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KMO-Langevin Equation and Fluctuation-Dissipation Theorem (I)

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§ 1. Introduction

In the course of theoretical and experimental investigations for confirming the **Alder-Wainwright effect** ([1], [2], [12] and [23]), it has become clear that Brownian motion with such an effect can be described by an equation treated in G. Stokes and J. Boussinesq ([24], [3], [4] and [13]), whose equation with a random force—**Stokes-Boussinesq-Langevin equation**—gives a precise description of the time evolution of Ornstein-Uhlenbeck's Brownian motion treated in A. Einstein and P. Langevin ([5], [14] and [25]).

In [22] we have then introduced two kinds of random forces for Stokes-Boussinesq-Langevin equation: one is a white noise and the other is a Kubo noise (cf. (8.13) in [22]) (A precise definition will be given in § 6 of this paper). And we have proved that the unique stationary solution for Stokes-Boussinesq-Langevin equation with a white noise or a Kubo noise as a random force has a qualitative nature of T-positivity and then satisfies an Alder-Wainwright effect—a long-time behavior ($\propto t^{-\frac{3}{2}}$) of velocity autocorrelation function. Next, as a generalization of Stokes-Boussinesq-Langevin equation, we have derived two kinds of Langevin equations—a first KMO-Langevin equation with a white noise as a random force and a second KMO-Langevin equation with a Kubo noise as a random force—which describe the time evolution of a real stationary Gaussian process with a qualitative nature of T-positivity. Furthermore we have clarified a mathematical stucture of the Kubo's fluctuation-dissipation theorem in his linear response theory in statistical physics ([8], [9], [10] and [11]), by proving a new type of fluctuation-dissipation theorem for the first KMO-Langevin equation and the Kubo's fluctuation-dissipation theorem for the second KMO-Langevin equation.

According to the so-called fluctuation-dissipation theorem in statistical physics, we know ([6]) that, in a physical linear system taking a reciprocal action with a microscopic and kinetic quantity which is itself doing a thermal motion, there exists a relation between the system function and the spectral density of the physical system.

In performing a data-treatment through the experiment and measurement for a physical system, a technological system, a statistical system, an economical system, a biological system and so on, it seems to be true that as its first step we can use only a correlation function or equivalently a spectral measure which can be measured. At this step we do not know what kind of equation describes the time evolution of the system concerned. Therefore, it seems to be useful, important and moreover fundamental in Kubo's linear response theory that, for a stationary curve A with a measured correlation function R as its covariance function, we construct such a linear system that A flows out as an output and a function $\chi_{(0,\infty)} R$ becomes a response function. Then we understand that the input in such a linear system is a Kubo noise. That is, we have to define a Kubo noise as an input in such a linear system and then derive an equation describing the time evolution of A with the Kubo noise as a random force. The author thinks that it is possible only after such a procedure that the Kubo's fluctuation-dissipation theorem in statistical physics has a physical meaning and a mathematical embodyment.

According to the spirit stated above, we shall state a content of this paper. Let R be any fixed correlation function. We define a holomorphic function $[R]$ on $\boldsymbol{C}^+$ by

$$(1.1) \qquad [R](\zeta)=\frac{1}{2\pi}\int_0^\infty e^{i\zeta t}R(t)\,dt.$$

In § 2 we shall recall Mori's theory of generalized Brownian motion which is applicable to a differentiable stationary curve A in a Hilbert space $\mathscr{H}$ with R as its covariance function ([15]):

$$(A(s),\ A(t))_{\mathscr{H}}=R(s-t) \quad (s,\ t\in\boldsymbol{R}).$$

Under the following further conditions (1.2), (1.3)[1] and (1.4):

$$(1.2) \qquad R(0)\neq 0$$

(1.3) there exists a null set Λ_R in $\boldsymbol{R}-\{0\}$ such that for any $\xi\in\boldsymbol{R}-\Lambda_R$

(1) Since the function $[R]$ is holomorphic on $\boldsymbol{C}^+$ with a positive real part, we can find from the theory of Herglotz function that there exists $\lim_{\eta\downarrow 0}[R](\xi+i\eta)$ for almost all $\xi\in\boldsymbol{R}$. Therefore, the essential part of condition (1.3) is that the $\lim_{\eta\downarrow 0}[R](i\eta)$ exists. The author would like to thank to the referee for pointing this fact to him.

$$\lim_{\eta\downarrow 0}\ [R](\xi+i\eta)\equiv[R](\xi+i0)\ \text{exists}$$

(1.4) there exist positive constants c and m such that

$$|[R](\xi)|\geq c(1+|\xi|^{m})^{-1}\ \text{for any}\ \xi\in\boldsymbol{C}^{+},$$

we shall in § 3 introduce a second KMO-Langevin data and then rewrite Mori's result in § 2 into such a form that we can prove Kubo's fluctuation-dissipation theorem for a non-differentiable stationary curve in a Hilbert space with R as its covariance function (Theorem 3.1). We note that Kubo's first fluctuation-dissipation theorem, in particular, Einstein relation does not hold for Mori's memory kernel equation and moreover a random force (called Mori noise) in Mori's memory kernel equation does not satisfy a condition of causality.

In § 4 we shall determine the function $[R]$ under only conditions (1.2), (1.3) and (1.4), by using Mori's result in § 2 through an approximation procedure (Theorem 4.1):

$$(1.5)\qquad [R](\xi)=\frac{\alpha}{\sqrt{2\pi}}\frac{1}{\beta-i\xi+\lim_{\varepsilon\downarrow 0}(-i\xi)K_{\varepsilon}(\xi)}\qquad(\xi\in\boldsymbol{C}^{+}).$$

Here

$$K_{\varepsilon}(\xi)=\int_{\boldsymbol{R}}\frac{1}{(\lambda-\xi-i\varepsilon)(\lambda-i\varepsilon)}\varkappa(d\lambda)$$

and $\alpha>0$, $\beta\in\boldsymbol{C}-\{0\}$ and $\varkappa$ is a Borel measure on $\boldsymbol{R}$.

We call the triple $(\alpha,\beta,\varkappa)$ the **second KMO-Langevin data** associated with R which can be represented by the following formulae (Theorem 4.2):

$$(1.6)\qquad \begin{cases} \alpha=\dfrac{R(0)}{\sqrt{2\pi}} \\ \beta=\dfrac{R(0)}{2\pi[R](0+i0)} \\ \varkappa(d\lambda)=\dfrac{R(0)}{2\pi^{2}}\mathrm{Re}\{([R](\lambda+i0))^{-1}\}d\lambda. \end{cases}$$

We will find that a kind of renormalization—an elimination of occurrence of infinity minus infinity—is taken in the passing to the limit in the proof of (1.5).

Through the relation (1.5), we shall in § 5 give a bijective correspondence between the set $\mathscr{R}$ of correlation functions with conditions (1.2), (1.3) and (1.4) and the set $\mathscr{L}$ of second KMO-Langevin data (Theorem 5.1). Furthermore we shall characterize two subclasses $\mathscr{L}_0$ (resp. $\mathscr{L}_1$) of $\mathscr{L}$ such that R has a spectral density (resp. R has a spectral density with finite second moment) (Theorems 5. 2 and 5. 3) We note that Theorem 5. 3 gives a characterization of second KMO-Langevin data introduced in § 3.

From § 6 to § 8, we shall treat any correlation function R of $\mathscr{L}_0$ and any stationary curve $A=(A(t)\,;\,t\in\boldsymbol{R})$ in a Hilbert space $\mathscr{H}$ with R as the covariance function of A. By using a spectral representation of A:

$$A(t)=\int_{\boldsymbol{R}} e^{-it\lambda}dE(\lambda)A(0),$$

where $(E(\lambda)\,;\,\lambda\in\boldsymbol{R})$ is a resolution of identity in $\mathscr{H}$, we shall define a **Kubo noise** $I=(I(\phi)\,;\,\phi\in\mathscr{S}(\boldsymbol{R}))$ as an $\mathscr{H}$-valued stationary tempered distribution by

$$(1.7)\qquad I(\phi)=\frac{1}{\sqrt{2\pi}}\int_{\boldsymbol{R}}\hat{\phi}(\lambda)([R](\lambda+i0))^{-1}dE(\lambda)A(0).$$

We shall show (Theorem 6. 1) that if R belongs to $L^1(\boldsymbol{R})$, then

$$(1.8)\qquad A(\phi)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}R(t)I(\phi(\cdot+t))dt \qquad (\phi\in\mathscr{S}(\boldsymbol{R})),$$

where $(A(\phi)\,;\,\phi\in\mathscr{S}(\boldsymbol{R}))$ is an $\mathscr{H}$-valued stationary tempered distribution defined by

$$(1.9)\qquad A(\phi)\equiv\int_{\boldsymbol{R}}\phi(t)A(t)dt=\int_{\boldsymbol{R}}\hat{\phi}(\lambda)dE(\lambda)A(0).$$

We understand that (1. 8) gives such a linear system that A, I and $\chi_{(0,\infty)}R$ can be regarded as its output, input and response function, respectively. Furthermore we shall represent a spectral density Δ'_I of the Kubo noise I in terms of the second KMO-Langevin data (α,β,κ) associated with R as follows (Theorem 6. 3):

$$(1.10)\qquad \Delta'_I(\xi)=\sqrt{\frac{2}{\pi}}\frac{1}{\alpha}\mathrm{Re}(\beta-i\xi+\lim_{\varepsilon\downarrow 0}(-i\xi)K_{\varepsilon}(\xi))$$
$$=\frac{\sqrt{2\pi}}{\alpha}\lim_{\eta\downarrow 0}\int_{\boldsymbol{R}}\frac{1}{\pi}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\kappa(d\lambda).$$

As a realization of the relation (1. 5), we shall in § 7 derive an equation describing the time evolution of A with the Kubo noise I as a random force (Theorem 7. 1): as $\mathscr{H}$-valued tempered distributions,

$$(1.11)\qquad \dot{A}=-\beta A-\lim_{\varepsilon\downarrow 0}\dot{\gamma}_{\varepsilon}*A+\alpha I.$$

Here $\gamma_{\varepsilon}(\varepsilon>0)$ are tempered distributions defined by

$$\gamma_\varepsilon = \frac{1}{2\pi}\hat{K}_\varepsilon.$$

We call equation (1.11) a **second KMO-Langevin equation.** We note that it gives a generalization of the equation (8.27) in [22] describing the time evolution of stationary Gaussian process with T-positivity.

On the basis of the second KMO-Langevin equation (1.11) describing the time evolution of a stationary curve A in a Hiblert space $\mathscr{H}$ with R as its covariance function, we shall find that (1.5), (1.6) and (1.10) imply Kubo's fluctuation-dissipation theorem (Theorems 8.1, 8.2 and 8.4). That is, we can see that a **complex mobility** μ of velocity A in a stationary state described by the second KMO-Langevin equation (1.11) is given by

$$(1.12)\qquad \mu(\zeta) = \frac{1}{\beta - i\zeta + \lim_{\varepsilon \downarrow 0}(-i\zeta)K_\varepsilon(\zeta)} \qquad (\zeta \in \boldsymbol{C}^+).$$

It then follows from (1.5), (1.6) and (1.10) that

(i) **Kubo's first fluctuation-dissipation theorem:**

$$\mu(\zeta) = \frac{1}{R(0)}\int_0^\infty e^{i\zeta t}R(t)\,dt \qquad (\zeta \in \boldsymbol{C}^+)$$

(ii) **Einstein relation** :

$$D\left(\equiv \lim_{\eta \downarrow 0}\int_0^\infty e^{-\eta t}R(t)\,dt\right) = \frac{R(0)}{\beta}$$

(iii) **Kubo's second fluctuation-dissipation theorem** :

$$\mathrm{Re}\left(\frac{1}{\mu(\xi + i0)}\right) = \frac{\pi}{R(0)}\Delta'_W(\xi) \qquad (a.\ e.\ \xi \in \boldsymbol{R}),$$

where Δ'_W is a spectral density of $W = \alpha I$.

