}(x, \eta', k) &= a_{-1}^{(1)}(x, \eta') + k a_{-1}^{(2)}(x, \eta') \\ &\quad + \left(W_1(x, \eta') - kW_2(x, \eta')\right)\left(g_{-1}(x, \eta') - kh_{-1}(x, \eta')\right), \end{aligned}$$

where g_{-1}, h_{-1} are homogeneous of degree -1, $-1-1/3$ in η' respectively and $a_{-1}^{(j)}(x, \eta')$, $j = 1, 2$, are the functions $b_j(x, \eta', \rho(x, \eta'))$ defined by (3.7) with $b(x, \eta', k) = a_{-1}^0(x, \eta', k)$, then (3.13) is reduced to the transport equations for g_0 and h_0:

(3.23) $$\begin{aligned} &W^*(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\, F_0(x, \eta', \mp\sqrt{\rho}) = 0 \\ &\qquad \text{for } \rho \geqq 0, \ = O(x_n^\infty) \text{ as } x_n \to +0 \text{ for } \alpha < 0\,, \end{aligned}$$

which will be solved in the following section. Analogously for $j = -1, -2, \cdots$ we can establish

$$P_1(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\, a_{j-1}(x, \eta', \mp\sqrt{\rho}) + F_j(x, \eta', \mp\sqrt{\rho})$$
$$= O\Big(W^*(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\, F_j(x, \eta', \mp\sqrt{\rho})\Big),$$

where

$$F_j(x, \eta', k) = P(x, D)\, a_j - iP_1(x, \rho_x)\, W_2 h_j,$$

and solve the transport equations for g_j and h_j:

$W^*(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\, F_j(x, \eta', \mp\sqrt{\rho}) = 0$ for $\rho \geqq 0$, $= O(x_n^\infty)$ as $x_n \to +0$ for $\alpha < 0$.

§ 4. Transport equations

In this section we shall look for $g_0(x, \eta')$ and $h_0(x, \eta')$ satisfying (3. 23). From (3. 9) and (3. 14) we have

$$(4.1) \qquad F_0(x, \eta', k) = \sum_{j=0}^{n} A_j(x)\,(W_1 - kW_2)\cdot(D_j g_0 - kD_j h_0) - ib_3 h_0$$
$$+ \Big(P(x, D)\,(W_1 - kW_2)\Big)\,(g_0 - kh_0).$$

Moreover, by (3. 10) and (3. 12), b_3 can be written as

$$b_3(x, \eta') = \pm\frac{1}{2\sqrt{\rho}} P_1(x, \rho_x)\, W(x, \theta_x \pm \sqrt{\rho}\,\rho_x)$$
$$\pm\frac{1}{2\sqrt{\rho}}\Big(b_2(x, \eta') + P_1(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\, W_2\Big).$$

Therefore

$$(4.2) \qquad F_0(x, \eta', \mp\sqrt{\rho}) = \sum_{j=0}^{n} A_j(x)\, W(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\, D_j(g_0 \pm \sqrt{\rho}\, h_0)$$
$$+ \Big(P(x, D)\,(W_1 - kW_2)\Big)\,(g_0 - kh_0)|_{k=\mp\sqrt{\rho}}$$
$$\mp\frac{i}{2\sqrt{\rho}}\Big(b_2(x, \eta') + P_1(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\, W_2\Big)\, h_0,$$

since

$$\sum_{j=0}^{n} A_j(x)\, W(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\, D_j(\pm\sqrt{\rho}) = \mp\frac{i}{2\sqrt{\rho}} P_1(x, \rho_x)\, W(x, \theta_x \pm \sqrt{\rho}\,\rho_x).$$

We shall first solve (3. 23) for $\rho \geqq 0$ and then extend the obtained g_0 and h_0 to the region $\rho < 0$ as in lemma 2. 2.

Let $\rho \geqq 0$. Then $b_2(x, \eta')$ and $(W^* P_1)(x, \theta_x \pm \sqrt{\rho}\,\rho_x)$ vanish according to (3. 10), (3. 11) and (3. 12). Thus (3. 23) becomes, by (4. 2),

$$(4.3)_\pm \qquad \sum_{j=0}^{n} A_j^\pm(x, \eta', \sqrt{\rho})\frac{\partial}{\partial x_j}(g_0 \pm \sqrt{\rho}\, h_0) + C^\pm(x, \eta', \sqrt{\rho})\,(g_0 \pm \sqrt{\rho}\, h_0) = 0\,,$$

where

$$(4.4) \qquad \begin{aligned} &A_j^\pm(x, \eta', t) = W^*(x, \theta_x \pm t\rho_x)\, A_j(x)\, W(x, \theta_x \pm t\rho_x), \quad j = 0, 1, \cdots, n\,, \\ &C^\pm(x, \eta', t) = iW^*(x, \theta_x \pm t\rho_x)\left(P(x, D)\, W(x, \theta_x \pm t\rho_x)\right). \end{aligned}$$

Furthermore it follows from (2.1), (2.2) and (2.24) that for small $\rho > 0$ the direction of differentiation in $(4.3)_+$ or $(4.3)_-$ coincides with the bicharacteristic curve of $\xi_n - \lambda(x, \xi') - \sqrt{\mu(x, \xi')}$ or $\xi_n - \lambda(x, \xi') + \sqrt{\mu(x, \xi')}$ respectively, i.e., with the incoming or outgoing bicharacteristic. (See for instance Ludwig [7]).

Let $|\eta'| = 1$. We shall show that the equations $(4.3)_\pm$ for $g_0 \pm \sqrt{\rho}\, h_0$ are uniquely solvable with data prescribed on the surface $\rho = 0$. To this end we make a change of variables $(x', x_n) \to (x', \rho)$, η' being regarded as a parameter, which is possible by virtue of (2.5). Set

$$a^\pm(x', \sqrt{\rho}, \eta') = g_0(x, \eta') \pm \sqrt{\rho}\, h_0(x, \eta')\,.$$

Then $(4.3)_\pm$ become

$$(4.5)_\pm \qquad \begin{aligned} &\left(\sum_{j=0}^{n} A_j^\pm(x, \eta', \sqrt{\rho})\,\rho_{x_j}\right)\frac{\partial a^\pm}{\partial \rho} + \sum_{j=0}^{n-1} A_j^\pm(x, \eta', \sqrt{\rho})\frac{\partial a^\pm}{\partial x_j} \\ &\quad + C^\pm(x, \eta', \sqrt{\rho})\, a^\pm = 0\,, \end{aligned}$$

where $x_n = x_n(x', \rho, \eta')$. As will be seen in (4.8) below, the coefficient of $\partial a^\pm / \partial \rho$ is singular for $\rho = 0$. So, we make once more a change of variables $(x', \rho) \to (x', t)$ by $t = \sqrt{\rho}$. Then $(4.5)_\pm$ are equivalent to

$$(4.6)_\pm \qquad C_n^\pm(x', t, \eta')\frac{\partial a^\pm}{\partial t} + \sum_{j=0}^{n-1} C_j^\pm(x', t, \eta')\frac{\partial a^\pm}{\partial x_j} + C_{n+1}^\pm(x', t, \eta')\, a^\pm = 0\,,$$

where

$$(4.7) \qquad \begin{aligned} &C_n^\pm(x', t, \eta') = \frac{1}{2t}\sum_{j=0}^{n} A_j^\pm(x, \eta', t)\,\rho_{x_j}\,, \\ &C_j^\pm(x', t, \eta') = A_j^\pm(x, \eta', t)\,, \quad j = 0, 1, \cdots, n-1\,, \\ &C_{n+1}^\pm(x', t, \eta') = C^\pm(x, \eta', t)\,. \end{aligned}$$

Notice that $a^-(x', t, \eta')$ satisfies the same equation as $a^+(x', -t, \eta')$, since by (4.4) and (4.7) $C_n^-(x', t, \eta') = -C_n^+(x', -t, \eta')$ and $C_j^-(x', t, \eta') = C_j^+(x', -t, \eta')$ for $j \neq n$. Moreover $C_j^\pm$ are hermitian for $j = 0, 1, \cdots, n$. Consequently $(4.6)_\pm$ have unique solutions smooth in (x', t, η') for t near zero provided smooth data are prescribed on $t = 0$, by virtue of

LEMMA 4.1. *$\mp(\text{sign}\,\xi_0^0)\,C_n^{\pm}(x', t, \eta')$ are smooth and positive definite for small $t \geqq 0$. Moreover*

$$W^*(x, \theta_x)\,P_1(x, \rho_x)\,W(x, \theta_x) = 0 \qquad \textit{for} \quad \rho = 0\,, \tag{4.8}$$

$$\begin{aligned} \lim_{t\to+0} C_n^{\pm}(x', t, \eta') \\ &= \mp \frac{1}{2}\,\rho_x(\partial_\xi W^*)(x, \theta_x)\,P_1(x, \theta_x)\,\rho_x(\partial_\xi W)\,(x, \theta_x)|_{\rho=0} \\ &= \pm\,\mathrm{Re}\Big\{W^*(x, \theta_x)\,P_1(x, \rho_x)\,\rho_x(\partial_\xi W)\,(x, \theta_x)\Big\}\Big|_{\rho=0}\,. \end{aligned} \tag{4.9}$$

