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Measurable norms and rotationally quasi-invariant cylindrical measures

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1. Introduction

The notion of measurable norm was introduced by Gross in 1962 ([5]). Successively, he defined the abstract Wiener space and obtained many remarkable properties concerning the Gaussian cylindrical measure ([6, 7]). In 1971, Dudley-Feldman-Le Cam introduced another notion of measurable norm ([4]). This induces one solution of the radonification problem with respect to general cylindrical measures. These two concepts of measurability are very significant respectively.

We are interested in the relation between these two measurabilities. In [9] the author investigated rotationally invariant cylindrical measures as a generalization of the Gaussian cylindrical measure and showed that these two measurabilities coincide with each other with respect to every rotationally invariant cylindrical measure. And also, Badrikian-Chevet offered the following problem:

"*Do these two measurabilities coincide with each other for every cylindrical measure?*" (cf. [1]).

The author showed the existence of the counter example to the above question ([10]).

The problem of finding a largest class of cylindrical measures for which two measurabilities are equivalent is unsolved and apparently difficult.

In this note we shall introduce a new concept "*rotationally quasi-invariant cylindrical measure* (which is shortened to the more convenient *RQI-cylindrical measure*)", and investigate the characterization of this cylindrical measure. And also, we shall show that two measurabilities are equivalent for each RQI-cylindrical measure.

2. Preliminaries

Let X be a real separable Banach space, X' its topological dual, $(\cdot,\cdot)$ the natural pairing between X and X' and $\mathscr{B}(X)$ the Borel σ-algebra of X. Let $\{\xi_1, \cdots, \xi_n\}$ be a finite system of elements of X'. Then by Ξ we denote the operator from X into R^n mapping $x\in X\mapsto((x, \xi_1), \cdots, (x, \xi_n))\in R^n$. A set $Z\subset X$ is said to be a *cylindrical set* if there are $\xi_1, \cdots, \xi_n\in X'$ and $B\in\mathscr{B}(R^n)$ such that $Z=\Xi^{-1}(B)$. If Y is a closed subspace of finite codimension of X contained in the kernel of Ξ, then Ξ factorizes into

$$X\xrightarrow{\pi_{X/Y}} X/Y \xrightarrow{\Xi'} R^n ,$$

where $\pi_{X/Y}$ is the canonical surjection. We say that $(\Xi')^{-1}(B)$ is a *base* of the cylindrical set Z. Let $\mathscr{C}_X$ denote the family of all cylindrical sets of X. A mp μ from $\mathscr{C}_X$ into $[0, 1]$ is called a *cylindrical measure* if it satisfies the following two conditions: (1) $\mu(X)=1$; (2) Restrict μ to the σ-algebra of cylindrical sets which are generated by a fixed finite system of functionals. Then each such restriction is countably additive.

By putting $\mu_{\xi_1,\cdots,\xi_n}(B)=\mu(\Xi^{-1}(B))$ each cylindrical measure μ defines a family of Radon probability measures. A finite Radon measure means a finite Borel measure with inner regularity (see [11]). Throughout this paper we shall use this expression $\mu_{\xi_1,\cdots,\xi_n}$.

Here we shall present two definitions of measurability. Let H be a real separable Hilbert space with norm $\|\cdot\|$, $FD(H)$ the family of all finite dimensional subspaces of H, μ a cylindrical measure on H and $p(\cdot)$ be a continuous semi-norm on H.

DEFINITION 1 (Gross [5]). We say that $p(\cdot)$ is *μ-measurable by projections* if for every $\varepsilon>0$, there exists $G\in FD(H)$ such that $\mu(N_\varepsilon\cap F+F^\perp)\geqq 1-\varepsilon$ whenever $F\in FD(H)$ and $F\perp G$, where $N_\varepsilon=\{x\in H: p(x)<\varepsilon\}$ and $F^\perp$ is the orthogonal complement of F.

DEFINITION 2 (Dudley-Feldman-Le Cam [4]). A continuous semi-norm $p(\cdot)$ is said to be *μ-measurable* if for every $\varepsilon>0$, there exists $G\in FD(H)$ such that $\mu(P_F(N_\varepsilon)+F^\perp)\geqq 1-\varepsilon$ whenever $F\in FD(H)$ and $F\perp G$, where P_F is the orthogonal projection of H onto F.

If $p(\cdot)$ is μ-measurable by projections, then it is also μ-measurable. However, the converse was the open problem ([1]). Recently the author solved it negatively ([10]).

Let E be the Banach space obtained from H by means of $p(\cdot)$ and i be the canonical map from H into E. If $p(\cdot)$ is μ-measurable, then the image of μ under the map i, write $i(\mu)$, is countably additive, i.e., $i(\mu)$ is extensible to a Radon probability measure on E, and also *vice versa* ([4]).

If $\mu(C)=\mu(u(C))$ for any $C\in\mathscr{C}_H$ and any unitary operator u of H, μ is called a rotationally invariant cylindrical measure. We denote by $RI(H)$ the family of all rotationally invariant cylindrical measures on H.

We have the following theorem.