In the final section § 9, we shall consider two typical examples: Ornstein-Uhlenbeck's Brownian motion and Mori's generalized Brownian motion. We shall find that the second KMO-Langevin equation (1.11) describing the time evolution of Ornstein-Uhlenbeck's Brownian motion is equal to the usual Langevin equation and then obtain a relation between the Mori noise and the Kubo noise appearing as random forces of Mori's memory kernel equation and the second KMO-Langevin equation describing the time evolution of Mori's generalized Brownian motion, respectively.

In a forthcoming paper, we shall give a representation theorem for the matrix valued function $[R]$ for the correlation matrix R of multi-dimensional stationary process of general type.

§ 2. Mori's theory of generalized Brownian motion

Let $\mathscr{H}$ be a Hilbert space and L be a self-adjoint operator on $\mathscr{H}$. We denote by $(U(t)\,;\, t\in\boldsymbol{R})$ a one-parameter group of unitary operators whose infinitesimal generator is equal to iL:

(2.1) $$U(t)=e^{itL}.$$

Furthermore we are given a vector $A_0\in\mathscr{H}$. Then we define a stationary curve $A=(A(t)\,;\, t\in\boldsymbol{R})$ in $\mathscr{H}$ by

(2.2) $$A(t)=U(t)A_0$$

and then a covariance function R_A on $\boldsymbol{R}$ by

(2.3) $$R_A(t)=(A(t),\ A(0))_{\mathscr{H}},$$

where $(\cdot,\cdot)_{\mathscr{H}}$ is an inner product in $\mathscr{H}$.

In this section we suppose the following conditions:

(2.4) $$A_0\neq 0$$

and

(2.5) $$A_0\in\mathscr{D}(L).$$

Then we know that $A(t)$ satisfies the following equation

(2.6) $$\dot{A}(t)=iLA(t)\quad (t\in\boldsymbol{R}),$$

where $\dot{A}(t)\equiv\frac{d}{dt}A(t)$.

We define a real number ω by

(2.7) $$\omega\equiv(LA(0),\ A(0))_{\mathscr{H}}(A(0),\ A(0))_{\mathscr{H}}^{-1}=i^{-1}\dot{R}_A(0)R_A(0)^{-1}.$$

Let $\mathscr{H}_0$ be the closed subspace generated by $A(0)$ and $\mathscr{H}_1$ be the orthogonal complementary subspace of $\mathscr{H}_0$ in $\mathscr{H}$. We denote by P_0 the projection operator on $\mathscr{H}_0$. Then we define a linear operator L_1 in $\mathscr{H}_1$ by

(2.8) $$\begin{cases}\mathscr{D}(L_1)\equiv\mathscr{H}_1\cap\mathscr{D}(L)\\ L_1u\equiv(1-P_0)Lu\quad (u\in\mathscr{D}(L_1)).\end{cases}$$

The following Lemma 2.1 is fundamental in Mori's theory.

LEMMA 2.1 ([15]) *L_1 is self-adjoint in $\mathscr{H}_1$.*

Then we can define a stationary curve $I_M=(I_M(t)\,;\, t\in\boldsymbol{R})$ in $\mathscr{H}_1$ by

(2.9) $$I_M(t)\equiv e^{itL_1}(1-P_0)\dot{A}(0)$$

and then a covariance function ϕ_M on $\boldsymbol{R}$ by

$$(2.10)\qquad \phi_M(t) \equiv (I_M(t),\ I_M(0))_{\mathscr{H}}(A(0),\ A(0))_{\mathscr{H}}^{-1}.$$

Now we can state Mori's theory of generalized Brownian motion. Concerning the covariance function R_A, we have

THEOREM 2.1 ([15])
(i) *For any* $t \in \boldsymbol{R}$

$$(2.11)\qquad \dot{R}_A(t) = i\omega R_A(t) - \int_0^t \phi_M(t-s) R_A(s)\, ds.$$

(ii) *For any* $\zeta \in \boldsymbol{C}^+$

$$(2.12)\qquad \int_0^\infty e^{i\zeta t} R_A(t)\, dt = R_A(0) \frac{1}{-i\omega - i\zeta + \int_0^\infty e^{i\zeta t} \phi_M(t)\, dt}.$$

Furthermore, the equation of motion which describes the time evolution of $(A(t)\,;\ t \in \boldsymbol{R})$ is given in

THEOREM 2.2 ([15]) *For any* $t \in \boldsymbol{R}$

$$(2.13)\qquad \dot{A}(t) = i\omega A(t) - \int_0^t \phi_M(t-s) A(s)\, ds + I_M(t).$$

DEFINITION 2.1 (i) We call ω, ϕ_M and I_M a frequency, memory function and Mori noise, respectively.
(ii) The equation (2.13) is said to be Mori's memory kernel equation.
For future use we note

LEMMA 2.2 *If* R_A *is a real valued function, then*
(i) $\omega = 0$
(ii) ϕ_M *is a real valued function.*

PROOF Since $R_A(t) = \overline{R_A(t)} = R_A(-t)$ in $\boldsymbol{R}$, (i) follows from (2.7). By substituting $\zeta = i\eta$ $(\eta > 0)$ in (2.17), we see that for any $\eta > 0$ $\eta + \int_0^\infty e^{-\eta t} \phi_M(t)\, dt$ is real and so by the uniqueness of Laplace transform $\phi_M(t) = \overline{\phi_M(t)}$ *in* $(0, \infty)$. Therefore, we have (ii) by noting that $\overline{\phi_M(t)} = \phi_M(-t)$ in $\boldsymbol{R}$ since ϕ_M is a non-negative definite function. (Q. E. D.)

§ 3. A second KMO-Langevin data (1) (regular case)

In this section we consider the same situation as § 2 and call it a regular case. Since R_A and ϕ_M are continuous and non-negative definite functions on

$\boldsymbol{R}$, there exist two bounded Borel measures Δ and κ on $\boldsymbol{R}$ such that

$$(3.1)\qquad R_A(t)=\int_{\boldsymbol{R}} e^{-it\lambda}\,\Delta(d\lambda)$$

and

$$(3.2)\qquad \phi_M(t)=\int_{\boldsymbol{R}} e^{-it\lambda}\kappa(d\lambda).$$

We define two functions $[R_A]$ and $[\phi_M]$ on $\boldsymbol{C}^+$ by

$$(3.3)\qquad [R_A](\zeta)\equiv\frac{1}{2\pi}\int_0^\infty e^{i\zeta t}R_A(t)\,dt=\frac{1}{2\pi i}\int_{\boldsymbol{R}}\frac{1}{\lambda-\zeta}\,\Delta(d\lambda)$$

and

$$(3.4)\qquad [\phi_M](\zeta)\equiv\frac{1}{2\pi}\int_0^\infty e^{i\zeta t}\phi_M(t)\,dt=\frac{1}{2\pi i}\int_{\boldsymbol{R}}\frac{1}{\lambda-\zeta}\kappa(d\lambda).$$

Besides conditions (2.4) and (2.5), we suppose the following conditions (3.5)[(2)] and (3.6):

(3.5) there exists a null set Λ_A in $\boldsymbol{R}-\{0\}$ such that for any $\xi\in\boldsymbol{R}-\Lambda_A$ $\lim_{\eta\downarrow 0}[R_A](\xi+i\eta)\equiv[R_A](\xi+i0)$ exists

(3.6) there exist positive constants c and m such that $|[R_A](\zeta)|\geq c(1+|\zeta|^m)^{-1}$ for any $\zeta\in\boldsymbol{C}^+$.

By condition (3.5), we can define a constant D by

$$(3.7)\qquad D\equiv 2\pi[R_A](0+i0)=\lim_{\eta\downarrow 0}\int_0^\infty e^{-\eta t}R_A(t)\,dt.$$

It then follows from (3.3) that

Lemma 3.1

(i) $D=\lim_{\eta\downarrow 0}\left(\int_{\boldsymbol{R}}\frac{\eta}{\lambda^2+\eta^2}\Delta(d\lambda)-i\int_{\boldsymbol{R}}\frac{\lambda}{\lambda^2+\eta^2}\Delta(d\lambda)\right)$

(ii) $\mathrm{Re}\,D=\lim_{\eta\downarrow 0}\int_{\boldsymbol{R}}\frac{\eta}{\lambda^2+\eta^2}\Delta(d\lambda)\geq 0.$

By (2.12), (3.5) and (3.6), we have

Lemma 3.2

(i) *For any* $\xi\in\boldsymbol{R}-\Lambda_A$ $\lim_{\eta\downarrow 0}[\phi_M](\xi+i\eta)\equiv[\phi_M](\xi+i0)$ *exists.*

(2) See footnote (1).

(ii) *For any* $\zeta\in\boldsymbol{C}^{+}$

$$[R_A](\zeta)=\frac{R_A(0)}{2\pi}\frac{1}{-i\omega-i\zeta+2\pi[\phi_M](\zeta)}.$$

(iii) *For any* $\xi\in\boldsymbol{R}-\Lambda_A$

$$[R_A](\xi+i0)=\frac{R_A(0)}{2\pi}\frac{1}{-i\omega-i\xi+2\pi[\phi_M](\xi+i0)}.$$

Immediately from (3.7) and Lemma 3.2 (iii), we have

LEMMA 3.3

$$-i\omega+2\pi[\phi_M](0+i0)=R_A(0)D_A^{-1}.$$

Next we define for each $\varepsilon>0$ a function γ_ε on $\boldsymbol{R}$ by

$$\gamma_\varepsilon(t)\equiv-\chi_{(0,\infty)}(t)\int_t^\infty e^{-\varepsilon s}\phi_M(s)ds. \tag{3.8}$$

A direct calculation gives

LEMMA 3.4

(i) *For each* $\varepsilon>0$ $\gamma_\varepsilon\in L^1(\boldsymbol{R})$.

(ii) *For any* $\zeta\in\boldsymbol{C}^{+}\cup\boldsymbol{R}$ *and any* $\varepsilon>0$

$$(-i\zeta)\int_0^\infty e^{i\zeta t}\gamma_\varepsilon(t)dt=\int_0^\infty e^{i(\zeta+i\varepsilon)t}\phi_M(t)dt-\int_0^\infty e^{-\varepsilon t}\phi_M(t)dt.$$

(iii) *For any* $\zeta\in\boldsymbol{C}^{+}\cup\boldsymbol{R}$ *and any* $\varepsilon>0$

$$\int_0^\infty e^{i\zeta t}\gamma_\varepsilon(t)dt=\int_{\boldsymbol{R}}\frac{1}{(\lambda-\zeta-i\varepsilon)(\lambda-i\varepsilon)}\varkappa(d\lambda).$$

Then we shall show

LEMMA 3.5

(i) *For any* $\zeta\in\boldsymbol{C}^{+}$

$$[\phi_M](\zeta)-[\phi_M](0+i0)=(-i\zeta)\lim_{\varepsilon\downarrow0}\tilde{\gamma}_\varepsilon(\zeta-i\varepsilon).$$

(ii) *For any* $\xi\in\boldsymbol{R}-\Lambda_A$

$$[\phi_M](\xi+i0)-[\phi_M](0+i0)=(-i\xi)\lim_{\varepsilon\downarrow0}\tilde{\gamma}_\varepsilon(\xi).$$

PROOF By (3.4), Lemma 3.2 (i) and (3.8), for any $\zeta\in\boldsymbol{C}^{+}$,

$$[\phi_M](\zeta)-[\phi_M](0+i0)$$
$$=(2\pi)^{-1}\lim_{\varepsilon\downarrow0}\{(i\zeta+\varepsilon)\int_0^\infty(\int_0^t e^{(i\zeta+\varepsilon)s}ds)e^{-\varepsilon t}\phi_M(t)dt\}$$

$$=(2\pi)^{-1}\lim_{\varepsilon\downarrow 0}(-i\zeta-\varepsilon)\int_0^\infty e^{(i\zeta+\varepsilon)s}\gamma_\varepsilon(s)\,ds,$$

which implies (i). Similarly, we have, for any $\xi\in\boldsymbol{R}-\Lambda_A$,

$$\begin{aligned}&[\phi_M](\xi+i0)-[\phi_M](0+i0)\\&=(2\pi)^{-1}\lim_{\varepsilon\downarrow 0}\{(i\xi)\int_0^\infty(\int_0^t e^{i\xi s}ds)e^{-\varepsilon t}\phi_M(t)\,dt\}\\&=(-i\xi)(2\pi)^{-1}\lim_{\varepsilon\downarrow 0}\int_0^\infty e^{i\xi s}\gamma_\varepsilon(s)\,ds,\end{aligned}$$

which gives (ii). (Q. E. D.)