PROOF. Let $\rho > 0$. Then from (4.4) and (4.7) we have

$$C_n^{\pm}(x', \sqrt{\rho}, \eta') = (2\sqrt{\rho})^{-1} W^*(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\,P_1(x, \rho_x)\,W(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\,,$$

and it follows from (3.10) and (3.11) that

$$\begin{aligned} &W^*(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\,P_1(x, \rho_x)\,W(x, \theta_x \pm \sqrt{\rho}\,\rho_x) \\ &\quad = \mp\sqrt{\rho}\,\rho_x(\partial_\xi W^*)\,(x, \theta_x)\,P_1(x, \theta_x)\,\rho_x(\partial_\xi W)\,(x, \theta_x) + O(\rho)\,, \end{aligned}$$

which implies (4.8) and (4.9). Therefore $C_n^{\pm}(x', t, \eta')$ are smooth in (x', t, η') with $t \geqq 0$. To prove that $\mp(\text{sign}\,\xi_0^0)\,C_n^{\pm}(x', t, \eta')$ are positive definite we suppose first $\xi_0^0 > 0$. Then it suffices to show that

$$(\partial_{\xi_n} W^*)\,(x, \theta_x)\,P_1(x, \theta_x)\,(\partial_{\xi_n} W)\,(x, \theta_x) \quad \text{is positive definite at} \quad (x, \theta_x) = (x^{0\prime}, \xi^0)\,, \tag{4.10}$$

since $C_n^{\pm}(x', t, \eta')$ are hermitian and $\rho_{x'}(x^{0\prime}, \eta^{0\prime}) = 0$, $\rho_{x_n}(x^{0\prime}, \eta^{0\prime}) > 0$. To do so we need

LEMMA 4.2. *Let $\xi_0^0 > 0$ and let $\xi_n^+(x, \xi')$, $\xi_n^-(x, \xi')$ be the outgoing, incoming root respectively of $(\xi_n - \lambda(x, \xi'))^2 - \mu(x, \xi') = 0$ i.e., $\xi_n^{\pm}(x, \xi') = \lambda(x, \xi') \mp \sqrt{\mu(x, \xi')}$ with $\sqrt{1} = 1$. Then, for $\xi_0 \geqq \mu_1(x, \xi'')$, the hermitian matrix $A(x)(\xi_n^+(x, \xi') - M(x, \xi'))$ restricted to the range of the projection*

$$\prod(x, \xi') = \frac{1}{2\pi i}\oint_{C(x,\xi')} \big(z - M(x, \xi')\big)^{-1} dz$$

has m_1 positive and m_1 zero eigenvalues, where $C(x, \xi')$ is a closed Jordam curve enclosing $\xi_n^{\pm}(x, \xi')$ only of the roots of $Q_1(x, \xi', \xi_n) = 0$.

PROOF. Let (x, ξ'') be fixed. Then $\xi_n^{\pm}(\xi_0) = \xi_n^{\pm}(x, \xi')$, being regarded as functions of ξ_0 only, are simple roots of $Q_1(x, \xi_0, \xi'', \xi_n) = 0$ for $\xi_0 > \mu_1(x, \xi'')$ and hence are analytic functions of ξ_0 which can be continued up to $\xi_0 \to +\infty$. On the other hand

$$N(\xi_0) = A(x)\left(\xi_n^+(\xi_0) - M(x, \xi')\right)\prod(x, \xi')$$

may be identified with a hermitian $2m_1\times 2m_1$ matrix of rank m_1 which has m_1 zero and m_1 nonzero real eigenvalues. Thus we must only show that the nonzero eigenvalues are positive. Since all eigenvalues of $N(\xi_0)$, say, $\gamma_1(\xi_0),\cdots,\gamma_{2m_1}(\xi_0)$ are real, they can be labelled so that $\gamma_1(\xi_0)\geqq\cdots\geqq\gamma_{2m_1}(\xi_0)$ and every $\gamma_j(\xi_0)$ is a single-valued continuous function for $\xi_0\geqq\mu_1(x,\xi'')$. Therefore it suffices to show that $\gamma_1(\xi_0),\cdots,\gamma_{m_1}(\xi_0)$ are positive. For ξ_0 large enough we have

$$A(x)\left(\xi_n^+(x,\xi_0,\xi'')-M(x,\xi_0,\xi'')\right)/\xi_0=A(x)\left(\xi_n^+(x,1,0)-M(x,1,0)+O(\xi_0^{-1})\right)$$

and $M(x,1,0)=-A(x)^{-1}$.
Here we may assume without loss of generality that $A(x)$ is diagonal and hence according to condition (i)

$$A(x)\prod(x,\xi')=\begin{bmatrix}\alpha(x)\,I_{m_1} & 0\\ 0 & \beta(x)\,I_{m_1}\end{bmatrix},$$

where $\alpha(x)>0$ and $\beta(x)<0$. We then observe that $\xi_n^+(x,1,0)=-\alpha(x)^{-1}$ and $\xi_n^-(x,1,0)=-\beta(x)^{-1}$, so

$$A(x)\left(\xi_n^+(x,1,0)-M(x,1,0)\right)\prod(x,1,0)=\begin{bmatrix}0 & 0\\ 0 & (1-\alpha(x)^{-1}\beta(x))\,I_{m_1}\end{bmatrix},$$

which proves the lemma, since the eigenvalues of $A(x)\,(\xi_n^+(x,\xi')-M(x,\xi'))$ are multi-valued continuous functions of (x,ξ').

END OF PROOF OF LEMMA 4.1. We shall construct another basis $\widetilde{W}(x,\xi)$ satisfying (4.10). Let us keep using the notations in Lemma 4.2 and its proof. Let $\gamma_1,\cdots,\gamma_{m_1}$ be the positive eigenvalues of $A(x^{0\prime})\,(\xi_n^0-M(x^{0\prime},\xi^{0\prime}))\prod(x^{0\prime},\xi^{0\prime})$ and let $\tilde{h}_1^0,\cdots,\tilde{h}_{m_1}^0$ be orthonormal eigenvectors of the matrix corresponding to $\gamma_1,\cdots,\gamma_{m_1}$ respectively. For $j=1,\cdots,m_1$ set

$$h_j(x,\xi')=\left(\lambda\left(x,\mu_1(x,\xi''),\xi''\right)-M(x,\xi')\right)\tilde{h}_j(x,\xi'), \tag{4.11}$$

with $\tilde{h}_j(x,\xi')=\prod(x,\xi')\,\tilde{h}_j^0$. Then $\tilde{h}_j(x^{0\prime},\xi^{0\prime})=\tilde{h}_j^0$ and it follows from Lemma 2.3 that

$$\left(\xi_n^+(x,\xi')-M(x,\xi')\right)^2\prod(x,\xi')=0 \qquad \text{for}\quad \xi_0=\mu_1(x,\xi''). \tag{4.12}$$

Therefore $h_j(x,\xi')$ are null vectors of $(\xi_n^+(x,\xi')-M(x,\xi'))\prod(x,\xi')$ for $\xi_0=\mu_1(x,\xi'')$ and hence $h_1,\cdots,h_{m_1}$, $\tilde{h}_1,\cdots,\tilde{h}_{m_1}$ are linearly independent. We shall now set

$$\widetilde{W}(x,\xi)=\begin{bmatrix}I_{2d}\\ -A_{\mathrm{II\,II}}^{-1}A_{\mathrm{II\,I}}\end{bmatrix}(x,\xi')\,W_{\mathrm{I}}(x,\xi),$$

and seek a $2d\times m_1$ matrix $W_{\mathrm{I}}(x,\xi)$ with maximal rank in the form

$$W_{\mathrm{I}}(x,\xi)=h(x,\xi')+\tilde{h}(x,\xi')\,S\big(x,\xi'',\lambda(x,\xi')-\xi_n\big) \tag{4.13}$$

such that

$$\big(\xi_n-M(x,\xi')\big)\,W_{\mathrm{I}}(x,\xi)=0 \qquad \text{for} \quad \xi_n=\xi_n^+(x,\xi'), \tag{4.14}$$

where $h=[h_1,\cdots,h_{m_1}]$, $\tilde{h}=[\tilde{h}_1,\cdots,\tilde{h}_{m_1}]$ and $S(x,\xi'',z)$ is a smooth $m_1\times m_1$ matrix, analytic in z, and $\partial S/\partial z=I_{m_1}$ for $z=0$. In this case it follows from (2.15) that $P_1(x,\xi)\,\widetilde{W}(x,\xi)=0$ for $\xi_n=\xi_n^+(x,\xi')$ and

$$\begin{aligned}&(\partial_{\xi_n}\widetilde{W}^*)(x^{0\prime},\xi^{0\prime})\,P_1(x^{0\prime},\xi^{0\prime})\,(\partial_{\xi_n}\widetilde{W})(x^{0\prime},\xi^0)\\ &\quad=(\tilde{h}^0)^*A(x^{0\prime})\big(\xi-M(x^{0\prime},\xi^{0\prime})\big)\prod(x^{0\prime},\xi^{0\prime})\,\tilde{h}^0=\begin{bmatrix}\gamma_1 & & 0\\ & \ddots & \\ 0 & & \gamma_{m_1}\end{bmatrix},\end{aligned}$$

which will give (4.10). To assure (4.14) with (4.13) we need to solve the linear equation for S

$$\begin{aligned}&\big(\lambda(x,\xi')-\sqrt{\mu(x,\xi')}-M(x,\xi')\big)\,\tilde{h}(x,\xi')\,S(x,\xi'',\sqrt{\mu})\\ &\qquad=-\big(\lambda(x,\xi')-\sqrt{\mu(x,\xi')}-M(x,\xi')\big)\,h(x,\xi').\end{aligned} \tag{4.15}$$

It follows from (4.11) that the ranks of $((\lambda-\sqrt{\mu}-M)\,[\tilde{h},h])(x,\xi')$ and $((\lambda-\sqrt{\mu}-M)\,\tilde{h})(x,\xi')$ are equal to m_1. Therefore (4.15) is uniquely solvable so that $S(x,\xi'',\sqrt{\mu})$ is smooth in $(x,\xi'',\sqrt{\mu})$ and analytic in $\sqrt{\mu}$. Moreover the right side of (4.15) vanishes for $\mu=0$ according to (4.11) and (4.12). Hence $S(x,\xi'',\sqrt{\mu})=O(\sqrt{\mu})$. Differentiate both sides of (4.15) with respect to $\sqrt{\mu}$ and set $\mu=0$. Then, since $\xi_0=\mu_1(x,\xi'')+O(\mu)$, we have

$$\big((\lambda-M)\,\tilde{h}\big)(x,\xi')\,(\partial S/\partial\sqrt{\mu})=h(x,\xi') \qquad \text{for} \quad \mu=0\,.$$

This and (4.11) yield

$$h(x,\xi')\,(\partial S/\partial\sqrt{\mu})=h(x,\xi') \qquad \text{for} \quad \mu=0\,.$$

Thus we find that $\partial S/\partial\sqrt{\mu})=I_{m_1}$ for $\mu=0$, since $h(x,\xi')$ is of rank m_1. This completes the proof of Lemma 4.1 when $\xi_0^0>0$. The other case may be analogously treated.