THEOREM A ([9]). *Let $p(\cdot)$ be a continuous semi-norm defined on H and $\mu\in RI(H)$. Then $p(\cdot)$ is μ-measurable if and only if $p(\cdot)$ is μ-measurable by projections.*

Moreover, let $\mu\in RI(H)\backslash\{\delta_0\}$ be given. If $p(\cdot)$ is μ-measurable, then $p(\cdot)$ is ν-measurable for every $\nu\in RI(H)\backslash\{\delta_0\}$.

The *Gaussian cylindrical measure* γ on H is the cylindrical measure defined as follows:

$$\gamma(Z)=(\sqrt{2\pi})^{-n}\int_B \exp\left(-\|x\|^2/2\right)dx$$

for $Z=\{x\in H: Px\in B\}$, where P is a finite dimensional orthogonal projection of H, $n=\dim PH$, $B\in\mathscr{B}(PH)$ and dx is the Lebesgue measure on PH. Let $p(\cdot)$ be a continuous norm on H, E be the completion of H with respect to $p(\cdot)$ and i be the inclusion map of H into E. If $p(\cdot)$ is γ-measurable by projections, then the triplet (i, H, E) is called an *abstract Wiener space.*

3. RQI-cylindrical measures

Let H be a real separable Hilbert space with norm $\|\cdot\|$ and U be the collection of all unitary operators of H.

(*) A cylindrical measure μ on H is said to be a *rotationally quasi-invariant cylindrical measure* (*RQI-cylindrical measure*) if for every $\varepsilon>0$, there exists a $\delta>0$ such that $\mu(A)<\delta$ implies $\mu(u(A))<\varepsilon$ for all $u\in U$ and all $A\in\mathscr{C}_H$.

In general, let λ and ν be two cylindrical measures on a Banach space X. We say that ν is *cylindrically absolutely continuous* with respect to λ, write $\nu\ll_c\lambda$, if for every $\varepsilon>0$, there exists a $\delta>0$ such that $\lambda(A)<\delta$ implies $\nu(A)<\varepsilon$ for all $A\in\mathscr{C}_X$. Two cylindrical measures λ and ν for which both $\nu\ll_c\lambda$ and $\lambda\ll_c\nu$ are called *cylindrically equivalent*, in symbols $\lambda\sim_c\nu$. If λ and ν are two Borel probability measures on X, then, by replacing $\mathscr{C}_X$ by $\mathscr{B}(X)$ in the above, we can define the notions of absolute continuity and equivalency between λ and ν, written by $\nu\ll\lambda$ and $\lambda\sim\nu$.

The definition (*) means that $\mu\sim_c u^{-1}(\mu)$ for all $u\in U$ and this relation holds uniformly with respect to $u\in U$. Notice that $u^{-1}(\mu)$ is the image of μ under the map u^{-1}.

Here, we are concerned with the characterization of RQI-cylindrical measures. Before getting in touch with the main subject, we shall explain the cylindrical measures to be of type 0.

Let X be a real Banach space, $\mathfrak{S}_X$ be the collection of all closed balls of X and μ be a cylindrical measure on X. We say that μ is of *type* 0 if for any $\varepsilon>0$ there exists a ball $A\in\mathfrak{S}_X$ such that $\mu_\xi(\xi(A))\geqq 1-\varepsilon$ for all $\xi\in X'$ (recall that $\mu_\xi=\mu\circ\xi^{-1}$).

For each cylindrical measure μ on X, the *characteristic function* of μ is the complex valued function $\phi(\mu,\cdot): X'\to \boldsymbol{C}$ defined by $\phi(\mu,\xi)=\int_X \exp\{i(x,\xi)\}\, d\mu(x)$.

An immediate calculation shows that

$$\int_X \exp\left\{i(x,\xi)\right\} d\mu(x)=\int_R \exp(it)\, d\mu_\xi(t)\,,$$

that is $\phi(\mu,\xi)=\phi(\mu_\xi, 1)$.

Also it is well known that μ determines the equivalence class of *linear random functions* $f: X'\to L^0(\Omega, m\,;\, R)$, where (Ω, m) is a Radon probability space and L^0 is the linear space of all real valued random variables, satisfying $\mu_\xi=f(\xi)(m)$, and *vice versa*. Here $f(\xi)(m)$ is the image measure of m under the map $f(\xi)$.

The following theorem shows the equivalence conditions for which μ is of type 0.