Now we define a positive constant α and a non-zero complex number β by

$$\alpha\equiv\frac{R_A(0)}{\sqrt{2\pi}}\tag{3.9}$$

and

$$\beta\equiv R_A(0)D_A^{-1}.\tag{3.10}$$

By Lemmas 3.2 (ii), 3.2 (iii), 3.3, 3.4, 3.5, (3.9), (3.10) and the fact that $\varkappa(\boldsymbol{R})$ is finite, we have

THEOREM 3.1

(i) *For any* $\zeta\in\boldsymbol{C}^+$

$$[R_A](\zeta)=\frac{\alpha}{\sqrt{2\pi}}\,\frac{1}{\beta-i\zeta+(-i\zeta)\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{1}{(\lambda-\zeta-i\varepsilon)(\lambda-i\varepsilon)}\varkappa(d\lambda)}.$$

(ii) *For any* $\xi\in\boldsymbol{R}-\Lambda_A$

$$[R_A](\xi+i0)=\frac{\alpha}{\sqrt{2\pi}}\,\frac{1}{\beta-i\xi+(-i\xi)\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{1}{(\lambda-\xi-i\varepsilon)(\lambda-i\varepsilon)}\varkappa(d\lambda)}.$$

By taking account of Definition 3.1 and Theorem 8.5 in [22], we shall give

DEFINITION 3.1 We call a triple $(\alpha,\beta,\varkappa)$ a second KMO-Langevin data associated with the covariance function R_A.

REMARK 3.1 After the study under a general setting in § 4, we shall in § 5 investigate the properties of the second KMO-Langevin data $(\alpha,\beta,\varkappa)$ (Theorem 5.3).

Concerning the regularity of the measure $\varkappa$, we shall show

LEMMA 3.6 *For any* $\xi \in \boldsymbol{R}-\Lambda_A$

$$\lim_{\varepsilon \downarrow 0} \int_{\boldsymbol{R}} \frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2} \kappa(d\lambda) = \frac{R_A(0)}{2\pi} \mathrm{Re}([R_A](\xi+i0)^{-1}).$$

PROOF By (3.4), for any $\xi \in \boldsymbol{R}$ and any $\varepsilon > 0$,

$$\text{(3.11)} \quad \int_{\boldsymbol{R}} \frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2} \kappa(d\lambda) = \mathrm{Re}(2\pi[\phi_M](\xi+i\varepsilon)).$$

On the other hand, by Lemma 3.2 (ii), we have

$$\text{(3.12)} \quad -i\omega - i(\xi+i\varepsilon) + 2\pi[\phi_M](\xi+i\varepsilon) = \frac{R_A(0)}{2\pi}[R_A](\xi+i\varepsilon)^{-1}$$

and so

$$\text{(3.13)} \quad \mathrm{Re}(2\pi[\phi_M](\xi+i\varepsilon)) = \frac{R_A(0)}{2\pi} \mathrm{Re}([R_A](\xi+i\varepsilon)^{-1}) - \varepsilon.$$

Therefore, we have Lemma 3.6 by (3.5), (3.6), (3.11) and (3.13).

(Q. E. D.)

By virtue of Lemma 3.6, we can define a function $P\kappa$ on $\boldsymbol{R}$ by

$$\text{(3.14)} \quad (P\kappa)(\xi) \equiv \begin{cases} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\boldsymbol{R}} \frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2} \kappa(d\lambda) & \text{for } \xi \in \boldsymbol{R}-\Lambda_A \\ 0 & \text{for } \xi \in \Lambda_A. \end{cases}$$

Then we shall shall show

LEMMA 3.7

(i) $0 \leq (P\kappa)(\xi) \leq \dfrac{R_A(0)}{2\pi^2} c^{-1}(1+|\xi|^m)$

(ii) $(P\kappa)(0) = \dfrac{\mathrm{Re}\beta}{\pi}$

(iii) $\kappa(d\lambda) = (P\kappa)(\lambda)d\lambda.$

PROOF (i) follows from (3.5), (3.6), Lemma 3.6 and (3.14). (ii) follows from (3.7), (3.10), Lemma 3.6 and (3.14). Take any $f \in C_0^\infty(\boldsymbol{R})$. Since κ is a bounded Borel measure, we have

$$\text{(3.15)} \quad \lim_{\varepsilon \downarrow 0} \int_{\boldsymbol{R}} f(\xi)\left(\frac{1}{\pi} \int_{\boldsymbol{R}} \frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2} \kappa(d\lambda)\right) d\xi = \int_{\boldsymbol{R}} f(\lambda)\kappa(d\lambda).$$

On the other hand, it follows from (3.6), (3.11) and (3.13) that for any $\varepsilon \in (0, 1)$

$$(3.16)\qquad \left|\frac{1}{\pi}\int_{\boldsymbol{R}}\frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2}\kappa(d\lambda)\right|\le\frac{R_A(0)}{2\pi^2}c^{-1}(1+|\xi|^m)+1.$$

Therefore, by (3.14) and (3.16), we can apply Lebesgue's convergence theorem to obtain

$$(3.17)\qquad \begin{aligned}&\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}f(\xi)\Big(\frac{1}{\pi}\int_{\boldsymbol{R}}\frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2}\kappa(d\lambda)\Big)d\xi\\ &=\int_{\boldsymbol{R}}f(\xi)(P\kappa)(\xi)d\xi.\end{aligned}$$

Thus, (iii) follows from (3.15) and (3.17). (Q. E. D.)

REMARK 3.2 Because of a relation in Lemma 3.7 (ii), we see that the second KMO-Langevin data (α, β, κ) can not be given independently one another.

The following Lemma 3.8 pays an important role when we derive a second KMO-Langevin equation describing the time evolution of $A=(A(t)\,;\, t\in\boldsymbol{R})$.

LEMMA 3.8 *There exist positive constants c_1 and m_1 such that for any $\xi \in \boldsymbol{C}^+\cup\boldsymbol{R}$*

$$\sup_{0<\varepsilon<1}\left|(-i\xi)\int_{\boldsymbol{R}}\frac{1}{(\lambda-\xi-i\varepsilon)(\lambda-i\varepsilon)}\kappa(d\lambda)\right|\le c_1(1+|\xi|^{m_1}).$$

PROOF By Lemmas 3.2 (ii), 3.4 (ii) and 3.4 (iii), for any $\xi\in\boldsymbol{C}^+$,

$$(3.18)\qquad \begin{aligned}&(-i\xi)\int_{\boldsymbol{R}}\frac{1}{(\lambda-\xi-i\varepsilon)(\lambda-i\varepsilon)}\kappa(d\lambda)\\ &=i\xi+\frac{R_A(0)}{2\pi}\{([R_A](\xi+i\varepsilon))^{-1}-([R_A](i\varepsilon))^{-1}\}.\end{aligned}$$

Therefore, by virtue of our condition (3.6), Lemma 3.8 follows from (3.18). (Q. E. D.)

§ 4. A second KMO-Langevin data (2) (general case)

In this section we shall consider a stationary curve $A=(A(t)\,;\, t\in\boldsymbol{R})$ in (2.2) with covariance function R_A in (2.3) satisfying conditions (2.4), (3.5) and (3.6) only. The difference between § 3 and § 4 is that we do not suppose condition (2.5) in § 2. For that reason we define for each $n\in\boldsymbol{N}$ a vector A_n in $\mathscr{H}$ by

(4.1) $$A_n \equiv n\int_0^\infty e^{-nt}U(t)A(0)\,dt.$$

It then follows from a general theory of semi-groups that

LEMMA 4.1
(i) $A_n \neq 0$ $(n \in \boldsymbol{N})$
(ii) $A_n \in \mathscr{D}(L)$ $(n \in \boldsymbol{N})$
(iii) $\lim_{n\to\infty} A_n = A(0)$ *in* $\mathscr{H}$.

For each $n \in \boldsymbol{N}$ we define a covariance function R_n on $\boldsymbol{R}$ and a function $[R_n]$ on $\boldsymbol{C}^+$ by

(4.2) $$R_n(t) = (U(t)A_n,\ A_n)_{\mathscr{H}}$$

and

(4.3) $$[R_n](\zeta) = \frac{1}{2\pi}\int_0^\infty e^{i\zeta t}R_n(t)\,dt.$$

Immediately from Lemma 4.1 (iii), we have

LEMMA 4.2
(i) *For any* $t \in \boldsymbol{R}$ $\qquad \lim_{n\to\infty} R_n(t) = R_A(t).$
(ii) *For any* $\zeta \in \boldsymbol{C}^+$ $\qquad \lim_{n\to\infty} [R_n](\zeta) = [R_A](\zeta).$

Now, by virtue of Lemmas 4.1 (i) and 4.1 (ii), we can apply Theorem 2.1 (ii) to find that there exist real numbers ω_n $(n \in \boldsymbol{N})$ and continuous and non-negative definite functions ϕ_n $(n \in \boldsymbol{N})$ such that for each $n \in \boldsymbol{N}$ and any $\zeta \in \boldsymbol{C}^+$

(4.4) $$[R_n](\zeta) = \frac{R_n(0)}{2\pi}\,\frac{1}{-i\omega_n - i\zeta + \int_0^\infty e^{i\zeta t}\phi_n(t)\,dt}$$

and so

(4.5) $$-i\omega_n - i\zeta + \int_0^\infty e^{i\zeta t}\phi_n(t)\,dt = \frac{R_n(0)}{2\pi}([R_n](\zeta))^{-1}.$$

Concerning a relation between $[R_A]$ and $[R_n]$, we shall show

LEMMA 4.3 *For each* $n \in \boldsymbol{N}$ *and any* $\zeta \in \boldsymbol{C}^+$ *with* Im $\zeta \neq n$,

$$[R_n](\zeta) = \frac{n}{2\pi(n - i\zeta)}\Big\{\frac{1}{2}\int_{\boldsymbol{R}} e^{-n|t|}R_A(t)\,dt +$$

$$+\frac{n}{n+i\xi}(2\pi[R_A](\xi)-\int_0^\infty e^{-nt}R_A(t)dt)\}.$$

PROOF By (2.3), (4.1) and (4.2), we see that for any $t\in\boldsymbol{R}$

$$R_n(t)=n\int_0^\infty e^{-ns}(U(t+s)A(0),\ A_n)_{\mathscr{H}}ds$$

and so for any $\xi\in\boldsymbol{C}^+$

$$(4.6)\qquad [R_n](\xi)=\frac{n^2}{2\pi}\int_0^\infty e^{-ns}(\int_0^\infty e^{i\xi t}\int_0^\infty e^{-n\tau}R_A(t+s-\tau)d\tau)dt)ds$$
$$=\frac{n^2}{2\pi}\int_0^\infty e^{-ns}(\int_0^\infty e^{(i\xi-n)t}(\int_0^\infty e^{n(t-\tau)}R_A(t+s-\tau)d\tau)dt)ds$$
$$=\frac{n^2}{2\pi}(\mathrm{I}+\mathrm{II}),$$

where

$$\mathrm{I}=\int_0^\infty e^{-ns}(\int_0^\infty e^{(i\xi-n)t}(\int_{-\infty}^0 e^{n\tau}R_A(s+\tau)d\tau)dt)ds$$

and

$$\mathrm{II}=\int_0^\infty e^{-ns}(\int_0^\infty e^{(i\xi-n)t}(\int_0^t e^{n\tau}R_A(s+\tau)d\tau)dt)ds.$$

Then we have

$$(4.7)\qquad \mathrm{I}=\frac{1}{n-i\xi}\int_0^\infty e^{-ns}(\int_{-\infty}^0 e^{n\tau}R_A(s+\tau)d\tau)ds$$
$$=\frac{1}{n-i\xi}\int_0^\infty e^{-2ns}(\int_{-\infty}^0 e^{n\tau}R_A(\tau)d\tau+\int_0^s e^{n\tau}R_A(\tau)d\iota)ds$$
$$=\frac{1}{2n(n-i\xi)}(\int_{-\infty}^0 e^{n\tau}R_A(\tau)d\tau+\int_0^\infty e^{-n\tau}R_A(\tau)d\tau)$$
$$=\frac{1}{2n(n-i\xi)}\int_{\boldsymbol{R}} e^{-n|\tau|}R_A(\tau)d\tau.$$

On the other hand,

$$(4.8)\qquad \mathrm{II}=\frac{1}{n-i\xi}\int_0^\infty e^{-ns}(\int_0^\infty e^{i\xi\tau}R_A(s+\tau)d\tau)ds$$
$$=\frac{1}{n-i\xi}\int_0^\infty e^{i\xi\tau}R_A(\tau)\frac{1-e^{-(n+i\xi)\tau}}{n+i\xi}d\tau$$
$$=\frac{1}{(n-i\xi)(n+i\xi)}(2\pi[R_A](\xi)-\int_0^\infty e^{-n\tau}R_A(\tau)d\tau).$$

Therefore, it follows from (4.6), (4.7) and (4.8) that Lemma 4.3 holds.