Now let $a^{\pm}(x',t,\eta')$ be the solutions of $(4.6)_{\pm}$ with smooth data on $t=0$ which will be specified in the next section so that (1.2) is solvable, and let

$$\hat{g}(x',\rho,\eta')=\frac{a^+(x',\sqrt{\rho},\eta')+a^+(x',-\sqrt{\rho},\eta')}{2},$$

$$\hat{h}(x', \rho, \eta') = \frac{a^+(x', \sqrt{\rho}, \eta') - a^+(x', -\sqrt{\rho}, \eta')}{2\sqrt{\rho}}.$$

Then $\hat{g}$ and $\hat{h}$ are smooth and $\hat{g} \pm \sqrt{\rho}\,\hat{h}$ satisfy $(4.5)_\pm$, since $a^\pm(x', t, \eta')$ are smooth for t near 0 and $a^-(x', t, \eta') = a^+(x', -t, \eta')$. Hence if we define g_0 and h_0 by

$$g_0(x, \eta') = \hat{g}\left(x', \rho(x, \eta'), \eta'\right), \quad h_0(x, \eta') = \hat{h}\left(x', \rho(x, \eta'), \eta'\right),$$

then $g_0 \pm \sqrt{\rho}\, h_0$ are solutions of $(4.3)_\pm$ or (3.23) for $\rho \geqq 0$.

Next we shall extend g_0 and h_0 to the region $\rho < 0$ so that (3.23) holds. To do so we eliminate $\sqrt{\rho}$. It follows from (4.1) and (4.4) that i times the left side of (3.23) is equal to

$$\sum_{j=0}^{n} A_j^\pm(x, \eta', \sqrt{\rho})\left(\frac{\partial g_0}{\partial x_j} \pm \sqrt{\rho}\,\frac{\partial h_0}{\partial x_j}\right) + C^\pm(x, \eta', \sqrt{\rho})\,(g_0 \pm \sqrt{\rho}\, h_0)$$
$$+ W^*(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\, b_3(x, \eta')\, h_0\,.$$

Moreover we can write

$$\begin{aligned} (4.16) \qquad & A_j^\pm(x, \eta', \sqrt{\rho}) = A_j^{(1)}(x', \rho, \eta') \pm \sqrt{\rho}\, A_j^{(2)}(x', \rho, \eta'), \\ & C^\pm(x, \eta', \sqrt{\rho}) = C^{(1)}(x', \rho, \eta') \pm \sqrt{\rho}\, C^{(2)}(x', \rho, \eta'), \\ & W^*(x, \theta_x \pm \sqrt{\rho}\,\rho_x)\, b_3(x, \eta') = b_3^{(1)}(x', \rho, \eta') \pm \sqrt{\rho}\, b_3^{(2)} x', \rho, \eta'), \end{aligned}$$

where $A_j^{(l)}$, $C^{(l)}$ and $b_3^{(l)}$ are smooth in (x', ρ, η'). Therefore for $\alpha < 0$ (3.23) becomes

$$(4.17) \qquad \left(\sum_{j=0}^{n} A_j^{(1)} \rho_{x_j}\right)\frac{\partial g_0}{\partial \rho} + \rho\left(\sum_{j=0}^{n} A_j^{(2)} \rho_{x_j}\right)\frac{\partial h_0}{\partial \rho} + \Phi_1 = O(x_n^\infty),$$

$$(4.18) \qquad \left(\sum_{j=0}^{n} A_j^{(2)} \rho_{x_j}\right)\frac{\partial g_0}{\partial \rho} + \left(\sum_{j=0}^{n} A_j^{(1)} \rho_{x_j}\right)\frac{\partial h_0}{\partial \rho} + \Phi_2 = O(x_n^\infty),$$

where

$$\Phi_1 = \sum_{j=0}^{n-1}\left(A_j^{(1)}\frac{\partial g_0}{\partial x_j} + \rho A_j^{(2)}\frac{\partial h_0}{\partial x_j}\right) + C^{(1)} g_0 + (\rho C^{(2)} + b_3^{(1)})\, h_0\,,$$

$$\Phi_2 = \sum_{j=0}^{n-1}\left(A_j^{(2)}\frac{\partial g_0}{\partial x_j} + A_j^{(1)}\frac{\partial h_0}{\partial x_j}\right) + C^{(2)} g_0 + (C^{(1)} + b_3^{(2)})\, h_0\,.$$

Note that the left sides of (4.17) and (4.18) vanish for $\rho \geqq 0$. Moreover from (4.4) and (4.16) we have for $\rho = 0$

$$\sum_{j=0}^{n} A_j^{(1)}(x', \rho, \eta')\, \rho_{x_j} = W^*(x, \theta_x)\, P_1(x, \rho_x)\, W(x, \theta_x),$$

$$\sum_{j=0}^{n} A_j^{(2)}(x', \rho, \eta')\, \rho_{x_j} = 2\mathrm{Re}\left\{W^*(x, \theta_x)\, P_1(x, \rho_x)\, \rho_x(\partial_\xi W)(x, \theta_x)\right\}.$$

Therefore Lemma 4. 1 implies that $\sum_{j=0}^{n} A_j^{(2)} \rho_{x_j}$ is nonsingular and $\rho^{-1}\sum_{j=0}^{n} A_j^{(1)} \rho_{x_j}$ is smooth. Hence (4. 17) is equivalent to

$$\left(\rho^{-1}\sum_{j=0}^{n} A_j^{(1)} \rho_{x_j}\right)\frac{\partial g_0}{\partial \rho} + \left(\sum_{j=0}^{n} A_j^{(2)} \rho_{x_j}\right)\frac{\partial h_0}{\partial \rho} + \rho^{-1}\Phi_1 = O(x_n^\infty), \tag{4. 19}$$

where $\rho^{-1}\Phi_1$ is also smooth. Furthermore the matrix

$$\begin{bmatrix} \rho^{-1}\sum_{j=0}^{n} A_j^{(1)} \rho_{x_j} & \sum_{j=0}^{n} A_j^{(2)} \rho_{x_j} \\ \sum_{j=0}^{n} A_j^{(2)} \rho_{x_j} & \sum_{j=0}^{n} A_j^{(1)} \rho_{x_j} \end{bmatrix}$$

is nonsingular. We shall now extend g_0 and h_0 arbitrarily to the region $\alpha<0$, $x_n=0$. Then, for $\alpha<0$ and $x_n=0$, all derivatives of g_0 and h_0 with respect to ρ (so all normal derivatives of g_0 and h_0) are uniquely determind so that (4. 18) and (4. 19) hold, as in the proof of Lemma 2. 2. Thus we obtain the desired extensions of g_0 and h_0 to the region $\rho<0$ by Whitney's extension theorem.

§ 5. Boundary conditions

The main task in this section is to solve (1. 2). It follows from (2. 25), (2. 26), (3. 2) and (3. 9) that the boundary value of $G^{(1)} v^{(1)}$ is

$$(G^{(1)} v^{(1)})(x') = \int_{R^n} e^{i\phi(x',\eta')}\left(c_1(x', \eta') - ic_2(x', \eta')\, K(\eta')\right)\chi(\eta')\, \hat{v}^{(1)}(\eta')\, d\eta', \tag{5. 1}$$

where $K(\eta')$ is the symbol defined by (2. 28),

$$\phi(x', \eta') = \theta(x', \eta') - \frac{2}{3}\rho(x', \eta')^{3/2} + \frac{2}{3}\alpha^{3/2}|\eta'|, \tag{5. 2}$$

$c_1 \in S_{1,0}^{0}$, $c_2 \in S_{1,0}^{-1/3}$ are classical symbols such that

$$\begin{aligned} c_1(x', \eta') &= W(x', \theta_x)\, g_0 + O(\alpha) \quad \mathrm{mod}\ S_{1,0}^{-1}, \\ c_2(x', \eta') &= \rho_x(\partial_\xi W)(x', \theta_x)\, g_0 + W(x', \theta_x)\, h_0 + O(\alpha|\eta'|^{-1/3}) \ \mathrm{mod}\ S_{1,0}^{-1-1/3}, \end{aligned} \tag{5. 3}$$

and $O(\alpha|\eta'|^q)$ denotes a symbol of the form $\alpha a(x', \eta')$ with $a \in S_{1,0}^q$. Therefore the restriction $G_0^{(1)}$ of $G^{(1)}$ to $\partial\Omega$ is a Fourier integral operator whose phase function is $\phi(x', \eta') - y'\cdot\eta'$, whose amplitude is $c_1 - ic_2 K$. Moreover the canonical transformation $x' = x'(y', \eta')$, $\xi' = \xi'(y', \eta')$ associated with $G_0^{(1)}$ maps $\alpha=0$ onto the glancing surface and is locally bijective according to (2. 3), (2. 4) and (5. 2). Let Σ be a conic neighborhood of $(x^{0'}, \xi^{0'})$ containing

$WF(f)$ and $\hat{\Sigma}$ be the inverse image of Σ under the canonical transformation. The cutoff function χ is then taken so that $\chi(\eta')=1$ if $(x'(y',\eta'), \xi'(y',\eta')) \in WF(f)$ for some y' and $\chi(\eta')=0$ if $(y',\eta')\notin\hat{\Sigma}$ for all y'.