THEOREM B ([11, p. 193, THEOREM 1; p. 197, COROLLARY; p. 265, THEOREM 2]). *Let μ be a cylindrical measure on X and $f: X'\to L^0(\Omega, m\,;\, R)$ be a linear random function associated with μ. The following conditions are equivalent.*

(1) *μ is of type 0;*

(2) *the* map $\xi\in X'\mapsto\mu_\xi\in M^1(R)$, *where $M^1(R)$ is the space of all finite Radon measures on R equipped with the vague topology, is continuous;*

(2)' *the* map $\xi\in X'\mapsto\mu_\xi\in M^1(R)$ *is continuous at the origin;*

(3) *the characteristic function $\phi(\mu,\cdot)$ is continuous;*

(3)' *$\phi(\mu,\cdot)$ is continuous at the origin;*

(4) *f is continuous if $L^0(\Omega, m\,;\, R)$ is equipped with the topology of convergence in probability;*

(4)' *f is continuous at the origin.*

REMARKS. (1) Let Y' be a subspace of X'. Restrict $\mathscr{C}_X$ to Y' and denote it by $\mathscr{C}_X[Y']$, i. e., $\mathscr{C}_X[Y']$ is the family of all cylindrical sets Z such that

$$Z=\left\{x\in X:\ \left((x,\xi_1),\cdots,(x,\xi_n)\right)\in B,\ B\in\mathscr{B}(R^n)\right\},$$

where $\{\xi_1,\cdots,\xi_n\}$ is an arbitrary finite system of elements of Y'. Restrict μ to $\mathscr{C}_X[Y']$. Then, we have the same result as in Theorem B which is replaced X' by Y'.

(2) Let $\mathfrak{S}$ be the collection of all closed balls of X and $L_{\mathfrak{S}}(X, R^n)$ be the linear space of all continuous linear maps from X into R^n which is equipped with the $\mathfrak{S}$-topology. If μ is of type 0, then the map $\Xi \in L_{\mathfrak{S}}(X, R^n) \mapsto \Xi(\mu) \in M^1(R^n)$ is continuous, where $M^1(R^n)$ is equipped with the vague topology.

Now we start with the next lemma. Notice that H and H' are identified.

LEMMA 1. *Every RQI-cylindrical measure on H is of type* 0.

PROOF. Let μ be any RQI-cylindrical measure on H. Take $x \in H$ such that $||x||=1$. For any $y \in H$ $(y \neq 0)$, put $y'=y/||y||$. Since $||x||=||y'||$, there is a $u \in U$ such that $y'=u^{-1}(x)$. By the definition (*), for every $\varepsilon>0$ there exists a $\delta>0$ such that $\mu(A) \geqq 1-\delta$ implies $\mu(v(A)) \geqq 1-\varepsilon$ for all $v \in U$ and all $A \in \mathscr{C}_H$. Since μ_x is a Radon probability measure on R, we have the ball $S_r=\{t \in H\colon ||t|| \leqq r\}$ satisfying that $\mu_x(x(S_r)) \geqq 1-\delta$. Hence

$$(3.1) \qquad \left(u(\mu)\right)_x \left(x(S_r)\right) \geqq 1-\varepsilon .$$

On the other hand, we have

$$(3.2) \qquad \begin{aligned} \mu_{y'}\left(y'(S_r)\right) &= \mu_{u^{-1}(x)}\left(u^{-1}(x)(S_r)\right) \\ &= \left(u(\mu)\right)_x \left(x\left(u(S_r)\right)\right) \\ &= \left(u(\mu)\right)_x \left(x(S_r)\right) . \end{aligned}$$

Also, it is easy to see that

$$(3.3) \qquad \mu_y\left(y(S)\right) = \mu_{y'}\left(y'(S)\right) \qquad \text{for all } S \in \mathfrak{S}_H .$$

It follows from (3.1), (3.2) and (3.3) that for every $\varepsilon>0$ there exists a ball S_r such that $\mu_y(y(S_r)) \geqq 1-\varepsilon$ for all $y \in H$. This means that μ is of type 0.

Let $\{e_n\}_{n=1,2,\cdots}$ be an orthonormal basis of H and N be the set of all positive integers. Let H_n $(n \in N)$ be the subspace of H generated by $\{e_1, \cdots, e_n\}$. For each $n \in N$, we denote by U_n the subset of U consisting of u satisfying the following two conditions: (1) Restrict u to the space H_n. Then such restriction is a unitary operator of H_n. (2) Restrict u to the space $H_n^{\perp}$. Then it is the identity operator on $H_n^{\perp}$. It is easy to see that U_n is isomorphic to the n-dimensional orthogonal group $O(n)$. Since $O(n)$ is the compact topological group, we have the normalized Haar measure on U_n, denote it by m_n.

LEMMA 2. *Let μ be an RQI-cylindrical measure on H, and H_n, U_n and m_n $(n \in N)$ be as in the above. Define*

$$\lambda_n(A)=\int_{U_n}\mu\Big(u(A)\Big)\,dm_n(u) \qquad \textit{for every } A\in\mathscr{C}_H.$$

Then λ_n is a cylindrical measure on H.

PROOF. Let $A\in\mathscr{C}_H$ be given. There exists $F\in FD(H)$ such that a base of A is on F. Let G be the finite dimensional subspace generated by $F\cup H_n$. Then we can see that a base of A is also on G. For any $u\in U_n$, its restriction to G is a unitary operator of G and its restriction to $G^{\perp}$ is the identity map on $G^{\perp}$. Therefore $u(A)$ has its base on G for every $u\in U_n$. If μ is restricted to the σ-algebra of cylindrical sets of which bases are in $\mathscr{B}(G)$, then such restriction is countably additive. And so we can consider such restriction of μ as a Radon probability measure on G. Therefore, $\lambda_n(A)$ is well defined for each $A\in\mathscr{C}_H$. Also we can conclude that λ_n is a cylindrical measure on H.