(Q. E. D.)

In particular, we see from condition (3. 5) and Lemma 4. 3 that

LEMMA 4. 4 *For each* $n \in \boldsymbol{N}$ *and any* $\xi \in \boldsymbol{R} - \Lambda_A$

$$
\begin{aligned}
&[R_n](\xi + i0) \equiv \lim_{\eta \downarrow 0} [R_n](\xi + i\eta) = \\
&= \frac{n}{2\pi(n - i\xi)} \{ \frac{1}{2} \int_{\boldsymbol{R}} e^{-n|t|} R_A(t)\, dt + \\
&+ \frac{n}{n + i\xi} (2\pi [R_A](\xi + i0) - \int_0^\infty e^{-nt} R_A(t)\, dt) \}.
\end{aligned}
$$

Furthermore we find from Lemma 4. 4 that

LEMMA 4. 5 *For any* $\xi \in \boldsymbol{R} - \Lambda_A$

$$
\lim_{n \to \infty} [R_n](\xi + i0) = [R_A](\xi + i0).
$$

On the other hand, we shall show

LEMMA 4. 6 *For any* $T > 0$, *there exists a positive number* n_T *such that for any* $n \in \boldsymbol{N} \cap (n_T, \infty)$

$$
[R_n](\xi + i0) \neq 0 \qquad \text{for any } \xi \in (\boldsymbol{R} - \Lambda_A) \cap [-T, T].
$$

PROOF We see from condition (3. 6) and Lemma 4. 4 that for any $\xi \in (\boldsymbol{R} - \Lambda_A) \cap [-T, T]$ and any $n \in \boldsymbol{N}$

$$
\begin{aligned}
(4.9) \qquad |[R_n](\xi + i0)| &\geq \frac{n}{2\pi |n - i\xi|} \{ \frac{2\pi n}{|n + i\xi|} |[R_A](\xi + i0)| \\
&- (\frac{1}{2} | \int_{\boldsymbol{R}} e^{-n|t|} R_A(t)\, dt | + \frac{n}{|n + i\xi|} | \int_0^\infty e^{-nt} R_A(t) |) \} \\
&\geq \frac{n}{2\pi(n + T)} \{ \frac{2\pi n c}{(n + T)(1 + T^m)} - 2 \frac{R_A(0)}{n} \}.
\end{aligned}
$$

Therefore, if we choose a positive number n_T such that for any $n \in \boldsymbol{N} \cap (n_T, \infty)$, $\frac{n+T}{n^2} < \frac{\pi c}{(1 + T^m) R_A(0)}$, we see from (4. 9) that Lemma 4. 6 holds.

(Q. E. D.)

By (3. 5) and Lemma 4. 4, we can define complex constants D and D_n $(n \in \boldsymbol{N})$ by

$$
(4.10) \qquad D \equiv 2\pi [R_A](0 + i0)
$$

and

$$
(4.11) \qquad D_n \equiv 2\pi [R_n](0 + i0).
$$

It then follows from (3. 5), (3. 6) and Lemma 4. 5 that

LEMMA 4.7

(i) $\lim_{n\to\infty} D_n = D$

(ii) *There exists* $n_0 \in \boldsymbol{N}$ *such that* $D_n \neq 0$ *for any* $n \in \boldsymbol{N} \cap [n_0, \infty)$.

Similarly as (3.9) and (3.10), we define positive constants α, α_n ($n \in \boldsymbol{N}$) and β, β_n ($n \in \boldsymbol{N} \cap [n_0, \infty)$) by

$$(4.12) \qquad \alpha \equiv \frac{R_A(0)}{\sqrt{2\pi}} \text{ and } \alpha_n \equiv \frac{R_n(0)}{\sqrt{2\pi}}$$

and

$$(4.13) \qquad \beta \equiv R_A(0) D^{-1} \text{ and } \beta_n \equiv R_n(0) D_n^{-1}.$$

By Lemmas 4.2 (i) and 4.7, we have

LEMMA 4.8

(i) $\lim_{n\to\infty} \alpha_n = \alpha$

c

(ii) $\lim_{n\to\infty} \beta_n = \beta$.

Similarly as (3.4), we define functions $[\phi_n]$ on $\boldsymbol{C}^+$ ($n \in \boldsymbol{N}$) by

$$(4.14) \qquad [\phi_n](\zeta) \equiv \frac{1}{2\pi} \int_0^\infty e^{i\zeta t} \phi_n(t)\, dt.$$

It then follows from (4.5), (4.11), Lemma 4.7 (ii) and (4.13) that

LEMMA 4.9 *For each* $n \in \boldsymbol{N} \cap [n_0, \infty)$

(i) $\lim_{\eta \downarrow 0} [\phi_n](i\eta) \equiv [\phi_n](0+i0)$ *exists*

(ii) $-i\omega_n + 2\pi[\phi_n](0+i0) = \beta_n$.

Furthermore we see from (4.5), Lemmas 4.6, 4.7 (ii), (4.12), (4.13) and Lemma 4.9 that

LEMMA 4.10

(i) *For each* $n \in \boldsymbol{N} \cap [n_0, \infty)$ *and any* $\zeta \in \boldsymbol{C}^+$

$$\beta_n - i\zeta + 2\pi([\phi_n](\zeta) - [\phi_n](0+i0)) = \frac{\alpha_n}{\sqrt{2\pi}}([R_n](\zeta))^{-1}.$$

(ii) *For any* $T > 0$, *each* $n \in \boldsymbol{N} \cap [n_T, \infty)$ *and any* $\xi \in (\boldsymbol{R} - \Lambda_A) \cap [-T, T]$

(a) $\lim_{\eta \downarrow 0} [\phi_n](\xi + i\eta) \equiv [\phi_n](\xi + i0)$ *exists*

(b) $\beta_n - i\xi + 2\pi([\phi_n](\xi+i0) - [\phi_n](0+i0))$
$= \frac{\alpha_n}{\sqrt{2\pi}}([R_n](\xi+i0))^{-1}$.

Since ϕ_n ($n \in \boldsymbol{N}$) are continuous and non-negative definite functions on $\boldsymbol{R}$, there exist bounded Borel measures $\varkappa_n$ on $\boldsymbol{R}$ ($n \in \boldsymbol{N}$) such that

$$(4.15) \qquad \phi_n(t) = \int_{\boldsymbol{R}} e^{-it\lambda} \varkappa_n(d\lambda).$$

We define symmetric and bounded Borel measures $\varkappa_n^{(s)}$ on $\boldsymbol{R}$ ($n \in \boldsymbol{N}$) by

$$(4.16) \qquad \varkappa_n^{(s)}(d\lambda) = \varkappa_n(d\lambda) + \varkappa_n(-d\lambda).$$

Then we shall show

LEMMA 4.11 *For each* $n \in \boldsymbol{N}$

(i) $\mathrm{Re}(\phi_n(t)) = \frac{1}{2} \int_{\boldsymbol{R}} e^{-it\lambda} \varkappa_n^{(s)}(d\lambda) \qquad (t \in \boldsymbol{R})$

(ii) $\frac{1}{2} \int_{\boldsymbol{R}} \frac{1}{1+\lambda^2} \varkappa_n^{(s)}(d\lambda) = \frac{\alpha_n}{\sqrt{2\pi}} \mathrm{Re}\{([R_n](i))^{-1}\} - 1.$

PROOF Since $\overline{\phi_n(t)} = \phi_n(-t)$ ($t \in \boldsymbol{R}$), we have (i). By substituting $\zeta = i$ into (4.5) and then noting (4.15), we have

$$-i\omega_n + 1 + \int_0^\infty e^{-t} \phi_n(t)\, dt = \frac{\alpha_n}{\sqrt{2\pi}}([R_n](i))^{-1}$$

and so

$$(4.17) \qquad \int_0^\infty e^{-t}\, \mathrm{Re}(\phi_n(t))\, dt = \frac{\alpha_n}{\sqrt{2\pi}} \mathrm{Re}\{[R_n](i))^{-1}\} - 1.$$

On the other hand, we see from (i) that

$$(4.18) \qquad \int_0^\infty e^{-t} \mathrm{Re}(\phi_n(t))\, dt = \frac{1}{2} \int_{\boldsymbol{R}} \frac{1}{1+i\lambda} \varkappa_n^{(s)}(d\lambda) = \frac{1}{2} \int_{\boldsymbol{R}} \frac{1}{1+\lambda^2} \varkappa_n^{(s)}(d\lambda),$$

noting that $\varkappa_n^{(s)}$ is a symmetric measure. Therefore, (ii) follows from (4.17) and (4.18). (Q. E. D.)

As a refinement of (4.18), we shall show

LEMMA 4.12 *For each* $n \in \boldsymbol{N}$ *and any* $\zeta \in \boldsymbol{C}^+$

$$\int_0^\infty e^{i\zeta t} \mathrm{Re}(\phi_n(t))\, dt = \frac{1}{2i} \int_{\boldsymbol{R}} \frac{\zeta}{(\lambda - \zeta)(\lambda + \zeta)} \varkappa_n^{(s)}(d\lambda).$$

PROOF By Lemma 4.11 (i), we have

$$\int_0^\infty e^{i\xi t}\mathrm{Re}(\phi_n(t))dt$$
$$=\frac{1}{2i}\int_{\boldsymbol{R}}\frac{1}{\lambda-\xi}\varkappa_n^{(s)}(d\lambda)$$
$$=\frac{1}{4i}\Big(\int_{\boldsymbol{R}}\frac{1}{\lambda-\xi}\varkappa_n^{(s)}(d\lambda)+\int_{\boldsymbol{R}}\frac{1}{\lambda-\xi}\varkappa_n^{(s)}(-d\lambda)\Big)$$
$$=\frac{1}{4i}\int_{\boldsymbol{R}}\Big(\frac{1}{\lambda-\xi}-\frac{1}{\lambda+\xi}\Big)\varkappa_n^{(s)}(d\lambda),$$

which gives Lemma 4.12. (Q. E. D.)