$G^{(2)}$ may be constructed by the eikonal method so that $G^{(2)}v^{(2)}$ satisfies (1.1) and is of the form

$$(G^{(2)}v^{(2)})(x)=\int_{R^n} e^{i\varphi^{(2)}}(x,\xi')\, a^{(2)}(x,\xi')\, \chi_1(\xi')\, \hat{v}^{(2)}(\xi')\, d\xi' .$$

Here $\varphi^{(2)}$ is a diagonal matrix whose elements are solutions of the initial value problems

$$\varphi_{x_n}(x,\xi')-\xi^+_{n,j}\big(x, \varphi_{x'}(x,\xi')\big)=0 \qquad \text{for} \quad x_n>0,\ j=1,\cdots,l,$$
$$\varphi(x,\xi')=x'\cdot\xi' \qquad \text{for} \quad x_n=0,$$

where $\xi^+_{n,j}(x,\xi')$ are the semisimple real roots of the equation $\det P_1(x,\xi',\xi_n)=0$ which are outgoing, i.e., $(\partial\xi^+_{n,j}/\partial\xi_0)(x^{0'},\xi^{0'})<0$ and $a^{(2)}(x,\xi')\in S^0_{1,0}$ is a classical symbol whose principal part $a^{(2)}_0(x,\xi')$ is a basis of the null space of $P_1(x,\xi',\xi^+_{n,j}(x,\xi'))$, $j=1,\cdots,l$. (See for instance [7]). The cutoff function χ_1 is taken so that $\chi_1(\xi'(y',\eta'))=1$ on $\hat{\Sigma}$.

$G^{(3)}$ may be constructed by the theory of elliptic pseudodifferential operators so that $G^{(3)}v^{(3)}$ satisfies (1.1) and its boundary value is

$$(G^{(3)}v^{(3)})(x')=\int_{R^n} e^{ix'\xi'}\, a^{(3)}(x',\xi')\, \chi_1(\xi')\, \hat{v}^{(3)}(\xi')\, d\xi' ,$$

where $a^{(3)}\in S^0_{1,0}$ is a classical symbol whose principal part $a^{(3)}_0$ is a basis of root subspace of $P_1(x',\xi',\xi^+_{n,j}(x',\xi'))$ corresponding to the roots $\xi^+_{n,j}(x',\xi')$ of $\det P_1(x',\xi',\xi_n)=0$ with $\operatorname{Im}\xi^+_{n,j}(x^{0'},\xi^{0'})>0$, $j=l+1,\cdots,d-m_1$. Recall that a Lopatinski determinant $R(x',\xi')$ of the mixed problem is defined by

$$R(x',\xi')=\det\Big(B(x')\big[W\big(x',\xi',\xi^+_n(x',\xi')\big),\ a^{(2)}_0(x',\xi'),\ a^{(3)}_0(x',\xi')\big]\Big), \tag{5.4}$$

where $\xi^+_n(x',\xi')$ is the outgoing root of $(\xi_n-\lambda(x',\xi'))^2-\mu(x',\xi')=0$.

We shall now solve (1.2). Let Φ be the elliptic Fourier integral operator with the same phase function as $G^{(1)}_0$ whose amplitude is equal to one, and let Φ^{-1} be an elliptic Fourier integral operator with canonical transformation inverting that associated to $G^{(1)}_0$ such that $\Phi\Phi^{-1}v=v$ (modulo C^∞) for $v\in\mathcal{E}'(\partial\Omega)$ with $WF(v)\in\Sigma$. Then (1.2) is equivalent (modulo C^∞) to

$$\Phi^{-1}B\Big(G^{(1)}_0 v^{(1)}+\sum_{j=2}^{3} G^{(j)}_0\Phi(\Phi^{-1}v^{(j)})\Big)=\Phi^{-1}f, \tag{5.5}$$

where $G^{(j)}_0$, $j=2,3$, denote the restrictions of $G^{(j)}$ to $\partial\Omega$. Note that $\Phi^{-1}BG^{(1)}_0$ and $\Phi^{-1}BG^{(j)}_0\Phi$, $j=2,3$, are pseudodifferential operators. Moreover it follows from (2. 14) and condition (ii) that B is of the form

$$B(x')=\left[B_{\mathrm{I}}(x'),\,0\right] \qquad \text{with } B_{\mathrm{I}} \text{ the } d\times 2d \text{ block.} \tag{5.6}$$

Let W_{I}, W_h and W_e be the matrices of the first $2d$ rows of W, $a_0^{(2)}$ and $a_0^{(3)}$ respectively. Note that the latters are of the same rank as the formers according to (2. 15). Then it follows from (5. 1), (5. 3) and (5. 6) that for $(x',\xi')=(x'(y',\eta'),\xi'(y',\eta'))$ the principal symbol in the amplitude of $\Phi^{-1}BG_0^{(1)}$ is

$$\begin{aligned} B_{\mathrm{I}}\left(W_{\mathrm{I}}(x',\theta_x)\,g_0+O(\alpha)\right)-iB_{\mathrm{I}}\Big\{&\rho_{x_\eta}(\partial_{\xi_n}W_{\mathrm{I}})(x',\theta_x)\,g_0\\ &+W_{\mathrm{I}}(x',\theta_x)\,h_0+O(\alpha|\eta'|^{-1/3})\Big\}\,K, \end{aligned} \tag{5.7}$$

and that of $\Phi^{-1}B[G_0^{(2)},G_0^{(3)}]\,\Phi$ is

$$B_{\mathrm{I}}[W_h,W_e]\,(x',\theta_{x'})+O(\alpha)\,. \tag{5.8}$$

Moreover the Lopatinski determinant defined by (5. 4) becomes

$$R(x',\xi')=\det\left(B_{\mathrm{I}}(x')\left[W_{\mathrm{I}}\left(x',\xi',\xi_n^+(x',\xi')\right),\,W_h(x',\xi'),\,W_e(x',\xi')\right]\right). \tag{5.9}$$

Now let $R(x^{0\prime},\xi^{0\prime})\neq 0$. In this case we shall prescribe the initial data for the transport equations $(4.6)_\pm$ so that $a^\pm|_{t=0}=g_0|_{\rho=0}=I_{m_1}$. Then (5. 5) is an elliptic pseudodifferential equation and hence is solvable, because the second term in (5. 7) is estimated by a constant times $(|\alpha|+|\eta'|^{-2/3})^{1/2}$ according to (2. 29).

In what follows we suppose $R(x^{0\prime},\xi^{0\prime})=0$. With $x'=x'(y',\eta')$ set for convenience

$$\begin{aligned} V(y',\eta')&=[V_1,\cdots,V_d]\,(y',\eta')\\ &=B_{\mathrm{I}}(x')\left[W_{\mathrm{I}}(x',\theta_x),\,W_h(x',\theta_{x'}),\,W_e(x',\theta_{x'})\right],\\ [\tilde V_1,\cdots,\tilde V_{m_1}]\,(y',\eta')&=|\eta'|\,B_{\mathrm{I}}(x')\,(\partial_{\xi_n}W_{\mathrm{I}})\,(x',\theta_x)\,. \end{aligned} \tag{5.10}$$

Then V, $\tilde V_j\in S_{1,0}^0$ and it follows from (2. 11) and (5. 9) that

$$R(x',\theta_{x'})=\det V(y',\eta') \qquad \text{for} \quad \alpha=0\,. \tag{5.11}$$

Therefore $R(x^{0\prime},\xi^{0\prime})=0$ becomes

$$\det V(y^{0\prime},\eta^{0\prime})=0 \tag{5.12}$$

with $y^{0\prime}=\phi_{\eta'}(x^{0\prime},\eta^{0\prime})$, and condition (iii) means

$$\sum_{j=0}^{m_1}\det\,[V_1,\cdots,V_{j-1},\tilde V_j,V_{j+1},\cdots,V_d]\neq 0 \qquad \text{at} \quad (y^{0\prime},\eta^{0\prime})\,. \tag{5.13}$$

For definiteness we assume that

(5.14) $\det[\tilde{V}_1, V_2, \cdots, V_d](y^{0\prime}, \eta^{0\prime}) \neq 0$

and set

(5.15) $\tilde{V}(y', \eta') = [\tilde{V}_1, V_2, \cdots, V_d](y', \eta')$.