Let $\{\nu_\iota\}_{\iota\in I}$ be a collection of cylindrical measures on a Banach space X. We say that $\{\nu_\iota\}_{\iota\in I}$ is *uniformly of type* 0 for $\iota\in I$ if for any $\varepsilon>0$ there exists a ball $S\in\mathfrak{S}_X$ such that $(\nu_\iota)_\xi(\xi(S))\geqq 1-\varepsilon$ for all $\xi\in X'$ and all $\iota\in I$.

LEMMA 3. *Notation is as in Lemma 2. Then $\mu\sim_c\lambda_n$ for all $n\in N$ and this relation holds uniformly with respect to $n\in N$.*

Furthermore, the class $\{\lambda_n\}_{n\in N}$ is uniformly of type 0 *for $n\in N$.*

PROOF. It follows from the definition (*) that for every $\varepsilon>0$, there exists a $\delta>0$ such that $\mu(A)<\delta$ implies $\mu(u(A))<\varepsilon$ for all $u\in U$ and all $A\in\mathscr{C}_H$. Therefore, $\mu(A)<\delta$ implies $\lambda_n(A)=\int_{U_n}\mu(u(A))\,dm_n(u)<\varepsilon$ for all $A\in\mathscr{C}_H$ and all $n\in N$.
Thus we have $\lambda_n\ll_c\mu$ uniformly for $n\in N$.

Next, assume that $\lambda_n(A)<\delta$ for some n. Then there exists a $v\in U_n$ such that $\mu(v(A))<\delta$. Therefore, $\mu(u(v(A)))<\varepsilon$ for all $u\in U$. Hence we have $\mu(A)<\varepsilon$. This means that $\mu\ll_c\lambda_n$. Thus we complete the proof of the former half. Also, it follows from Lemma 1 that μ is of type 0. Therefore the rest is an immediate consequence of the former.

LEMMA 4. *Let ξ and η be in H. If $||\xi||=||\eta||$, then for every $\varepsilon>0$, there exists a positive integer m^ε and there exists $v_m^\varepsilon\in U_m$ for each $m\geqq m^\varepsilon$ such that $||v_m^\varepsilon(\xi)-\eta||<\varepsilon$.*

PROOF. We denote by $\langle\cdot,\cdot\rangle$ the inner product on H. Then we can express $\xi=\sum_{n=1}^{\infty}\langle\xi,e_n\rangle e_n$ and $\eta=\sum_{n=1}^{\infty}\langle\eta,e_n\rangle e_n$. Let $\xi_N=\sum_{n=1}^{N}\langle\xi,e_n\rangle e_n$ and $\eta_N=\sum_{n=1}^{N}\langle\eta,e_n\rangle e_n$.

For every $\varepsilon>0$, there exists a positive integer m^ε such that $|\,||\xi_N||-||\eta_N||\,|$

$<\varepsilon/3$, $\|\xi-\xi_N\|<\varepsilon/3$ and $\|\eta-\eta_N\|<\varepsilon/3$ for every $N\geqq m^\varepsilon$.

We can assume that $\|\xi_N\|\neq 0$ and $\|\eta_N\|\neq 0$. Let $\xi'_N=\frac{\|\eta_N\|}{\|\xi_N\|}\xi_N$. Then we have $\|\xi'_N\|=\|\eta_N\|$. Also, ξ'_N, $\eta_N\in H_N$.

Therefore, we have $v^\varepsilon_N\in U_N$ such that $v^\varepsilon_N(\xi'_N)=\eta_N$. Then,

$$\begin{aligned}&\left\|v^\varepsilon_N(\xi_N)-\eta_N\right\|\\&\quad=\left\|\frac{\|\xi_N\|}{\|\eta_N\|}\eta_N-\eta_N\right\|\\&\quad=\left|\|\xi_N\|-\|\eta_N\|\right|<\varepsilon/3\,.\end{aligned}$$

On the other hand, $\xi=\xi_N+\sum_{n=N+1}^{\infty}\langle\xi, e_n\rangle e_n$. Hence $v^\varepsilon_N(\xi)=v^\varepsilon_N(\xi_N)+\sum_{n=N+1}^{\infty}\langle\xi, e_n\rangle e_n$ $=v^\varepsilon_N(\xi_N)+\xi-\xi_N$, i.e., $v^\varepsilon_N(\xi-\xi_N)=\xi-\xi_N$. Thus we have

$$\begin{aligned}&\left\|v^\varepsilon_N(\xi)-\eta\right\|\\&\quad=\left\|v^\varepsilon_N(\xi_N)+\xi-\xi_N-\eta\right\|\\&\quad\leqq\left\|v^\varepsilon_N(\xi_N)-\eta_N\right\|+\|\eta_N-\eta\|+\|\xi-\xi_N\|\\&\quad<\varepsilon\,.\end{aligned}$$

Let $\{\nu_i\}$ be a sequence of cylindrical measures on a Banach space X. We say that the sequence $\{\nu_i\}$ *converges cylindrically* to ν if $\{(\nu_i)_{\xi_1,\cdots,\xi_n}\}$ converges vaguely to $\nu_{\xi_1,\cdots,\xi_n}$ for every finite system $\{\xi_1,\cdots,\xi_n\}\subset X'$.