We define bounded Borel measures $\varkappa_n^{(s)}$ on $[-\infty,\infty]$ by

(4.19) $\tilde{\varkappa}_n^{(s)}(d\lambda)=\varkappa_n^{(s)}((-\infty,\infty)\cap d\lambda).$

It then follows from (3.6), Lemmas 4.2 (ii), 4.8 (i) and 4.11 that

(4.20) $\sup_{n\in\boldsymbol{N}}\tilde{\varkappa}_n^{(s)}([-\infty,\infty])<\infty$

and so there exist a subsequence $(n_k;\ k\in\boldsymbol{N})$ $(\lim_{k\to\infty} n_k=\infty)$ and a bounded Borel measure $\tilde{\varkappa}$ on $[-\infty,\infty]$ such that

(4.21) $w-\lim_{k\to\infty}\varkappa_n^{(s)}=\tilde{\varkappa}\ on\ [-\infty,\infty].$

We define a function g on $\boldsymbol{R}\times\boldsymbol{C}^+$ by

(4.22) $g(\lambda,\xi)=\frac{(-i\xi)}{2}\frac{\lambda^2+1}{(\lambda-\xi)(\lambda+\xi)}.$

Then we note that

(4.23) $g(\cdot,\xi)\in C((-\infty,\infty))$

(4.24) $g(\pm\infty,\xi)=\frac{-i\xi}{2}.$

Therefore, by Lemma 4.12, (4.21), (4.23) and (4.24), we have

LEMMA 4.13 *For any* $\xi\in\boldsymbol{C}^+$

$$\lim_{k\to\infty}\int_0^\infty e^{i\xi t}\,\mathrm{Re}(\phi_{n_k}(t))dt=\int_{[-\infty,\infty]}g(\lambda,\xi)\tilde{\varkappa}(d\lambda).$$

We claim

LEMMA 4.14
(i) $\tilde{\varkappa}(\{-\infty, \infty\})=0$
(ii) $\tilde{\varkappa}(\{0\})=0.$

PROOF By substituting $\zeta=i\eta$ $(\eta>0)$ into (4.5), we have, similarly as (4.17),

$$(4.25)\qquad \int_0^\infty e^{-\eta t}\mathrm{Re}(\phi_n(t))dt=\frac{\alpha_n}{\sqrt{2\pi}}\mathrm{Re}\{([R_n](i\eta)^{-1}\}-\eta.$$

By Lemmas 4.2 (i), 4.8 (i), (4.22), (4.24), Lemma 4.13 and (4.25), we have

$$\frac{\eta}{2}\int_R \frac{\lambda^2+1}{\lambda^2+\eta^2}\tilde{\varkappa}(d\lambda)+\tilde{\varkappa}(\{-\infty, \infty\})\frac{\eta}{2}$$
$$=\frac{\alpha}{\sqrt{2\pi}}\mathrm{Re}\{[R_A](i\eta)^{-1}\}-\eta$$

and so

$$(4.26)\qquad \frac{1}{2}\int_R \frac{\lambda^2+1}{\lambda^2+\eta^2}\tilde{\varkappa}(d\lambda)+\frac{1}{2}\tilde{\varkappa}(\{-\infty, \infty\})$$
$$=\frac{\alpha}{\sqrt{2\pi}}\mathrm{Re}\{(\eta[R_A](i\eta))^{-1}\}-1.$$

On the other hand, we note by (3.3) and (4.12) that

$$(4.27)\qquad \lim_{\eta\to\infty}\eta[R_A](i\eta)=\frac{\alpha}{\sqrt{2\pi}}.$$

Therefore, by letting η tend to infinity in (4.26), we have (i). Furthermore, it follows from (i) and (4.26) that for any $\eta>0$

$$(4.28)\qquad \tilde{\varkappa}(\{0\})\leq 2\frac{\alpha}{\sqrt{2\pi}}\eta\ \mathrm{Re}\{([R_A](i\eta))^{-1}\}.$$

By letting η tend to zero in (4.28), we see from (3.5) and (3.6) that (ii) holds. (Q. E. D.)

Next we define two bounded Borel measures $\tilde{\varkappa}_n^{(+)}$ and $\tilde{\varkappa}_n^{(-)}$ on $[0,\infty]$ $(n\in \boldsymbol{N})$ by

$$(4.29)\qquad \tilde{\varkappa}_n^{(+)}(B)=\int_{B\cap[0,\infty]}\frac{1}{1+\lambda^2}\varkappa_n(d\lambda)$$

and

$$(4.30)\qquad \tilde{\varkappa}_n^{(-)}(B)=\int_{B\cap[0,\infty]}\frac{1}{1+\lambda^2}\varkappa_n(-d\lambda)$$

for any Borel set B of $[0,\infty]$.
Then we note by (4.16) and (4.19) that

$$(4.31)\qquad \tilde{\varkappa}_n^{(+)}(\{\infty\})=\tilde{\varkappa}_n^{(-)}(\{\infty\})=0$$

$$(4.32)\qquad \tilde{\varkappa}_n^{(+)}(B)+\tilde{\varkappa}_n^{(-)}(B)\leq\tilde{\varkappa}_n^{(s)}(B)\qquad \text{for any Borel set } B \text{ of } [0,\infty]$$

$$(4.33)\qquad \varkappa_n(\{0\})=\frac{1}{2}\tilde{\varkappa}_n^{(s)}(\{0\}).$$

By (4.20), (4.32) and (4.33), we find that there exist a subsequence $(m_k\,;\,k\in\boldsymbol{N})$ of $(n_k\,;\,k\in\boldsymbol{N})$ $(\lim_{k\to\infty} m_k=\infty)$ and two bounded Borel measures $\tilde{\varkappa}^{(+)}$ and $\tilde{\varkappa}^{(-)}$ on $[0,\infty]$ such that

$$(4.34)\qquad w-\lim_{k\to\infty}\tilde{\varkappa}_{m_k}^{(+)}=\tilde{\varkappa}^{(+)}\qquad \text{on } [0,\infty]$$

$$(4.35)\qquad w-\lim_{k\to\infty}\tilde{\varkappa}_{m_k}^{(-)}=\tilde{\varkappa}^{(-)}\qquad \text{on } [0,\infty]$$

$$(4.36)\qquad \lim_{k\to\infty}\varkappa_{m_k}(\{0\}) \text{ exists.}$$

We claim that

LEMMA 4.15

(i) $\lim_{k\to\infty}\varkappa_{m_k}(\{0\})=0$

(ii) $\tilde{\varkappa}^{(+)}(\{\infty\})=\tilde{\varkappa}^{(-)}(\{\infty\})=0$

(iii) $\tilde{\varkappa}^{(+)}(\{0\})=\tilde{\varkappa}^{(-)}(\{0\})=0.$

PROOF By (4.21), Lemma 4.14 (ii) and (4.33),

$$\lim_{k\to\infty}\varkappa_{m_k}(\{0\})\leq\frac{1}{2}\overline{\lim_{k\to\infty}}\,\tilde{\varkappa}_{m_k}^{(s)}(\{0\})\leq\frac{1}{2}\tilde{\varkappa}(\{0\})=0,$$

which gives (i). Take any bounded and non-negative continuous function f on $[-\infty,\infty]$. It then follows from (4.21), (4.32), (4.34) and (4.35) that

$$(4.37)\qquad \begin{aligned}\int_{[-\infty,\infty]}fd\tilde{\varkappa}&=\lim_{k\to\infty}\int_{[-\infty,\infty]}fd\tilde{\varkappa}_{m_k}^{(s)}\geq\underline{\lim_{k\to\infty}}\int_{[0,\infty]}fd\tilde{\varkappa}_{m_k}^{(s)}\\&\geq\underline{\lim_{k\to\infty}}\Big(\int_{[0,\infty]}fd\tilde{\varkappa}_{m_k}^{(+)}+\int_{[0,\infty]}fd\tilde{\varkappa}_{m_k}^{(-)}\Big)\\&=\int_{[0,\infty]}fd\tilde{\varkappa}^{(+)}+\int_{[0,\infty]}fd\tilde{\varkappa}^{(-)}.\end{aligned}$$

By taking a sequence of functions f_n of $C([-\infty,\infty])$ such that $0\leq f_n\leq 1$ and

$\lim_{n\to\infty} f_n = \chi_{\{0,\infty\}}$, we see from Lemma 4.14 and (4.37) that (ii) and (iii) holds. (Q. E. D.)

Next we define bounded Borel measures $\tilde{\varkappa}_n$ $(n\in \boldsymbol{N})$ and $\tilde{\tilde{\varkappa}}$ on $[-\infty,\infty]$ by

$$(4.38)\qquad \tilde{\varkappa}_n(B)=\int_{B\cap[-\infty,\infty]}\frac{1}{1+\lambda^2}\varkappa_n(d\lambda)$$

and

$$(4.39)\qquad \tilde{\tilde{\varkappa}}(B)=\tilde{\varkappa}^{(+)}([0,\infty]\cap B)+\tilde{\varkappa}^{(-)}([0,\infty]\cap(-B))$$

for any Borel set B of $[-\infty,\infty]$.
Then we shall show

LEMMA 4.16

$$w-\lim_{k\to\infty}\tilde{\varkappa}_{m_k}=\tilde{\tilde{\varkappa}}\qquad \text{on } [-\infty,\infty].$$

PROOF Take any $f\in C([-\infty,\infty])$. We see from (4.29), (4.30) and (4.38) that

$$\int_{[-\infty,\infty]} fd\tilde{\varkappa}_{m_k}$$
$$=\int_{[0,\infty]} f(-\lambda)\tilde{\varkappa}_{m_k}^{(-)}(d\lambda)+\int_{[0,\infty]} fd\tilde{\varkappa}_{m_k}^{(+)}-f(0)\varkappa_{m_k}(\{0\})$$

and so by (4.34), (4.35), Lemma 4.15 and (4.39)

$$\lim_{k\to\infty}\int_{[-\infty,\infty]} fd\tilde{\varkappa}_{m_k}$$
$$=\int_{[0,\infty]} f(-\lambda)\tilde{\varkappa}^{(-)}(d\lambda)+\int_{[0,\infty]} f(\lambda)\tilde{\varkappa}^{(+)}(d\lambda)$$
$$=\int_{[-\infty,\infty]} f(\lambda)\tilde{\tilde{\varkappa}}(d\lambda),$$

which gives Lemma 4.16. (Q. E. D.)

Now we define a Borel measure $\varkappa$ on $\boldsymbol{R}$ by

$$(4.40)\qquad \varkappa(d\lambda)=(1+\lambda^2)\tilde{\tilde{\varkappa}}(d\lambda).$$

It then follows from Lemma 4.15 (iii), (4.39) and (4.40) that

$$(4.41)\qquad \varkappa(\{0\})=0$$

and

$$(4.42)\qquad \int_{\boldsymbol{R}}\frac{1}{1+\lambda^2}\varkappa(d\lambda)<\infty.$$

We define functions $C_{n,\varepsilon}$ on $C^{+}\cup \boldsymbol{R}$ $(n\in \boldsymbol{N}, \varepsilon\in(0,\infty))$ by

$$(4.44)\qquad C_{n,\varepsilon}(\zeta)=\frac{(-i\zeta)}{2\pi}\int_{\boldsymbol{R}}\frac{1}{(\lambda-\zeta-i\varepsilon)(\lambda-i\varepsilon)}\varkappa_n(d\lambda).$$

By virtue of (a) in Lemma 4.10 (ii), we have, similarly as Lemmas 3.4 (iii) and 3.5 (ii),

LEMMA 4.17 *For any* $T>0$ *and each* $n\in\boldsymbol{N}\cap[n_T,\infty)$,

(i) $[\phi_n](\zeta)-[\phi_n](0+i0)=\lim_{\varepsilon\downarrow 0} C_{n,\varepsilon}(\zeta)$

for any $\zeta\in \boldsymbol{C}^{+}$

(ii) $[\phi_n](\xi+i0)-[\phi_n](0+i0)=\lim_{\varepsilon\downarrow 0} C_{n,\varepsilon}(\xi)$

for any $\xi\in(\boldsymbol{R}-\Lambda_A)\cap[-T,T]$.