Let $\tilde{V}(y', D_{y'})$ be the elliptic pseudodifferential operator with symbol $\tilde{V}(y', \eta')$ and let $\tilde{V}^{-1}$ be a microlocal parametrix for $\tilde{V}$. Then by (5.8), (5.10) and (5.15) the principal symbol of $\tilde{V}(y', D_{y'})^{-1}\Phi^{-1}B[G_0^{(2)}, G_0^{(3)}]\Phi$ is

(5.16) $$\begin{bmatrix} O(\alpha) \\ I_{d-m_1} \end{bmatrix} \mod S_{1,0}^{-1},$$

since

(5.17) $\tilde{V}(y', \eta')^{-1} V(y', \eta') = [a_{jk}\,;\, j \downarrow 1, \cdots, d,\ k \rightarrow 1, \cdots, d]$,

where $a_{jk} = \delta_{jk}$ for $j \geqq 1$, $k \geqq 2$, $a_{11} = (\det V)/\det \tilde{V}$, $a_{j1} = -\det(V|_{V_j \rightarrow \tilde{V}_1})/\det \tilde{V}$ for $j \geqq 2$, and $V|_{V_j \rightarrow \tilde{V}_1}$ denotes the matrix V with V_j replaced by $\tilde{V}_1$. Moreover according to (5.7) the principal symbol of $\tilde{V}^{-1}\Phi^{-1}BG_0^{(1)}$ is of the form

(5.18) $\Psi - i\tilde{\Psi}K \mod S_{1/3,0}^{-1}$,

where $\Psi \in S_{1,0}^{0}$, $\tilde{\Psi} \in S_{1,0}^{-1/3}$, and for $\alpha = 0$

(5.19) $\Psi(y', \eta') = [a_{jk}\,;\, j \downarrow 1, \cdots, d, k \rightarrow 1, \cdots, m_1]\, g_0$,

(5.20) $$\tilde{\Psi}(y', \eta') = |\eta'|^{-1} \rho_{x_n} \tilde{V}^{-1}[\tilde{V}_1, \cdots, \tilde{V}_{m_1}]\, g_0 + [a_{jk}\,;\, j \downarrow 1, \cdots, d, k \rightarrow 1, \cdots, m_1]\, h_0 .$$

Therefore, setting

$$\tilde{V}^{-1}\Phi^{-1}BG_0^{(1)} = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} \chi, \quad \tilde{V}^{-1}\Phi^{-1}B[G_0^{(2)}, G_0^{(3)}]\,\Phi = \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix},$$

where B_{11} and B_{22} are $m_1 \times m_1$ and $(d-m_1) \times (d-m_1)$ matrices of operators respectively, we find from (5.16) that $\sigma(B_{22}) \in S_{1,0}^{0}$ is elliptic and $\sigma(B_{12}) \in S_{1,0}^{0}$ is $O(\alpha) \mod S_{1,0}^{-1}$, $\sigma(B_{jk})$ being the symbol of B_{jk}, while $\sigma(B_{11})$, $\sigma(B_{21}) \in S_{1/3,0}^{0}$. Consequently (5.5) is equivalent (modulo C^∞) to

(5.21) $$(B_{11} - B_{12}B_{22}^{-1}B_{21})\,\chi v^{(1)} = f^{(1)} - B_{12}B_{22}^{-1} \begin{bmatrix} f^{(2)} \\ f^{(3)} \end{bmatrix} \equiv F$$

with

$$\begin{bmatrix} v^{(2)} \\ v^{(3)} \end{bmatrix} = \Phi B_{22}^{-1} \left(\begin{bmatrix} f^{(2)} \\ f^{(3)} \end{bmatrix} - B_{21}\chi v^{(1)} \right),$$

where B_{22}^{-1} is a microlocal parametrix for B_{22} and $\tilde{V}^{-1}\Phi^{-1}f = {}^t[{}^t f^{(1)}, {}^t f^{(2)}, {}^t f^{(3)}]$.

Now let us consider (5. 21). Since $\sigma(B_{12}B_{22}^{-1})=O(\alpha)$ mod $S_{1,0}^{-1}$, the principal symbol of $B_{11}-B_{12}B_{22}^{-1}B_{21}$ is

$$\Psi^{(1)}-i\widetilde{\Psi}^{(1)}K \bmod S_{1/3,0}^{-1}, \tag{5. 22}$$

where $\Psi^{(1)}$, $\widetilde{\Psi}^{(1)}$ are the uppermost $m_1\times m_1$ blocks of Ψ, $\widetilde{\Psi}$ respectively when $\alpha=0$. It follows from (5. 17) and (5. 19) that

$$\Psi^{(1)}=\begin{bmatrix} a_{11}g_{11}, & a_{11}g_{12}, \cdots, & a_{11}g_{1m_1} \\ a_{21}g_{11}+g_{21}, \cdots\cdots & & \\ \vdots & & \\ a_{m_1 1}g_{11}+g_{m_1 1}, \cdots & & \end{bmatrix} \quad \text{for} \quad \alpha=0,$$

where we have set $g_0=[g_{jk}\,;\ j\downarrow 1,\cdots,m_1,\ k\rightarrow 1,\cdots,m_1]$. For the transport equations $(4.6)_{\pm}$ we shall prescribe the initial data on $t=0$ so that $g_{jk}|_{\rho=0}=\delta_{jk}$ for $j\geqq 1$, $k\geqq 2$, $g_{11}\neq 0$, say, $g_{11}|_{\rho=0}=\det\tilde{V}|_{\alpha=0}$ and

$$a_{j1}g_{11}+g_{j1}=0 \qquad \text{for} \quad \rho=0,\ j=2,\cdots,m_1. \tag{5. 23}$$

Then g_0 is nonsignular by virtue of (5. 14) and it follows that

$$\Psi^{(1)}(y',\eta')=\begin{bmatrix} \det V(y',\eta') & 0 \\ 0 & I_{m_1-1} \end{bmatrix} \quad \text{for} \quad \alpha=0. \tag{5. 24}$$

Moreover we find that the uppermost left entry $\widetilde{\Psi}_{11}$ of $\widetilde{\Psi}^{(1)}$ is

$$\widetilde{\Psi}_{11}(y',\eta')=|\eta'|^{-1}\rho_{x_n}\sum_{j=1}^{m_1}\det(V|_{V_j\to\tilde{V}_j})+|\eta'|^{-1/3}O(\det V) \quad \text{for } \alpha=0, \tag{5. 25}$$

which does not vanish by (2. 5), (5. 12) and (5. 13). In fact, setting $\tilde{V}^{-1}[\tilde{V}_1,\cdots,\tilde{V}_{m_1}]=[b_{jk}\,;\ j\downarrow 1,\cdots,d,\ k\rightarrow 1,\cdots,m_1]$ we have

$$\tilde{V}_j=b_{1j}\tilde{V}_1+\sum_{k=2}^{d}b_{kj}V_k,\ j=1,\cdots,m_1,$$

where $b_{11}=1$ and $b_{k1}=0$ for $k\geqq 2$. Therefore it follows from (5. 23) that for $\alpha=0$

$$\sum_{j=1}^{m_1}\det(V|_{V_j\to\tilde{V}_j})=\sum_{j=1}^{m_1}b_{1j}g_{j1}+(\det V)\sum_{j=2}^{m_1}b_{jj}.$$

Thus we obtain (5. 25) by (5. 20), since $a_{1k}=O(\det V)$ for $k\geqq 1$.

Now set

$$v^{(1)}=\begin{bmatrix} v_1 \\ v' \end{bmatrix},\ F=\begin{bmatrix} F_1 \\ F' \end{bmatrix},\ \Psi^{(1)}=\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix},\ \widetilde{\Psi}^{(1)}=\begin{bmatrix} \widetilde{\Psi}_{11} & \widetilde{\Psi}_{12} \\ \widetilde{\Psi}_{21} & \widetilde{\Psi}_{22} \end{bmatrix},$$

where v_1, F_1, Ψ_{11} and $\widetilde{\Psi}_{11}$ are scalars, and write (5. 21) as

$$C_{11}\chi v_1+C_{12}\chi v'=F_1$$
$$C_{21}\chi v_1+C_{22}\chi v'=F'.$$

Then according to (5. 22) and (5. 24) the principal symbol c_{ij} of C_{ij} are such that

$$\begin{aligned} c_{11}&=\Psi_{11}-i\widetilde{\Psi}_{11}K,\ \Psi_{11}=\det V+O(\alpha),\\ c_{12}&=-i\widetilde{\Psi}_{12}K+O(\alpha),\\ c_{21}&=-i\widetilde{\Psi}_{21}K+O(\alpha),\\ c_{22}&=I_{m_1-1}-i\widetilde{\Psi}_{22}K+O(\alpha).\end{aligned} \tag{5. 26}$$

Since $\widetilde{\Psi}_{22}\in S_{1,0}^{-1/3}$ and (2. 29) implies

$$|\eta'|^{-1/3}\left|K(\eta')\right|\leqq C(|\alpha|+|\eta'|^{-2/3})^{1/2}, \tag{5. 27}$$

it follows that $c_{22}\in S_{1/3,0}^{0}$ is elliptic. Thus (5. 21) is equivalent (modulo C^∞) to

$$(C_{11}-C_{12}C_{22}^{-1}C_{21})\chi v_1=F_1-C_{12}C_{22}^{-1}F'\equiv\tilde{F}_1 \tag{5. 28}$$

with

$$v'=C_{22}^{-1}(F'-C_{21}\chi v_1),$$

where C_{22}^{-1} denotes a microlocal parametrix for C_{22}. We shall show that for $\alpha=0$ and $|\eta'|=1$

$$\Psi_{11}=D(x',\theta_{x''})\,\widetilde{\Psi}_{11}\sqrt{\mu_2(x',\theta_{x'})}/\rho_{x_n}+O\left(D(x',\theta_{x''})^2\right). \tag{5. 29}$$