Let $M^1(X)$ be the linear space of all finite Radon measures on X. The *narrow topology* on $M^1(X)$ is defined as the topology of pointwise convergence on bounded continuous functions.

Now we shall show the main theorem.

THEOREM 1. *A cylindrical measure μ on H is rotationally quasi-invariant if and only if there exists a rotationally invariant cylindrical measure λ on H such that $\mu\sim_c\lambda$.*

PROOF. Suppose that μ is an RQI-cylindrical measure. Let T be an injective Hilbert-Schmidt operator of H, $p(x)=\|Tx\|$ and (i, H, E) be the abstract Wiener space induced by the norm $p(\cdot)$. Let $\{e_n\}_{n=1,2,\cdots}$ be an orthonormal basis of H. Regarding this orthonormal basis $\{e_n\}_{n=1,2,\cdots}$, we define H_n, U_n, m_n *and* λ_n as in the preceding lemmas. Since (i, H, E) is the abstract Wiener space induced by the Hilbert-Schmidt operator T, we can see that for every cylindrical measure of type 0, each image under the map i is countably additive, i.e. extensible to a Radon probability measure on E (cf. [12, XXV. 1, PROPOSITION (XXV, 1; 1)]). It follows from Lemmas

1 and 3 that the images of μ and λ_n's are extensible to Radon probability measures on E, denote them by $\bar{\mu}$ and $\bar{\lambda}_n$ $(n\in N)$.

It is easy to see that $\bar{\mu}\sim\bar{\lambda}_n$ for all $n\in N$ and this relation holds uniformly for $n\in N$. Since $\bar{\mu}$ is the Radon probability measure on E, for every $\varepsilon>0$, there exists a compact subset K of E such that $\bar{\mu}(K)\geqq 1-\varepsilon$. The above-mentioned uniform equivalency among $\bar{\mu}$ and $\bar{\lambda}_n$ $(n\in N)$ shows that for every $\varepsilon>0$, there exists a compact subset K of E such that $\bar{\lambda}_n(K)\geqq 1-\varepsilon$ for all $n\in N$. This implies that the sequence $\{\bar{\lambda}_n\}$ contains a convergent subsequence $\{\bar{\lambda}_{n_j}\}$ with respect to the narrow topology, denote the limit by $\bar{\lambda}$ (cf. [11, p. 381, THEOREM 4; 12, III. 5, PROPOSITION (III; 4, 1)]).

(I) First we shall show that there uniquely exists a cylindrical measure λ of type 0 on H such that the extension of its image under i to the σ-algebra $\mathscr{B}(E)$ coincides with $\bar{\lambda}$, and also that λ_{n_j} converges cylindrically to λ as $j\to\infty$.

The restriction of $\bar{\lambda}$ to $\mathscr{C}_E$ is the cylindrical measure of type 0 on E. Theorem B says that there exists the continuous linear random function $\bar{f}$: $E'\to L^0(\Omega, m\,;\, R)$ associated with the restriction of $\bar{\lambda}$.

Let $\langle\cdot,\cdot\rangle$ be the inner product on H and $(\cdot,\cdot)$ be the natural pairing between E and E'. Let $\mathscr{C}_H[i^*(E')]$ be the collection of cylindrical sets Z such that

$$Z=\left\{x\in H\colon \left(\left\langle x, i^*(\xi_1)\right\rangle, \cdots, \left\langle x, i^*(\xi_n)\right\rangle\right)\in B,\ B\in\mathscr{B}(R^n)\right\}$$

for every finite system $\{\xi_1, \cdots, \xi_n\}\subset E'$, where i^* is the adjoint of i.

Define λ' as follows:

$$\lambda'(Z)=\bar{\lambda}(\bar{Z})\quad \text{for all } Z\in\mathscr{C}_H\left[i^*(E')\right],$$

where $\bar{Z}=\{y\in E\colon ((y,\xi_1), \cdots, (y,\xi_n))\in B\}$ if

$$Z=\left\{x\in H\colon \left(\left\langle x, i^*(\xi_1)\right\rangle, \cdots, \left\langle x, i^*(\xi_n)\right\rangle\right)\in B,\ B\in\mathscr{B}(R^n)\right\}.$$