Next we shall show

LEMMA 4.18

(i) *For any* $\varepsilon\in(0,1)$ *and any* $\zeta\in\boldsymbol{C}^{+}\cup\boldsymbol{R}$

$$\lim_{k\to\infty} C_{m_k,\varepsilon}(\zeta)=\frac{(-i\zeta)}{2\pi}\int_{\boldsymbol{R}}\frac{1}{(\lambda-\zeta-i\varepsilon)(\lambda-i\varepsilon)}\varkappa(d\lambda).$$

(ii) *For any* $\zeta\in\boldsymbol{C}^{+}\cup(\boldsymbol{R}-\Lambda_A)$

$$\lim_{\varepsilon\downarrow 0}(\lim_{n\to\infty} C_{n,\varepsilon}(\zeta))=\lim_{n\to\infty}(\lim_{\varepsilon\downarrow 0} C_{n,\varepsilon}(\zeta)).$$

PROOF By Lemma 4.15 (ii), (4.39), Lemma 4.16 and (4.40), we have (i). Since by (4.15) and (4.44)

$$C_{n,\varepsilon}(\zeta)=[\phi_n](\zeta+i\varepsilon)-[\phi_n](i\varepsilon),$$

we find from Lemma 4.10 (i) that

$$(4.45)\qquad 2\pi C_{n,\varepsilon}(\zeta)=\frac{\alpha_n}{\sqrt{2\pi}}\{([R_n](\zeta+i\varepsilon))^{-1}-([R_n](i\varepsilon))^{-1}\}+i\zeta.$$

By applying Lemmas 4.2 (ii) and 4.8 (i) to (4.45), we have

$$(4.46)\qquad 2\pi\lim_{n\to\infty} C_{n,\varepsilon}(\zeta)=\frac{\alpha}{\sqrt{2\pi}}\{([R_A](\zeta+i\varepsilon))^{-1}-([R_A](i\varepsilon))^{-1}\}+i\zeta$$

and then by our conditions (3.5) and (3.6)

$$(4.47)\qquad 2\pi\lim_{\varepsilon\downarrow 0}(\lim_{n\to\infty} C_{n,\varepsilon}(\zeta))=\frac{\alpha}{\sqrt{2\pi}}\{([R_A](\zeta+i0))^{-1}-([R_A](0+i0))^{-1}\}$$
$$+i\zeta.$$

On the other hand, by applying Lemmas 4.4 and 4.6 to (4.45) at first, we see that

$$(4.48)\qquad 2\pi \lim_{\varepsilon \downarrow 0} C_{n,\varepsilon}(\zeta)=\frac{\alpha_n}{\sqrt{2\pi}}\{([R_n](\zeta+i0))^{-1}-([R_n](0+i0))^{-1}\} + i\zeta$$

and then by using Lemmas 4.2 (ii), 4.5 and 4.8 (i), we have

$$(4.49)\qquad 2\pi \lim_{n\to\infty}(\lim_{\varepsilon \downarrow 0} C_{n,\varepsilon}(\zeta))=\frac{\alpha}{\sqrt{2\pi}}\{([R_A](\zeta+i0))^{-1}-([R_A](0+i0))^{-1}\} + i\zeta.$$

Therefore, (ii) follows from (4.47) and (4.49). (Q. E. D.)

After the above investigation, we shall show the following important

LEMMA 4.19

(i) *For any* $\zeta\in \boldsymbol{C}^+$

$$[R_A](\zeta)=\frac{\alpha}{\sqrt{2\pi}}\frac{1}{\beta-i\zeta+(-i\zeta)\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{1}{(\lambda-\zeta-i\varepsilon)(\lambda-i\varepsilon)}\varkappa(d\lambda)}.$$

(ii) *For any* $\xi\in \boldsymbol{R}-\Lambda_A$

$$[R_A](\xi+i0)=\frac{\alpha}{\sqrt{2\pi}}\frac{1}{\beta-i\xi+(-i\xi)\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{1}{(\lambda-\xi-i\varepsilon)(\lambda-i\varepsilon)}\varkappa(d\lambda)}.$$

PROOF (i) follows from Lemmas 4.2 (ii), 4.8, 4.10 (i), 4.17 (i) and 4.18. (ii) follows from Lemmas 4.5, 4.8, 4.10 (ii), 4.17 (ii) and 4.18. (Q. E. D.)

Next we shall investigate the regularity of the measure $\varkappa$. For that purpose we define functions $K_\varepsilon(\varepsilon>0)$ on $\boldsymbol{C}^+\cup\boldsymbol{R}$ by

$$(4.50)\qquad K_\varepsilon(\zeta)=\int_{\boldsymbol{R}}\frac{1}{(\lambda-\zeta-i\varepsilon)(\lambda-i\varepsilon)}\varkappa(d\lambda).$$

We note that for any $\zeta=\xi+i\eta\in\boldsymbol{C}^+\cup\boldsymbol{R}$ and any $\varepsilon>0$

$$\begin{aligned}(4.51)\qquad (-i\zeta)K_\varepsilon(\zeta)&=\frac{1}{i}\int_{\boldsymbol{R}}\Big(\frac{1}{\lambda-\zeta-i\varepsilon}-\frac{1}{\lambda-i\varepsilon}\Big)\varkappa(d\lambda)\\ &=\frac{1}{i}\int_{\boldsymbol{R}}\Big(\frac{\lambda-\xi}{(\lambda-\xi)^2+(\varepsilon+\eta)^2}-\frac{\lambda}{\lambda^2+\varepsilon^2}\Big)\varkappa(d\lambda)\\ &\quad+\int_{\boldsymbol{R}}\Big(\frac{\varepsilon+\eta}{(\lambda-\xi)^2+(\varepsilon+\eta)^2}-\frac{\varepsilon}{\lambda^2+\varepsilon^2}\Big)\varkappa(d\lambda).\end{aligned}$$

By our condition (3.6), Lemma 4.18 (i) and (4.46), we have

LEMMA 4.20 *There exist positive constants c_1 and m_1 such that for any $\zeta \in \boldsymbol{C}^+ \cup \boldsymbol{R}$*

$$\sup_{0<\varepsilon<1} |(-i\zeta)K_\varepsilon(\zeta)| \leq c_1(1+|\zeta|^{m_1}).$$

In particular we can see from (4.50) and Lemma 4.20 that

LEMMA 4.21

(i) *K_ε is holomorphic in $\boldsymbol{C}^+$ and continuous on $\boldsymbol{C}^+ \cup \boldsymbol{R}$.*

(ii) *For each $\varepsilon > 0$ there exists a positive constant c_ε such that*

$$|K_\varepsilon(\zeta)| \leq c_\varepsilon(1+|\zeta|^{m_1}) \text{ for any } \zeta \in \boldsymbol{C}^+ \cup \boldsymbol{R}.$$

Furthermore it follows from Lemma 4.18 and (4.47) that

LEMMA 4.22

(i) *For any $\zeta \in \boldsymbol{C}^+$*

$$\lim_{\varepsilon \downarrow 0}(-i\zeta)K_\varepsilon(\zeta) = \frac{\alpha}{\sqrt{2\pi}}\{([R_A](\zeta))^{-1} - ([R_A](0+i0))^{-1}\} + i\zeta.$$

(ii) *For any $\xi \in \boldsymbol{R} - \Lambda_A$*

$$\lim_{\varepsilon \downarrow 0}(-i\xi)K_\varepsilon(\xi) = \frac{\alpha}{\sqrt{2\pi}}\{([R_A](\xi+i0))^{-1} - ([R_A](0+i0))^{-1}\} + i\xi.$$

Next we shall show the following Lemma 4.23 carefully, which corresponds to Lemma 3.6, because κ is not always bounded in our case.

LEMMA 4.23

(i) $\displaystyle \lim_{\varepsilon \downarrow 0}\int_{\boldsymbol{R}} \frac{\varepsilon}{\lambda^2+\varepsilon^2}\kappa(d\lambda) = \frac{\alpha}{\sqrt{2\pi}}\mathrm{Re}\{([R_A](0+i0))^{-1}\}$

(ii) *For any $\xi \in \boldsymbol{R} - \Lambda_A$*

$$\lim_{\varepsilon \downarrow 0}\int_{\boldsymbol{R}} \frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2}\kappa(d\lambda) = \frac{\alpha}{\sqrt{2\pi}}\mathrm{Re}\{([R_A](\xi+i0))^{-1}\}.$$

PROOF By (4.15), Lemma 4.15 (ii), (4.39), Lemma 4.16 and (4.40), we have for any $\varepsilon > 0$

$$(4.52) \quad \int_{\boldsymbol{R}} \frac{\varepsilon}{\lambda^2+\varepsilon^2}\kappa(d\lambda) = \lim_{k\to\infty}\int_{\boldsymbol{R}} \frac{\varepsilon}{\lambda^2+\varepsilon^2}\kappa_{m_k}(d\lambda)$$
$$= \lim_{k\to\infty}\int_0^\infty e^{-\varepsilon t}\mathrm{Re}(\phi_{m_k}(t))dt.$$

Therefore, by Lemmas 4.2 (ii), 4.8 (i), (4.25) and (4.52), we have

$$(4.53)\qquad \int_{\boldsymbol{R}} \frac{\varepsilon}{\lambda^2+\varepsilon^2}\varkappa(d\lambda) = \frac{\alpha}{\sqrt{2\pi}}\mathrm{Re}\{([R_A](i\varepsilon))^{-1}\} - \varepsilon.$$

By letting ε tend to zero in (4.53), we have (i). On the other hand, we see from Lemma 4.22 (ii) that for any $\xi \in \boldsymbol{R} - \Lambda_A$

$$(4.54)\qquad \lim_{\varepsilon\downarrow 0} \mathrm{Re}\{(-i\xi)K_\varepsilon(\xi)\}$$
$$= \frac{\alpha}{\sqrt{2\pi}}\mathrm{Re}\{([R_A](\xi+i0))^{-1} - ([R_A](0+i0))^{-1}\}.$$

By (4.51),

$$(4.55)\qquad \mathrm{Re}\{(-i\xi)K_\varepsilon(\xi)\}$$
$$= \int_{\boldsymbol{R}} \frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2}\varkappa(d\lambda) - \int_{\boldsymbol{R}} \frac{\varepsilon}{\lambda^2+\varepsilon^2}\varkappa(d\lambda).$$

Therefore, (ii) follows from (i), (4.54) and (4.55). (Q. E. D.)