Let $\alpha=0$, $|\eta'|=1$ and for convenience set $\xi'=\theta_{x'}(x',\eta')$. Then (1. 5) and (2. 12) imply $\xi_0=\mu_1(x',\xi'')$. Hence from (1. 6)

$$\tilde{R}(x',\xi'',0)=-D(x',\xi'')\,\tilde{R}^{(1)}(x',\xi'',0)$$

and

$$\tilde{R}^{(1)}(x',\xi'',0)=(\partial\tilde{R}/\partial z)(x',\xi'',0)+O\left(D(x',\xi'')\right).$$

Moreover it follows from (5. 9) and (5. 10) that

$$(\partial\tilde{R}/\partial z)(x',\xi'',0)=-\sqrt{\mu_2(x',\xi'')}\sum_{j=1}^{m_1}\det(V|_{V_j\to\tilde{V}_1})$$

since $\xi_n^+(x',\xi')=\lambda(x',\xi')-\sqrt{\mu(x',\xi')}$. Therefore we obtain (5. 29) by virtue of (5. 11),

First we consider a special case where $D(x',\xi'')$ vanishes identically. Then $\Psi_{11}=O(\alpha)$ and hence

$$c_{11}(y',\eta')=-i\widetilde{\Psi}_{11}K+O(\alpha).$$

Set, as in [17],

$$v_1 = |D_{y'}|^{1/3} K(D_{y'})^{-1} \tilde{v}_1$$

and to both sides of (5.28) apply $i\widetilde{\Psi}_{11}(y', D_{y'})^{-1}|D_{y'}|^{-1/3}$ which is an elliptic pseudodifferential operator of order zero according to (5.25). Then (5.28) is equivalent (modulo C^∞) to

$$(I+\tilde{C}_{11})\, \chi \tilde{v}_1 = i\widetilde{\Psi}_{11}^{-1}|D_{y'}|^{-1/3}\tilde{F}_1$$

where the principal symbol $\tilde{c}_{11}$ of $\tilde{C}_{11}$ is

$$\begin{aligned} \tilde{c}_{11}(y', \eta') &= |\eta'|^{1/3} K^{-1} O(\alpha) \\ &\quad + \left(\widetilde{\Psi}'_{12}|\eta'|^{-1/3}K + O(\alpha)\right) c_{22}^{-1} \left(\widetilde{\Psi}'_{21} + |\eta'|^{1/3}K^{-1}O(\alpha)\right) \quad \bmod S^{-1/3}_{1/3,0} \end{aligned}$$

with $\widetilde{\Psi}'_{12}$, $\widetilde{\Psi}'_{21} \in S^0_{1,0}$. Therefore we find from (2.33) and (5.27) that $1+\tilde{c}_{11} \in S^0_{1/3,0}$ is elliptic. Consequently (5.28) or (5.21) is solvable modulo C^∞ with $||K\chi v_1||_{-\frac{1}{3}} \leqq \text{const.}\, ||\tilde{F}_1||$.

Now we shall consider the general case. Assume that condition (iv) holds. In this case we use a device, due to Imai and Shirota [4], which is based on

LEMMA 5.1. ([4]). *The real and imaginary parts of $A'(s)/A(s)$ are negative for real s.*

We shall first derive the a priori estimate for (5.28):

$$||\chi v_1||_0 \leqq \text{const.}\, ||\tilde{F}_1||_{1/3} \tag{5.30}$$

taking δ in the cutoff function χ small enough. Applying the elliptic operator $-e^{i\delta_1}\widetilde{\Psi}_{11}^{-1} \in S^{1/3}_{1,0}$ to both sides of (5.28) we see that the equation is equivalent to

$$(-e^{i\delta_1}\widetilde{\Psi}_{11}^{-1}C_{11} + e^{i\delta_1}\widetilde{\Psi}_{11}^{-1}C_{12}C_{22}^{-1}C_{21})\,\chi v_1 = -e^{i\delta_1}\widetilde{\Psi}_{11}^{-1}\tilde{F}_1 \tag{5.31}$$

and by (5.26) the principal symbol of $-e^{i\delta_1}\widetilde{\Psi}_{11}^{-1}C_{11}$ is

$$ie^{i\delta_1}K - e^{i\delta_1}\widetilde{\Psi}_{11}^{-1}\Psi_{11} \qquad \bmod S^{-2/3}_{1/3,0}\,.$$

It follows from (5.29) and condition (iv) that

$$\mathrm{Re}\,(-e^{i\delta_1}\widetilde{\Psi}_{11}^{-1}\Psi_{11})(y', \eta') \geqq 0 \qquad \text{for} \quad \alpha = 0\,, \tag{5.32}$$

while (2.27), (2.29) and Lemma 5.1 imply that for some positive constant C_0

$$\mathrm{Re}\left(ie^{i\delta_1}K(\eta')\right) \geqq 4C_0\left|K(\eta')\right|. \tag{5.33}$$

Therefore by virtue of the sharp form of Gårding's inequality we have

$$4C_0\left\| |K|^{1/2}\chi v_1\right\|_0^2 \leqq \mathrm{Re}\,(-e^{i\theta_1}\widetilde{\Psi}_{11}^{-1}C_{11}\chi v_1, \chi v_1)$$
$$+\mathrm{Re}\Big(a(y', D_{y'})\, b(D_{y'})\,\chi v_1, \chi v_1\Big)+\mathrm{const.}\|\chi v_1\|_{-1/3}^2$$

where $a(y', \eta')\in S_{1,0}^{1/3}$ and $b(\eta')=\alpha$. Moreover the second term on the right side is estimated by $C_0\| |K|^{1/2}\chi v_1\|_0^2$. In fact, write

$$(ab\chi v_1, \chi v_1)=\Big(|K|^{-1/2}ab|K|^{-1/2}(|K|^{1/2}\chi v_1), |K|^{1/2}\chi v_1\Big).$$

Then by means of (2. 31) and (2. 33) we get for small $\varepsilon>0$

$$\Big|(ab\chi v_1, \chi v_1)\Big| \leqq \varepsilon\left\| |K|^{1/2}\chi v_1\right\|_0^2.$$

Therefore by (5. 31) we obtain

$$2C_0\left\| |K|^{1/2}\chi v_1\right\|_0^2 \leqq C\left\| |K|^{-1/2}\widetilde{\Psi}_{11}^{-1}C_{12}C_{22}^{-1}C_{21}\chi v_1\right\|_0^2 \tag{5. 34}$$
$$+C'\|\tilde{F}_1\|_{1/3}^2+C''\|\chi v_1\|_{-1/3}^2$$

with some constants C, C' and C''. Furthermore we claim that for small $\varepsilon>0$

$$\left\| |K|^{-1/2}\widetilde{\Psi}_{11}^{-1}C_{12}C_{22}^{-1}C_{21}\chi v_1\right\|_0^2 \leqq \varepsilon\left\| |K|^{1/2}\chi v_1\right\|_0^2+\mathrm{const.}\left\| |K|^{1/2}\chi v_1\right\|_{-1/3}^2. \tag{5. 35}$$

To this end we write $\chi v_1=|K|^{1/2}|K|^{-1}(|K|^{1/2}\chi v_1)$ and

$$|K|^{-1/2}\widetilde{\Psi}_{11}^{-1}C_{12}C_{22}^{-1}C_{21}|K|^{1/2}|K|^{-1}=\widetilde{\Psi}_{11}^{-1}C_{12}C_{22}^{-1}C_{21}|K|^{-1} \tag{5.36}$$
$$+|K|^{-1/2}\Big\{\Big[\widetilde{\Psi}_{11}^{-1}C_{12}C_{22}^{-1}, |K|^{1/2}\Big]C_{21}+\widetilde{\Psi}_{11}^{-1}C_{12}C_{22}^{-1}\Big[C_{21}, |K|^{1/2}\Big]\Big\}|K|^{-1}.$$

It follows from (2. 33) and (5. 26) that $C_{21}|K|^{-1}$ is of order $-1/3$. Moreover C_{12} is an operator of order zero whose norm is small according to (5. 26) and (5. 27). Therefore the first term on the right side in (5. 36) is an operator of order zero whose norm is small. On the other hand the second term is of order $-1/3$. In fact we have for example

$$i\Big[C_{21}, |K|^{1/2}\Big]|K|^{-1}=\Big[\widetilde{\Psi}_{21}, |K|^{1/2}\Big]K|K|^{-1}+\Big[a, |K|^{1/2}\Big]|K|^{-1},$$

where $a(y', \eta')=O(\alpha)$. Hence Corollary 2.5 implies that $[C_{21}, |K|^{1/2}]\,|K|^{-1}$ is of order $-2/3$. Thus we get (5. 35), which and (5. 34) yield (5. 30).

Now in order to solve (5. 28) it suffices to derive the a priori estimate analogous to (5. 30) for the adjoint problem of (5. 31). This can be accomplished by a producre similar to that derived (5. 30), since it is clear that the analogues to (5. 32) and (5. 33) hold. Consequently (5. 5) is solvable with $WF(v^{(1)})\subset WF(\Phi^{-1}f)$ and $WF(v^{(j)})\subset WE(f)$ for $j=2, 3$. Therefore $WF(G_0^{(j)}v^{(j)})\subset WF(f)$ for $j=1, 2, 3$ and it is known that for some neighborhood

U of $x^{0\prime}$ in R^{n+1} $G^{(2)}v^{(2)}$ and $G^{(3)}v^{(3)}$ satisfy (1. 3), $G^{(3)}v^{(3)}\in C^{\infty}(\Omega\cap U)$ and $WF(G^{(2)}v^{(2)})$ as a subset of $T^{*}(\Omega\cap U)$ has the required property. Furthermore it is shown in [16] that, according to the choice (2. 25) of Airy function, $G^{(1)}v^{(1)}$ also satisfies (1. 3) and $WF(G^{(1)}v^{(1)})$ has the required property. (See also [3]).