Also, define f' on $i^*(E')$ as follows: $f'(i^*(\xi))=\bar{f}(\xi)$ for all $\xi\in E'$. The subspace $i^*(E')\subset H$ is dense in H and also the linear space $L^0(\Omega, m\,;\, R)$ is the complete metric space. Therefore, if we can show the continuity of f' at the origin of H, then we have the extension of f' to the space H, denote it by f. It is easy to see that the associated cylindrical measure with f is of type 0 and coincides with λ' on $\mathscr{C}_H[i^*(E')]$. In other words, we have the extension of λ' to $\mathscr{C}_H$, and denote it by λ. Obviously we can check that the extension of $i(\lambda)$ to $\mathscr{B}(E)$ is $\bar{\lambda}$, and such λ is uniquely defined because λ is of type 0 (see [11, p. 200, PROPOSITION 6]). Then we have to show the continuity of f' at the origin of H. By Lemma 3, the family $\{\lambda_{n_j}\}_{j=1,2,\cdots}$ is uniformly of type 0, i. e., for every $\varepsilon>0$ there exists a positive number r such that $(\lambda_{n_j})_x(x(S_r))\geqq 1-\varepsilon$ for all $x\in H$ and all $j\in N$, where $S_r=\{t\in H: ||t||\leqq r\}$ as before. Obviously, the sequence of Radon probability measures $\{(\lambda_{n_j})_x\}_{j=1,2,\cdots}$ vaguely converges to λ'_x for all $x\in i^*(E')$. Therefore, we have $\lambda'_x(x(S_r))\geqq 1-\varepsilon$ for all $x\in i^*(E')$. Applying the remarks of Theorem B, we can conclude that f' is continuous at the origin of H.

It remains to show that $\lambda_{n_j}\to\lambda$ cylindrically as $j\to\infty$, i. e., $(\lambda_{n_j})_{x_1,\cdots,x_n}\to\lambda_{x_1,\cdots,x_n}$ vaguely as $j\to\infty$ for every finite system $\{x_1,\cdots,x_n\}\subset H$. Since $i^*(E')$ is dense in H, we can choose $\xi_k^m\in i^*(E')$ ($k=1,2,\cdots,n$; $m=1,2,\cdots$) such that $\xi_k^m\to x_k$ as $m\to\infty$ for each k ($k=1,2,\cdots,n$). For each m,

$$(3.4)\qquad (\lambda_{n_j})_{\xi_1^m,\cdots,\xi_n^m}\longrightarrow\lambda_{\xi_1^m,\cdots,\xi_n^m}\qquad\text{vaguely as } j\to\infty\,.$$

Since λ is of type 0, we can use the remarks of Theorem B and we have

$$(3.5)\qquad \lambda_{\xi_1^m,\cdots,\xi_n^m}\longrightarrow\lambda_{x_1,\cdots,x_n}\qquad\text{vaguely as } m\to\infty\,.$$

Also, since the family $\{\lambda_{n_j}\}$ is uniformly of type 0 for $j\in N$, we can check that

$$(3.6)\qquad (\lambda_{n_j})_{\xi_1^m,\cdots,\xi_n^m}\longrightarrow(\lambda_{n_j})_{x_1,\cdots,x_n}\qquad\text{vaguely as } m\to\infty$$

and this relation holds uniformly for $j\in N$.

The facts (3. 4), (3. 5) and (3. 6) implies that $(\lambda_{n_j})_{x_1,\cdots,x_n}\to\lambda_{x_1,\cdots,x_n}$ vaguely as $j\to\infty$. Hence we have the desired result.

(II) Next, we shall show that λ is a rotationally invariant cylindrical measure on H.

We have to prove $\lambda=u^{-1}(\lambda)$ for all $u\in U$. It is sufficient to show that $\phi(\lambda,\xi)=\phi(u^{-1}(\lambda),\xi)=\phi(\lambda,u(\xi))$ for every $\xi\in H$ (cf. [2, 11]).

Write $\eta=u(\xi)$.

Since $\lambda_{n_j}\to\lambda$ cylindrically as $j\to\infty$, we can see that $(\lambda_{n_j})_\xi\to\lambda_\xi$ narrowly as $j\to\infty$. Therefore, $\phi((\lambda_{n_j})_\xi,1)$ converges to $\phi(\lambda_\xi,1)$ as $j\to\infty$, i. e.,

$$(3.7)\qquad \phi(\lambda_{n_j},\xi)\longrightarrow\phi(\lambda,\xi)\qquad\text{as}\quad j\to\infty\,.$$

Similarly, we have

$$(3.8)\qquad \phi(\lambda_{n_j},\eta)\longrightarrow\phi(\lambda,\eta)\qquad\text{as}\quad j\to\infty\,.$$

Since $||\xi||=||\eta||$, we can use Lemma 4. Take $\varepsilon=1$ in Lemma 4. There exists a positive integer m^1 and there exists $v_m^1\in U_m$ for each $m\geqq m^1$ such that $||v_m^1(\xi)-\eta||<1$. We have j_1 such that $\max(m^1,1)<n_{j_1}$, and put $v^1_{n_{j_1}}=\bar{v}_1$. Next, take $\varepsilon=1/2$. We have j_2 such that $\max(m^{1/2},n_{j_1})<n_{j_2}$, and put $v^{1/2}_{n_{j_2}}=\bar{v}_2$. Repeating this process, we get the subsequence $\{n_{j_k}\}$ of $\{n_j\}$ satisfying that $n_{j_1}<\cdots<n_{j_k}<n_{j_{k+1}}<\cdots$ and $||\bar{v}_k(\xi)-\eta||<1/k$ for each $k\in N$.
Therefore,