In particular, we see from Lemmas 4.20, 4.23 (i) and (4.55) that

LEMMA 4.24 *There exists a positive constant c_2 such that for any $\xi \in \boldsymbol{R}$*

$$\sup_{0<\varepsilon<1} \int_{\boldsymbol{R}} \frac{1}{\pi} \frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2}\varkappa(d\lambda) \le c_2(1+|\xi|^{m_1}).$$

Concerning the continuity of $\varkappa$, we shall show

LEMMA 4.25 *For any $\xi \in \boldsymbol{R} - \Lambda_A$*

$$\lim_{\eta\downarrow 0}(\lim_{\varepsilon\downarrow 0} \int_{\boldsymbol{R}} (\frac{\lambda-\xi}{(\lambda-\xi)^2+\eta^2} - \frac{\lambda-\xi}{(\lambda-\xi)^2+\varepsilon^2})\varkappa(d\lambda)) = 0.$$

PROOF By Lemma 4.19 and (4.50), we have

$$(4.56)\qquad \lim_{\eta\downarrow 0}(\lim_{\varepsilon\downarrow 0}(-i(\xi+i\eta))K_\varepsilon(\xi+i\eta)) = \lim_{\varepsilon\downarrow 0}(-i\xi)K_\varepsilon(\xi)$$

for any $\xi \in \boldsymbol{R} - \Lambda_A$. Furthermore, by (4.51),

$$(4.57)\qquad (-i(\xi+i\eta))K_\varepsilon(\xi+i\eta) = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\mathrm{I} = \frac{1}{i}\int_{\boldsymbol{R}} (\frac{\lambda-\xi}{(\lambda-\xi)^2+(\eta+\varepsilon)^2} - \frac{\lambda-\xi}{(\lambda-\xi)^2+\eta^2})\varkappa(d\lambda)$$
$$\mathrm{II} = \frac{1}{i}\int_{\boldsymbol{R}} (\frac{\lambda-\xi}{(\lambda-\xi)^2+\eta^2} - \frac{\lambda}{\lambda^2+\varepsilon^2})\varkappa(d\lambda)$$

and

$$\mathrm{III}=\int_{\boldsymbol{R}}\left(\frac{\eta+\varepsilon}{(\lambda-\xi)^2+(\eta+\varepsilon)^2}-\frac{\varepsilon}{\lambda^2+\varepsilon^2}\right)\kappa(d\lambda).$$

Since

$$\left|\frac{\lambda-\xi}{(\lambda-\xi)^2+(\eta+\varepsilon)^2}-\frac{\lambda-\xi}{(\lambda-\xi)^2+\eta^2}\right|\leq\varepsilon(\varepsilon+2\eta)\left|\frac{\lambda-\xi}{((\lambda-\xi)^2+\eta^2)^2}\right|,$$

it follows from (4.43) that

$$\lim_{\varepsilon\downarrow 0}\mathrm{I}=0. \tag{4.58}$$

By (4.43) and Lemma 4.23, we see that

$$\lim_{\varepsilon\downarrow 0}\mathrm{III}=\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\kappa(d\lambda)-\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{\varepsilon}{\lambda^2+\varepsilon^2}\kappa(d\lambda)$$

and so

$$\lim_{\eta\downarrow 0}(\lim_{\varepsilon\downarrow 0}\mathrm{III})=\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\left(\frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2}-\frac{\varepsilon}{\lambda^2+\varepsilon^2}\right)\kappa(d\lambda). \tag{4.59}$$

On the other hand, by (4.51), we have

$$(-i\xi)K_\varepsilon(\xi)=\frac{1}{i}\int_{\boldsymbol{R}}\left(\frac{\lambda-\xi}{(\lambda-\xi)^2+\varepsilon^2}-\frac{\lambda}{\lambda^2+\varepsilon^2}\right)\kappa(d\lambda)+\int_{\boldsymbol{R}}\left(\frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2}-\frac{\varepsilon}{\lambda^2+\varepsilon^2}\right)\kappa(d\lambda). \tag{4.60}$$

Therefore, by (4.56), (4.57), (4.58), (4.59) and (4.60), we have Lemma 4.25. (Q. E. D.)

By virtue of Lemma 4.23, we can define a function $P\kappa$ on $\boldsymbol{R}$ by

$$(P\kappa)(\xi)\equiv\begin{cases}\lim_{\varepsilon\downarrow 0}\dfrac{1}{\pi}\displaystyle\int_{\boldsymbol{R}}\frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2}\kappa(d\lambda) & \text{for } \xi\in\boldsymbol{R}-\Lambda_A\\ 0 & \text{for } \xi\in\Lambda_A.\end{cases} \tag{4.61}$$

Similarly as Lemma 3.7, we shall show

LEMMA 4.26

(i) $0\leq(P\kappa)(\lambda)\leq c_2(1+|\lambda|^{m_1})$ $(\lambda\in\boldsymbol{R})$

(ii) $(P\kappa)(0)=\dfrac{\mathrm{Re}\beta}{\pi}$

(iii) $\kappa(d\lambda)=(P\kappa)(\lambda)d\lambda$.

For that purpose we prepare

LEMMA 4.27 *Let m be any Borel measure on $\boldsymbol{R}$ such that*

$$\int_{\boldsymbol{R}} \frac{1}{1+\xi^2} m(d\xi) < \infty.$$

Then for any $f \in C_0(\boldsymbol{R})$

$$\lim_{\varepsilon \downarrow 0} \int_{\boldsymbol{R}} f(\lambda) \left(\frac{1}{\pi} \int_{\boldsymbol{R}} \frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2} m(d\xi) \right) d\lambda = \int_{\boldsymbol{R}} f(\lambda) m(d\lambda).$$

PROOF We may assume that $f \geq 0$ and $f \in C_0(|\lambda| < T)$ with some $T \in (0, \infty)$. We divide

$$(4.62) \quad \int_{\boldsymbol{R}} f(\lambda) \left(\frac{1}{\pi} \int_{\boldsymbol{R}} \frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2} m(d\xi) \right) d\lambda = \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon,$$

where

$$\mathrm{I}_\varepsilon = \int_{|\xi| < T+1} \left(\int_{\boldsymbol{R}} \frac{1}{\pi} \frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2} f(\lambda) d\lambda \right) m(d\xi)$$

and

$$\mathrm{II}_\varepsilon = \int_{|\xi| \geq T+1} \left(\int_{\boldsymbol{R}} \frac{1}{\pi} \frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2} f(\lambda) d\lambda \right) m(d\xi).$$

Since $m((-T-1, T+1)) < \infty$, we have

$$(4.63) \quad \lim_{\varepsilon \downarrow 0} \mathrm{I}_\varepsilon = \int_{|\xi| < T+1} f(\xi) m(d\xi).$$

On the other hand, we see that for any $\xi \in \boldsymbol{R} - (-T-1, T+1)$ and any $\lambda \in (-T, T)$ $|\xi - \lambda| \geq |\xi| - |\lambda| \geq |\xi| - T$ and so

$$\left| \int_{\boldsymbol{R}} \frac{1}{\pi} \frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2} f(\lambda) d\lambda \right| \leq \left(\int_{|\lambda| < T} f(\lambda) d\lambda \right) \frac{\varepsilon}{\pi} \frac{1}{(|\xi| - T)^2}.$$

Since $\int_{|\xi| \geq T+1} \frac{1}{(|\xi| - T)^2} m(d\xi) < \infty$, we can apply Lebesgue's convergence theorem to get

$$(4.64) \quad \lim_{\varepsilon \downarrow 0} \mathrm{II}_\varepsilon = \int_{|\xi| \geq T+1} f(\xi) m(d\xi).$$

Therefore Lemma 4.27 follows from (4.62), (4.63) and (4.64).

(Q. E. D.)

PROOF OF LEMMA 4.26 (i) follows from Lemma 4.24 and (4.61). (ii) follows from (4.10), (4.12), (4.13), Lemma 4.23 (i) and (4.61).

Take any $f\in C_0(\boldsymbol{R})$. By Lemma 4.24 and (4.61),

$$(4.65)\quad \lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}} f(\lambda)\Big(\int_{\boldsymbol{R}}\frac{1}{\pi}\frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2}\varkappa(d\xi)\Big)d\lambda=\int_{\boldsymbol{R}} f(\lambda)(P\varkappa)(\lambda)d\lambda.$$

On the other hand, we find from Lemma 4.26 that the left-hand side of (4.65) is equal to $\int_{\boldsymbol{R}} f(\lambda)\varkappa(d\lambda)$, which gives (iii). (Q. E. D.)

Next let Δ be a spectral measure of R_A, that is, Δ is a bounded Borel measure on $\boldsymbol{R}$ such that

$$(4.66)\quad R_A(t)=\int_{\boldsymbol{R}} e^{-it\xi}\Delta(d\xi).$$

A direct calculation gives

LEMMA 4.28 *For any $\eta>0$ and $\xi\in\boldsymbol{R}$,*

$$\frac{1}{\pi}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\Delta(d\lambda)=2\operatorname{Re}([R_A](\xi+i\eta)).$$

Then we shall show

LEMMA 4.29

$$\sup_{\eta>0}\Bigg(\int_{\boldsymbol{R}}\frac{\eta+\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\varkappa(d\lambda)}{|\eta+\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\varkappa(d\lambda)|^2+|\operatorname{Im}\beta-\xi-\int_{\boldsymbol{R}}(\frac{\lambda-\xi}{(\lambda-\xi)^2+\eta^2}-\frac{\lambda}{\lambda^2+\eta^2})\varkappa(d\lambda)|^2}d\xi\Bigg)<\infty.$$

PROOF By Lemma 4.28, we have

$$(4.67)\quad \sup_{\eta>0}\int_{\boldsymbol{R}}|\operatorname{Re}([R_A](\xi+i\eta))|\,d\xi<\infty.$$

On the other hand, we see from Lemma 4.19 (i) that for any $\zeta=\xi+i\eta\in\boldsymbol{C}^+$

$$(4.68)\quad \operatorname{Re}([R_A](\zeta))=\frac{\alpha}{\sqrt{2\pi}}\frac{\operatorname{Re}(\beta-i\zeta+\lim_{\varepsilon\downarrow 0}(-i\zeta)K_\varepsilon(\zeta))}{|\beta-i\zeta+\lim_{\varepsilon\downarrow 0}(-i\zeta)K_\varepsilon(\zeta)|^2}$$

and moreover by (4.55), (4.61) and Lemma 4.26 (ii),

$$(4.69)\quad \operatorname{Re}(\beta-i\zeta+\lim_{\varepsilon\downarrow 0}(-i\zeta)K_\varepsilon(\zeta))=\eta+\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\varkappa(d\lambda)$$

and

(4.70) $$\mathrm{Im}(\beta - i\xi + \lim_{\varepsilon \downarrow 0}(-i\xi)(K_\varepsilon(\xi)) = \mathrm{Im}\beta - \xi - \int_{\boldsymbol{R}} \left(\frac{\lambda - \xi}{(\lambda - \xi)^2 + \eta^2} - \frac{\lambda}{\lambda^2 + \eta^2}\right) \varkappa(d\lambda).$$

Therefore Lemma 4.29 follows from (4.67), (4.68), (4.69) and (4.70). (Q. E. D.)

Finally, by collecting the results which we have investigated, we shall show the following main

THEOREM 4.1 *Let R_A be any covariance function of the form* (2.3) *satisfying conditions* (2.4), (3.5) *and* (3.6). *Then there exists a unique triple* $(\alpha, \beta, \varkappa)$ *such that for any* $\zeta \in \boldsymbol{C}^+ \cup (\boldsymbol{R} - \Lambda_A)$

(4.71) $$[R_A](\zeta + i0) = \frac{\alpha}{\sqrt{2\pi}} \frac{1}{\beta - i\zeta + (-i\zeta)\lim_{\varepsilon \downarrow 0} \int_{\boldsymbol{R}} \frac{1}{(\lambda - \zeta - i\varepsilon)(\lambda - i\varepsilon)} \varkappa(d\lambda)}.$$

Here

(i) $\alpha > 0$

(ii) $\beta \in \boldsymbol{C} - \{0\}$

(iii) $\varkappa$ *is a Borel measure on* $\boldsymbol{R}$ *satisfying the following conditions* (a), (b), (c), (d), (e), (f), (g) *and* (h):

(a) $\varkappa(\{0\}) = 0$

(b) $\int_{\boldsymbol{R}} \frac{1}{\lambda^2 + 1} \varkappa(d\lambda) < \infty$

(c) $\lim_{\varepsilon \downarrow 0} \int_{\boldsymbol{R}} \frac{1}{\pi} \frac{\varepsilon}{\lambda^2 + \varepsilon^2} \varkappa(d\lambda) = \frac{1}{\pi} \mathrm{Re}\, \beta$

(d) *there exist positive constants c_1 and m_1 such that for any* $\zeta \in \boldsymbol{C}^+ \cup (\boldsymbol{R} - \Lambda_A)$

$$\sup_{0 < \varepsilon < 1} |(-i\zeta) K_\varepsilon(\zeta)| \leq c_1 (1 + |\zeta|^{m_1}),$$

where

$$K_\varepsilon(\zeta) = \int_{\boldsymbol{R}} \frac{1}{(\lambda - \zeta - i\varepsilon)(\lambda - i\varepsilon)} \varkappa(d\lambda)$$

(e) *for any* $\zeta \in \boldsymbol{C}^+ \cup (\boldsymbol{R} - \Lambda_A)$

$$\lim_{\varepsilon \downarrow 0} (-i\zeta) K_\varepsilon(\zeta) \text{ exists}$$