§ 6. Remarks and examples

Let $(x^{0\prime}, \xi^{0\prime})\in T^{*}(\partial\Omega)\backslash 0$ be a diffractive point and let $R(x^{0\prime}, \xi^{0\prime})=0$. In this case we had to choose the initial data $g_{0}|_{\rho=0}$ for the transport equations $(4.6)_{\pm}$ so that (5. 23) holds for $x_{n}=0$ if $a_{j1}|_{\alpha=0}\neq 0$ for some $j=2, \cdots, m_{1}$. Hereafter we keep using the notations in (5. 4), (5. 9), (5. 10), (5. 15) and (5. 17). We shall show that $a_{j1}(y^{0\prime}, \eta^{0\prime})$, $j=2, \cdots, m_{1}$, are proportional to the corresponding *reflection coefficients* $c_{j1}(x^{0\prime}, \xi^{0\prime})$ (which will be defined below) with a nonzero ratio and hence $g_{0}|_{\rho=x_{n}=0}$ must be so taken as to depend on $B(x')$ if $c_{j1}(x^{0\prime}, \xi^{0\prime})\neq 0$ for some $j=2, \cdots, m_{1}$ and for any basis $W(x', \xi)$ of the null space of $P_{1}(x', \xi)$ with $Q_{1}(x', \xi)=0$ which satisfies (5. 14) and is independent of $B(x')$. In fact, such situations appear for example in the case of Maxwell's equations or the linear elastic equations in an isotropic medium with certain energy conserving boundary conditions which satisfy condition (iv), as will be seen below.

Recall that for $\xi_{0}>\mu_{1}(x', \xi'')$ the reflection coefficients c_{jk} associated with a basis of the null space of P_{1} are defined by

$$\begin{aligned}(6.1)\qquad &\left[c_{jk}(x', \xi')\,;\ j\downarrow 1, \cdots, d,\ k\rightarrow 1, \cdots, m_{1}\right]\\ &=\left\{B(x')\left[W\left(x', \xi', \xi_{n}^{+}(x', \xi')\right),\ a_{0}^{(2)}(x', \xi'),\ a_{0}^{(3)}(x', \xi')\right]\right\}^{-1}\\ &\quad B(x')\,W\left(x', \xi', \xi_{n}^{-}(x', \xi')\right),\end{aligned}$$

where $\xi_{n}^{-}(x', \xi')$ is the incoming root of $(\xi_{n}-\lambda(x', \xi'))^{2}-\mu(x', \xi')=0$. (See [5]; the other c_{jk}, $k=m_{1}+1, \cdots, d$, can be defined analogously, although they do not be used here.) Note that $c_{jk}(x', \xi')$ are well defined for such (x', ξ') since $|R(x', \xi')|$ is from below by a nonzero constant times $(\xi_{0}-\mu_{1}(x', \xi''))^{1/2}$ according to condition (iv). For $\xi_{0}=\mu_{1}(x', \xi'')$ these are defined to be the limits as ξ_{0} tends to $\mu_{1}(x', \xi'')$.

An interpretation of $c_{jk}(x^{0\prime}, \xi')$ for $\xi_{0}>\mu_{1}(x^{0\prime}, \xi'')$ is as follows. Consider the frozen (constant coefficients) problem at $x^{0\prime}$:

$$\begin{aligned}&P(x^{0}, D)\,u=0 \quad \text{in } \Omega\,,\\ &B(x^{0\prime})\,u=f \quad \text{on } \partial\Omega\,,\\ &u(x)=0 \quad \text{in } \Omega\cap\{x_{0}<0\}\,.\end{aligned}$$

Let u_k^- be an incoming solution of $P(x^{0\prime}, D)u=0$ defined by

$$u_k^-(x)=\int_{R^n} e^{i\phi^-(x,\xi')}\chi(\xi')\, W_k\Big(x^{0\prime}, \xi', \xi_n^-(x^{0\prime}, \xi')\Big)\, d\xi' ,$$

where $W(x', \xi)=[W_1(x', \xi), \cdots, W_{m_1}(x', \xi)]$, $\phi^\pm(x, \xi')=(x'-x^{0\prime})\cdot\xi'+x_n\xi_n^\pm(x^{0\prime}, \xi')$ and $\chi\in C_0^\infty(R^n)$ is a cutoff function supported in a small neighborhood of $\xi^{0\prime}$ with $\xi_0>\mu_1(x^{0\prime}, \xi'')$. Note that sing supp u_k^- is contained in the bicharacteristic lines of $\xi_n-\xi_n^-(x^{0\prime}, \xi')$ with $\xi'\in\operatorname{supp}\chi$ which hit $\partial\Omega$ transversally at $x'=x^{0\prime}$. Let $f=B(x^{0\prime})\,u_k^-|_{\partial\Omega}$ and $x_0^0>0$. Assume for simplicity that det $P_1(x, \xi)=Q_1(x, \xi)^{m_1}\tilde{Q}(x, \xi)$ and Q_1 is of the second order. Then a parametrix u for the frozen problem is given by

$$u(x)=\int_{R^n} e^{i\phi^+(x,\xi')}\chi(\xi')\sum_{j=1}^{m_1} a_j(x^{0\prime}, \xi')\, W_j\Big(x^{0\prime}, \xi', \xi_n^+(x^{0\prime}, \xi')\Big)\, d\xi' ,$$

where, for $\xi'\in\operatorname{supp}\chi$, a_j are determined by

$$\sum_{j=1}^{m_1} a_j(x^{0\prime}, \xi')\, B(x^{0\prime})\, W_j\Big(x^{0\prime}, \xi', \xi_n^+(x^{0\prime}, \xi')\Big)=B(x^{0\prime})\, W_k\Big(x^{0\prime}, \xi', \xi_n^+(x^{0\prime}, \xi')\Big) .$$

Thus we have $a_j(x^{0\prime}, \xi')=c_{jk}(x^{0\prime}, \xi')$. Roughly speaking, the "k-th incident wave" creates $c_{jk}(x^{0\prime}, \xi')$ times the "j-th reflected wave".

Now let $\xi_0^0=\mu_1(x^{0\prime}, \xi^{0\prime\prime})>0$. We shall then prove that for $j=2, \cdots, m_1$ and $|\eta^{0\prime}|=1$

$$c_{j1}(x^{0\prime}, \xi^{0\prime})=a_{j1}(y^{0\prime}, \eta^{0\prime})\det \tilde{V}(y^{0\prime}, \eta^{0\prime})\lim_{\xi_0\to\xi_0^0+0}\Big((\xi_n^+-\xi_n^-)/R\Big)(x^{0\prime}, \xi_0, \xi^{0\prime\prime}) , \tag{6.2}$$

where the coefficient of $a_{j1}(y^{0\prime}, \eta^{0\prime})$ is a nonzero constant by virtue of (5.14) and (1.6). For convenience set $\xi'=\theta_{x'}(x', \eta')$ and denote by $V_j^+(x', \xi')$ the j-th coiumn of the matrix in (5.9): $B_{\mathrm{I}}(x')\,[W_{\mathrm{I}}(x', \xi', \xi_n^+(x', \xi')), W_h(x', \xi'), W_e(x', \xi')]$ and by V_1^- the vector V_1^+ with ξ_n^+ replaced by ξ_n^-. Then from (5.10) we have $V_1^+(x', \xi')=V_1(y', \eta')$ for $\alpha=0$. Let $\alpha>0$, $|\eta'|=1$ and set $U_1=(V_1^+-V_1^-)/(\xi_n^+-\xi_n^-)$. Then $\lim_{\alpha\to+0} U_1(x', \xi')=\tilde{V}_1(y', \eta')|_{\alpha=0}$ and it follows from (6.1) and (5.6) that

$$V_1^-(x', \xi')=\sum_{j=1}^{d} c_{j1}(x', \xi')\, V_j^+(x', \xi') .$$

Hence

$$(\xi_n^+-\xi_n^-)\, U_1=(1-c_{11})\, V_1^+-\sum_{j=2}^{d} c_{j1} V_j^+ .$$

Therefore we find that for $j=2, \cdots, m_1$

$$c_{j1}(x', \xi')=b_{j1}(x', \xi')\,(\xi_n^+-\xi_n^-)/R(x', \xi') ,$$

where $b_{j1}=-\det[V_1^+,\cdots,V_{j-1}^+,U_1,V_{j+1}^+,\cdots,V_d^+]$ tends, by (5.17), to $a_{j1}\det\tilde{V}$ as $\alpha\to+0$. Thus we obtain (6.2).

To illustrate the arguments above we shall treat the linear elastic equations in a homogeneous, isotropic medium defined by $\partial_{x_0}{}^2 w_j=\sum_{k=1}^{3}\partial_{x_k}\sigma_{jk}$, $j=1,2,3$, where $[\sigma_{jk}\,;\,j,\,k=1,2,3]$ is the stress tensor and $w=(w_1,w_2,w_3)$ the displacement vector. On the boundary we prescribe a condition of the form

$$b\times\partial_{x_0}w=0\quad\text{and}\quad\sum_{j,k=1}^{3}b_j\sigma_{jk}n_k=0\,,$$

where $n=(n_1,n_2,n_3)$ is the inward unit normal and $b=(b_1,b_2,b_3)$ are real valued functions with $n\cdot b\neq0$. (See [15]). Since the equations are invariant under rotations and the Lopatinski determinant and reflection coefficients may be obtained from the frozen problems we shall in what follows a constant coefficients problem in the half space $\Omega=\{x=(x_0,x_1,x_2,x_3)\,;\,x_3>0\}$.