$$\lim_{k\to\infty} \bar{v}_k(\xi) = \eta\,. \tag{3.9}$$

And also, the family $\{\lambda_{n_{j_k}}\}$ is uniformly of type 0 for $k \in N$. Hence, it follows from (3. 8) and (3. 9) that for every $\varepsilon > 0$ there exists a positive integer k_0 such that

$$\begin{aligned} &\left|\phi\left(\lambda_{n_{j_k}}, \bar{v}_k(\xi)\right) - \phi(\lambda, \eta)\right| \\ &\quad \leqq \left|\phi\left(\lambda_{n_{j_k}}, \bar{v}_k(\xi)\right) - \phi(\lambda_{n_{j_k}}, \eta)\right| + \left|\phi(\lambda_{n_{j_k}}, \eta) - \phi(\lambda, \eta)\right| \\ &\quad < \varepsilon \qquad \text{for all } k \geqq k_0\,. \end{aligned}$$

Thus we have

$$\phi\left(\lambda_{n_{j_k}}, \bar{v}_k(\xi)\right) \longrightarrow \phi(\lambda, \eta) \qquad \text{as } k \to \infty\,. \tag{3.10}$$

On the other hand, the definition of λ_n shows that

$$\phi(\lambda_n, \xi) = \phi\left(\lambda_n, u_n(\xi)\right) \qquad \text{for all } u_n \in U_n\,.$$

Therefore, using (3. 7), we have

$$\phi\left(\lambda_{n_{j_k}}, \bar{v}_k(\xi)\right) = \phi(\lambda_{n_{j_k}}, \xi) \longrightarrow \phi(\lambda, \xi) \qquad \text{as } k \to \infty\,. \tag{3.11}$$

By (3. 10) and (3. 11), we have $\phi(\lambda, \xi) = \phi(\lambda, \eta)$. This completes the proof of (II).

(III) Finally, we have to show that $\mu \sim_c \lambda$. However, this is the immediate consequence of that $\lambda_{n_j} \to \lambda$ cylindrically as $j \to \infty$ and $\mu \sim_c \lambda_{n_j}$ uniformly for $j \in N$.

Thus we conclude that if μ is an RQI-cylindrical measure, then there exists a rotationally invariant cylindrical measure λ such that $\mu \sim_c \lambda$.

The converse is trivial. Then we complete the full proof.

Using the method of the above proof, we have the following corollary.

COROLLARY 1. *Notation is as in Theorem* 1. *Let ν be a cylindrical measure of type* 0 *on H. If ν is invariant for every $u \in \bigcup_{n=1}^{\infty} U_n$, then ν is rotationally invariant.*

Let RQI(H) be the family of all RQI-cylindrical measures on H.

THEOREM 2. *Let μ be an RQI-cylindrical measure on H and $p(\cdot)$ be a continuous semi-norm defined on H. Then $p(\cdot)$ is μ-measurable if and only if $p(\cdot)$ is μ-measurable by projections.*

Moreover, let $\mu \in RQI(H) \backslash \{\delta_0\}$ be given. If $p(\cdot)$ is μ-measurable, then $p(\cdot)$ is ν-measurable for every $\nu \in RQI(H) \backslash \{\delta_0\}$.

PROOF. Assume that $p(\cdot)$ is μ-measurable. Theorem 1 says that there exists a rotationally invariant cylindrical measure λ such that $\mu\sim_c\lambda$. Therefore, $p(\cdot)$ is λ-measurable. Using Theorem A, we can say that $p(\cdot)$ is λ-measurable by projections. Hence, using the fact $\mu\sim_c\lambda$ again, we conclude that $p(\cdot)$ is μ-measurable by projections. Thus we have the proof of the former half. Similarly, using Theorems A and 1, we have the latter half.

Thus we have a generalization of Theorem A.

4. Examples and application

In this section, first we shall show some examples of RQI-cylindrical measure.

Let H be a real separable Hilbert space as ever and μ be a cylindrical measure on H. We say that μ is cylindrically equivalent to the Lebesgue measure if each $\mu_{e_1,\cdots,e_n}$ is equivalent to the Lebesgue measure on R^n for any finite orthonormal system $\{e_1, \cdots, e_n\}\subset H$.

EXAMPLE 1. Let λ be a rotationally invariant cylindrical measure on H which is cylindrically equivalent to the Lebesgue measure. Define the translation of λ as follows: for each $x\in H$, $\lambda^x(A)=\lambda(A+x)$ for all $A\in\mathscr{C}_H$. Then every λ^x is an RQI-cylindrical measure.

Indeed, we have $\lambda^x\sim_c\lambda$ for every $x\in H$ (cf. [8]).

EXAMPLE 2. Let $\{e_i\}_{i=1,2,\cdots}$ be an orthonormal basis of H and ϕ be the bounded continuous positive function defined on R^n. Let λ be a rotationally invariant cylindrical measure on H and $f(x)=\phi(\langle x, e_1\rangle, \cdots, \langle x, e_n\rangle)$. If $\int_H f(x)\,d\lambda=1$, then we can define the cylindrical measure μ as follows:

$$\mu(A)=\int_A f(x)\,d\lambda \qquad \text{for } A\in\mathscr{C}_H.$$

It is easy to check that $\mu\sim_c\lambda$ and μ is an RQI-cylindrical measure.