Set $u={}^t(\sigma_{11},\sigma_{22},\sigma_{33},\sigma_{23},\sigma_{13},\sigma_{12},\partial_{x_0}w)$. Then the equations can be written as

$$P(D)\,u=\sum_{j=0}^{3}A_jD_ju=0\qquad\text{in}\quad\Omega$$

and the boundary condition as

$$Bu=f\qquad\text{on}\quad\partial\Omega$$

with

$$B=\begin{bmatrix}0&0&0&0&0&0&0&b_3&-b_2\\0&0&0&0&0&0&b_3&0&-b_1\\0&0&b_3&b_2&b_1&0&0&0&0\end{bmatrix},\quad b_3\neq0\,.$$

Here

$$A_0^{-1}=\begin{bmatrix}E_0^{-1}&0&0\\0&\mu I_3&0\\0&0&I_3\end{bmatrix},\quad E_0^{-1}=\begin{bmatrix}\lambda+2\mu&\lambda&\lambda\\\lambda&\lambda+2\mu&\lambda\\\lambda&\lambda&\lambda+2\mu\end{bmatrix},$$

where λ, μ are the Lamé parameters of the medium, and

$$\sum_{j=1}^{3}A_j\xi_j=-\begin{bmatrix}0&C(\xi)\\{}^tC(\xi)&0\end{bmatrix},\quad {}^tC(\xi)=\begin{bmatrix}\xi_1&0&0&0&\xi_3&\xi_2\\0&\xi_2&0&\xi_3&0&\xi_1\\0&0&\xi_3&\xi_2&\xi_1&0\end{bmatrix},$$

because ${}^t(\sigma_{11},\sigma_{22},\sigma_{33})=E_0^{-1}e^{(1)}$ and ${}^t(\sigma_{23},\sigma_{13},\sigma_{12})=2\mu e^{(2)}$, where $e^{(1)}={}^t(e_{11},e_{22},e_{33})$, $e^{(2)}={}^t(e_{23},e_{13},e_{12})$ and $e_{jk}=(\partial_{x_k}w_j+\partial_{x_j}w_k)/2$. Noting that the eigenvalues of

E_0^{-1} are 2μ, 2μ and $3\lambda+2\mu$ we assume that $\mu>0$ and $3\lambda+2\mu>0$. Then A_0 is positive definite, so $P(D)$ is hyperbolic. Moreover we have $\det A_0^{-1}P(\xi)=Q_1(\xi)^2 Q_2(\xi)\tilde{Q}(\xi)$, where $Q_1(\xi)=\xi_0^2-\mu(\xi_1^2+\xi_2^2+\xi_3^2)$, $Q_2(\xi)=\xi_0^2-(\lambda+2\mu)(\xi_1^2+\xi_2^2+\xi_3^2)$ and $\tilde{Q}(\xi)=\xi_0^3$. Furthermore the boundary condition $Bu=0$ is energy conserving (i. e., $u\cdot A_3 u=0$ for $u\in\ker B$) with respect to the quadratic form $u\cdot A_0 u/2$ which equals the classical energy density: $e^{(1)}\cdot E_0^{-1}e^{(1)}/2+2\mu|e^{(2)}|^2+|\partial_{x_0}w|^2/2$.

Now let $\xi^{0\prime}\in R^3\backslash 0$ be a point such that $Q_1(\xi^{0\prime},\xi_3)=0$ has the real double root $\xi_3=0$, say, $\xi_0^0=\sqrt{\mu}\,|\xi^{0\prime\prime}|$ and $\xi_1^0\neq 0$, and let ξ' belong to a conic neighborhood of $\xi^{0\prime}$. Then a basis $W(\xi)=[W_1(\xi),W_2(\xi)]/|\xi''|^2$ of the null space of $P(\xi)$ with $Q_1(\xi)=0$ is given by $W_1(\xi)={}^t(2\mu\xi_1\xi_3, 0, -2\mu\xi_1\xi_3, -\mu\xi_1\xi_2, \mu(\xi_3^2-\xi_1^2), \mu\xi_2\xi_3, \xi_0\xi_3, 0, -\xi_0\xi_1)$, $W_2(\xi)={}^t(-2\mu\xi_1\xi_2, 2\mu\xi_1\xi_2, 0, \mu\xi_1\xi_3, -\mu\xi_2\xi_3, \mu(\xi_1^2-\xi_2^2), -\xi_0\xi_2, \xi_0\xi_1, 0)$ and a null vector $W_3(\xi)$ of $P(\xi)$ with $Q_2(\xi)=0$ by $W_3(\xi)={}^t(\xi_0^2-2\mu(\xi_2^2+\xi_3^2), \xi_0^2-2\mu(\xi_1^2+\xi_3^2), \xi_0^2-2\mu(\xi_1^2+\xi_2^2), 2\mu\xi_2\xi_3, 2\mu\xi_1\xi_3, 2\mu\xi_1\xi_2, \xi_0\xi_1, \xi_0\xi_2, \xi_0\xi_3)$. Let $r_1^+(\xi')$ be the outgoing root of $Q_1(\xi',\xi_3)=0$, i. e., $r_1^+(\xi')=-(\xi_0^2/\mu-|\xi''|^2)^{1/2}$ and $r_2^+(\xi')$ be the root of $Q_2(\xi',\xi_3)=0$ with $\operatorname{Im} r_2^+>0$, i. e., $r_2^+(\xi')=i(|\xi''|^2-\xi_0^2/(\lambda+2\mu))^{1/2}$. Taking $a_0^{(2)}=0$ and $a_0^{(3)}=W_3(\xi',r_2^+(\xi'))/|\xi''|^2$ in (5.4) we find that modulo a nonzero factor

$$R(\xi')=r_1^+\left\{b_3^2\xi_0^2/\mu+(b_2\xi_1-b_1\xi_2)^2+r_1^+r_2^+(b_1^2+b_2^2)\right\}+r_2^+(b_1\xi_1+b_2\xi_2)^2.$$

Let $R(\xi^{0\prime})=0$. Then $b_1\xi_1^0+b_2\xi_2^0=0$ and condition (iii) holds, since $r_1^+(\xi')=-z(2\sqrt{\mu}\,|\xi''|+z^2)^{1/2}/\sqrt{\mu}$ with $z=(\xi_0-\sqrt{\mu}\,|\xi''|)^{1/2}$. Moreover we observe that condition (iv) is satisfied with $\arg D(\xi'')=\pi/2$. In fact, $iR(\xi')$ is an analytic function of $-iz$ with real coefficients and $ir_2^+(\xi')<0$, so if $R(\xi')=0$ then $-iz\geqq 0$.

We shall examine reflection coefficients. Let first $b_1=b_2=0$ and $b_3=1$. Then we find that for $\xi_0>\sqrt{\mu}\,|\xi''|$ $c_{21}(\xi')=c_{12}(\xi')=0$, $c_{11}(\xi')=-1$ and $c_{22}(\xi')=1$. Thus the boundary condition does not couple two shear waves. Suppose next for example that $b_2=b_1\neq 0$ and $\xi_2^0=-\xi_1^0$. We then have

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}(\xi^{0\prime})=(b_3^2-2b_1^2)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}+2b_1b_3\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

which one can never bring to a triangular matrix by any similar transformation independent of b_1/b_3.

We can also treat similarly Maxwell's equations defined by

$$P(D)\begin{bmatrix} E \\ H \end{bmatrix}=\left(D_0+\frac{1}{i}\begin{bmatrix} 0 & -\mathrm{curl} \\ \mathrm{curl} & 0 \end{bmatrix}\right)\begin{bmatrix} E \\ H \end{bmatrix}=0 \quad \text{in} \quad \Omega,$$

where $E(H)$ is the electric (magnetic) field vector respectively, and $\det P(\xi)$

$=\xi_0^2(\xi_0^2-\xi_1^2-\xi_2^2-\xi_3^2)^2$. For the system a class of energy conserving boundary conditions is given in the form: $n\times E=b(n\times H)$, where b is a real valued function. (See for instance [13]). Note that $b=0$ corresponds to the classical boundary condition.

Let $\xi^{0\prime}\in R^3\backslash 0$ be a point such that $\xi_0^0=|\xi^{0\prime\prime}|>0$. Then for ξ' near $\xi^{0\prime}$ a basis $W(\xi)=[W_1(\xi), W_2(\xi)]/|\xi''|^2$ of the null space of $P(\xi)$ with $\xi_3^2=\xi_0^2-\xi_1^2-\xi_2^2$ is given by $W_1(\xi)={}^t(-\xi_1\xi_3, -\xi_2\xi_3, \xi_1^2+\xi_2^2, -\xi_0\xi_2, \xi_0\xi_1, 0)$ and $W_2(\xi)={}^t(\xi_0\xi_2, -\xi_0\xi_1, 0, -\xi_1\xi_3, -\xi_2\xi_3, \xi_1^2+\xi_2^2)$. Therefore we have $R(\xi')=\xi_0\xi_3^+(\xi')\cdot$(non-zero factor). (See for instance [5]). Consequently condition (iii) and (iv) with $D(\xi'')=0$ hold, while for $\xi_0>|\xi''|$

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}(\xi')=\frac{1-b^2}{1+b^2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}+\frac{2b}{1+b^2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus if $b=0$ then the boundary condition does not couple two waves, while if $b\neq 0$ then one can never bring $[c_{jk}]$ to a triangular matrix by any similar transformation independent of b. We may also show that condition (iv) with $\arg D(\xi'')=-\pi$ holds for each maximal dissipative boundary condition $B\begin{bmatrix} E \\ H \end{bmatrix}=0$ with B real valued which satisfies condition (iii). (See [5]).

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