REMARK. Clearly, we can choose ϕ such that μ is not any translation of a rotationally invariant cylindrical measure.

EXAMPLE 3. Let $\{e_i\}_{i=1,2,\cdots}$ be an orthonormal basis of H, for each $n\in N$, F_n be the 1-dimensional subspace of H generated by $\{e_n\}$, $H^n=(F_1\oplus\cdots\oplus F_n)^\perp$ and γ be the Gaussian cylindrical measure on H. Put $f(x)=(\sqrt{2\pi})^{-1}\exp(-x^2/2)$ for $x\in R$. Let $\{f_n(x)\}_{n=1,2,\cdots}$ be the sequence of positive real valued continuous functions defined on R satisfying that

$$\int_{-\infty}^{\infty} f_n(x)\,dx=1 \quad\text{and}\quad \int_{-\infty}^{\infty}\left|f_n(x)-f(x)\right|dx<2^{-n} \quad \text{for all } n\in N.$$

We define probability measures $\nu^n=f_n(x)\,dx$ $(n\in N)$ on R. For each $n\in N$, using the mapping $t \in R \mapsto te_n \in F_n$, we have the image measure of ν^n on F_n and denote by ν_n.

Since $H = F_1 \oplus \cdots \oplus F_n \oplus H^n$, we can define a cylindrical measure $\mu_n = \nu_1 \otimes \cdots \otimes \nu_n \otimes \gamma^n$ on H, where γ^n is the Gaussian cylindrical measure on H^n.

Using the method of Theorem 1, we have the result that there is a subsequence $\{\mu_{n_j}\}$ of $\{\mu_n\}$ which converges cylindrically to some cylindrical measure, denote it by μ. It is easy to check that μ is an RQI-cylindrical measure. Indeed we have $\mu \sim_c \gamma$.

Thus we conclude that there exist many RQI-cylindrical measures which are neither rotationally invariant cylindrical measures nor translations of them.

Let μ be a cylindrical measure on a Banach space X. Suppose that $\mu^x \sim_c \mu$ for every $x \in X$. Then μ is said to be quasi-invariant ([8]).

PROPOSITION 1. *If μ is an RQI-cylindrical measure on H which is cylindrically equivalent to the Lebesgue measure, then μ is quasi-invariant.*

PROOF. Since μ is an RQI-cylindrical measure, there exists a rotationally invariant cylindrical measure λ such that $\mu \sim_c \lambda$. It is easy to see that λ is cylindrically equivalent to the Lebesgue measure. The result in [8] says that λ is quasi-invariant. The fact $\mu \sim_c \lambda$ implies that $\mu^x \sim_c \lambda^x$ for every $x \in H$. Therefore, μ is quasi-invariant.

This proposition and Examples 2 and 3 say that every quasi-invariant cylindrical measure on H is not necessary the translation of a rotationally invariant cylindrical measure.

REMARK. Linde offered in [8] the following problem: "Is every quasi-invariant cylindrical measure on a Hilbert space the translation of a rotationally invariant cylindrical measure?"

Chevet solved negatively this problem by the counter example ([3]). Here, we showed another method of construction of counter examples.

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References

[1] A. BADRIKIAN and S. CHEVET: Mesures cylindriques, espaces de Wiener et fonctions aleatoires gaussiennes, Lecture Notes in Math. 379, Springer Verlag, Berlin-Heidelberg-New York, 1974.

[2] P. BAXENDALE: Gaussian measures on function spaces, Amer. J. of Math., 98 (1976), 891–952.

[3] S. CHEVET: Quelques nouveaux resultats sur les mesures cylindriques, Lecture Notes in Math. 644, Springer Verlag, Berlin-Heidelberg-New York, 1978.

[4] R. M. DUDLEY, J. FELDMAN and L. Le CAM: On semi norms and probabilities, and abstract Wiener spaces, Ann. of Math., 93 (1971), 390–408.

[5] L. GROSS: Measurable functions on Hilbert space, Trans. Amer. Math. Soc., 105 (1962), 372–390.

[6] L. GROSS: Abstract Wiener spaces, Proc. Fifth Berkeley Symp. on Math. Stat. and Prob., 2 (1965), 31–42.

[7] L. GROSS: Potential theory on Hilbert space, J. Functional Analysis, 1 (1967), 123–181.

[8] W. LINDE: Quasi-invariant cylindrical measures, Z. Wahrscheinlichkeitstheorie verw. Gebiete, 40 (1977), 91–99.

[9] M. MAEDA: Rotationally invariant cylindrical measures I, to appear.

[10] M. MAEDA: Some examples on measurable norms, to appear.

[11] L. SCHWARTZ: Radon measures on arbitrary topological spaces and cylindrical measures, Oxford University press, Bombay, 1973.

[12] L. SCHWARTZ: Applications radonifiantes, Séminaire L. Schwartz 1969–1970, (Ecole Polytechnique, Paris).

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