(f) *for any* $\xi \in \boldsymbol{R} - \Lambda_A$

$$\lim_{\eta\downarrow 0}(\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}(\frac{\lambda-\xi}{(\lambda-\xi)^2+\eta^2}-\frac{\lambda-\xi}{(\lambda-\xi)^2+\varepsilon^2})\varkappa(d\lambda))=0$$

(g) *for any* $\xi\in\boldsymbol{R}-\Lambda_A$

$$\beta-i\xi+\lim_{\varepsilon\downarrow 0}(-i\xi)K_\varepsilon(\xi)\neq 0$$

(h)

$$\sup_{\eta>0}(\int_{\boldsymbol{R}}\frac{\eta+\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\varkappa(d\lambda)}{|\eta+\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\varkappa(d\lambda)|^2+|\operatorname{Im}\beta-\xi-\int_{\boldsymbol{R}}(\frac{\lambda-\xi}{(\lambda-\xi)^2+\eta^2}-\frac{\lambda}{\lambda^2+\eta^2})\varkappa(d\lambda)|^2}d\xi)<\infty.$$

PROOF The existence of a triple $(\alpha,\beta,\varkappa)$ satisfying relation (4.71) with conditions (i), (ii) and (iii) follows from Lemma 4.19, (4.12), (4.13), (4.42), (4.43), (4.61), Lemma 4.25 (ii), (4.50), Lemmas 4.20, 4.22, 4.25 and 4.28. Next we shall prove the uniqueness of such a triple $(\alpha,\beta,\varkappa)$. By noting that $0\notin\Lambda_A$ and then substituting $\zeta=0$ into (4.71) we have

$$[R_A](0+i0)=\frac{\alpha}{\sqrt{2\pi}\,\beta}. \tag{4.72}$$

On the other hand, by substituting $\zeta=i\eta$ $(\eta>0)$ into (4.71) and then using (b), (c) and (e), we have

$$\begin{aligned}\operatorname{Re}\{([R_A](i\eta))^{-1}\}&=\frac{\sqrt{2\pi}}{\alpha}\operatorname{Re}(\beta+\eta+\lim_{\varepsilon\downarrow 0}\eta K_\varepsilon(i\eta))\\&=\frac{\sqrt{2\pi}}{\alpha}\eta(1+\int_{\boldsymbol{R}}\frac{1}{\lambda^2+\eta^2}\varkappa(d\lambda)).\end{aligned} \tag{4.73}$$

Dividing (4.73) by η and then letting η to infinity, we see that

$$\alpha=\frac{R_A(0)}{\sqrt{2\pi}} \tag{4.74}$$

and so (4.72) and (4.74) determine a pair (α,β) uniquely. Finally, by substituting $\zeta=\xi\in\boldsymbol{R}-\Lambda_A$ into (4.71) and then using (c) and (e), we find that

$$\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\varkappa(d\lambda)=\frac{\alpha}{\sqrt{2\pi}}\operatorname{Re}\{([R_A](\xi+i0))^{-1}\}-\eta \tag{4.75}$$

and so by (3.6), Lemma 4.27 and (4.75)

(4.76) $$\varkappa(d\xi)=\frac{R_A(0)}{2\pi^2}\mathrm{Re}\{([R_A](\xi+i0))^{-1}\}d\xi,$$

which determines the measure $\varkappa$ uniquely. (Q. E. D.)

By taking account of Definition 3. 1, we shall give

DEFINITION 4. 1 We call a triple $(\alpha, \beta, \varkappa)$ determined uniquely through relation (4. 71) a second KMO-Langevin data associated with the covariance function R_A.

In particular, it follows from (4. 72), (4. 74) and (4. 76) that

THEOREM 4. 2 *The second KMO-Langevin data* $(\alpha, \beta, \varkappa)$ *associated with the covariance function* R_A *is given by the following formulae* :

(i) $\alpha=\dfrac{R_A(0)}{\sqrt{2\pi}}$

(ii) $\beta=\dfrac{R_A(0)}{2\pi[R_A](0+i0)}$

(iii) $\varkappa(d\xi)=\dfrac{R_A(0)}{2\pi^2}\mathrm{Re}\{([R_A](\xi+i0))^{-1}\}d\xi.$

§ 5. A second KMO-Langevin data(3)

In the previous sections § 3 and § 4, we have considered a stationary curve $A=(A(t)\,;\,t\in\boldsymbol{R})$ in a Hilbert space and then introduced a second KMO-Langevin data $(\alpha, \beta, \varkappa)$ associated with its covariance function R_A. In this section, apart from stationary curves in Hilbert spaces, we shall give a bijective correspondence between non-negative definite functions and second KMO-Langevin data.

We define $\mathscr{R}$ and $\mathscr{L}$ by

(5.1) $\mathscr{R}=\{R:\boldsymbol{R}\to\boldsymbol{C}\,;$

(i) $R(0)\neq 0$

(ii) continuous and non-negative definite

(iii)[3] there exists a null set $\Lambda=\Lambda_R$ in $\boldsymbol{R}-\{0\}$ such that for any $\xi\in\boldsymbol{R}-\Lambda$

$$\lim_{\varepsilon\downarrow 0}[R](\xi+i\varepsilon) \text{ exists}$$

(iv) there exist positive constants c and m such that for any $\zeta\in\boldsymbol{C}^+$

$$|[R](\zeta)|\geq c(1+|\zeta|^m)^{-1},$$

where $[R](\zeta)=\dfrac{1}{2\pi}\displaystyle\int_0^\infty e^{i\zeta t}R(t)\,dt\}$

(3) See footnote (1).

and

(5.2) $\mathscr{L}=\{(\alpha,\beta,\varkappa)$;

(i) $\alpha>0$

(ii) $\beta\in\boldsymbol{C}-\{0\}$

(iii) $\varkappa$ is a Borel measure on $\boldsymbol{R}$ such that

(a) $\varkappa(\{0\})=0$

(b) $\displaystyle\int_{\boldsymbol{R}}\frac{1}{\lambda^{2}+1}\varkappa(d\lambda)<\infty$

(c) $\displaystyle\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{\varepsilon}{\lambda^{2}+\varepsilon^{2}}\varkappa(d\lambda)=\operatorname{Re}\beta$

(d) there exist positive constants c and m such that

$$\sup_{0<\varepsilon<1}|(-i\zeta)K_{\varepsilon}(\zeta)|\leq c(1+|\zeta|^{m})\text{ for any }\zeta\in\boldsymbol{C}^{+},$$

where

$$K_{\varepsilon}(\zeta)=\int_{\boldsymbol{R}}\frac{1}{(\lambda-\zeta-i\varepsilon)(\lambda-i\varepsilon)}\varkappa(d\lambda)\quad(\zeta\in\boldsymbol{C}^{+}\cup\boldsymbol{R}) \tag{5.3}$$

(e) there exists a null set $\Lambda=\Lambda_{(\alpha,\beta,\varkappa)}$ in $\boldsymbol{R}-\{0\}$ such that

$$\lim_{\varepsilon\downarrow 0}(-i\zeta)K_{\varepsilon}(\zeta)\text{ exists for any }\zeta\in\boldsymbol{C}^{+}\cup(\boldsymbol{R}-\Lambda)$$

(f) for any $\xi\in\boldsymbol{R}-\Lambda$

$$\lim_{\eta\downarrow 0}(\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}(\frac{\lambda-\xi}{(\lambda-\xi)^{2}+\eta^{2}}-\frac{\lambda-\xi}{(\lambda-\xi)^{2}+\varepsilon^{2}})\varkappa(d\lambda))=0$$

(g) for any $\xi\in\boldsymbol{R}-\Lambda$

$$\beta-i\xi+\lim_{\varepsilon\downarrow 0}(-i\xi)K_{\varepsilon}(\xi)\neq 0$$

(h) $\displaystyle\sup_{\eta>0}\int_{\boldsymbol{R}}M_{\eta}(\xi)d\xi<\infty,$

where

(5.4)

$$M_{\eta}(\xi)=\frac{\eta+\displaystyle\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^{2}+\eta^{2}}\varkappa(d\lambda)}{|\eta+\displaystyle\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^{2}+\eta^{2}}\varkappa(d\lambda)|^{2}+|\operatorname{Im}\beta-\xi-\displaystyle\int_{\boldsymbol{R}}(\frac{\lambda-\xi}{(\lambda-\xi)^{2}+\eta^{2}}-\frac{\lambda}{\lambda^{2}+\eta^{2}})\varkappa(d\lambda)|^{2}}\}.$$

THEOREM 5.1 *There exists a bijective mapping Φ from $\mathscr{R}$ and $\mathscr{L}$ such that for any $R\in\mathscr{R}$, $\Phi(R)=(\alpha,\beta,\varkappa)$ and $\zeta\in\boldsymbol{C}^{+}\cup(\boldsymbol{R}-\Lambda)$*

$$(5.5)\qquad [R](\zeta)=\frac{\alpha}{\sqrt{2\pi}}\frac{1}{\beta-i\zeta+\lim\limits_{\varepsilon\downarrow 0}(-i\zeta)\int_{\boldsymbol{R}}\frac{1}{(\lambda-\zeta-i\varepsilon)(\lambda-i\varepsilon)}\varkappa(d\lambda)}.$$

REMARK 5.1 Let $R\in\mathscr{R}$ and $(\alpha,\beta,\varkappa)\in\mathscr{L}$ such that $\Phi(R)=(\alpha,\beta,\varkappa)$. Then we can choose null sets Λ_R and $\Lambda_{(\alpha,\beta,\varkappa)}$ such that $\Lambda_R=\Lambda_{(\alpha,\beta,\varkappa)}$.

PROOF OF THEOREM 5.1

Let R be any element of $\mathscr{R}$. We take a complex-valued stationary process $X=(X(t)\,;\,t\in\boldsymbol{R})$ on a probability space $(\Omega,\mathscr{B},P)$ such that $E(X(t))=0$ and $E(X(t)\overline{X(s)})=R(t-s)$ $(t,s\in\boldsymbol{R})$. Then we define a Hilbert space $\mathscr{H}$ and a one-parameter group of unitary operators $\{U(t)\,;\,t\in\boldsymbol{R}\}$ by

$$(5.6)\qquad \mathscr{H}=\text{the closed linear hull of }\{X(t)\,;\,t\in\boldsymbol{R}\}\text{ in }L^2(\Omega,\mathscr{B},P)$$

and

$$(5.7)\qquad U(t)(X(s))=X(s+t).$$

Since R satisfies conditions (2.4), (3.5) and (3.6) in the situation of § 4, we can apply Theorem 4.1 to obtain a mapping Φ from $\mathscr{R}$ into $\mathscr{L}$ satisfying relation (5.5). By Lemmas 4.6 and 4.7, we see that Φ is injective. Next, let $(\alpha,\beta,\varkappa)$ be any element of $\mathscr{L}$. We define a function Z_0 on $\boldsymbol{C}^+$ by

$$(5.8)\qquad Z_0(\zeta)=\beta-i\zeta+\lim_{\varepsilon\downarrow 0}(-i\zeta)K_\varepsilon(\zeta).$$

By conditions (b) and (c) in $\mathscr{L}$, we see that for any $\zeta=\xi+i\eta\in\boldsymbol{C}^+$

$$(5.9)\qquad \mathrm{Re}(Z_0(\xi+i\eta))=\eta+\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\varkappa(d\lambda)$$

and

$$(5.10)\qquad \mathrm{Im}(Z_0(\xi+i\eta))=\mathrm{Im}\,\beta-\xi-\int_{\boldsymbol{R}}\left(\frac{\lambda-\xi}{(\lambda-\xi)^2+\eta^2}-\frac{\lambda}{\lambda^2+\eta^2}\right)\varkappa(d\lambda).$$

In particular, by (5.9),

$$(5.11)\qquad Z_0(\zeta)\neq 0\qquad\qquad \text{for any }\zeta\in\boldsymbol{C}^+.$$

Furthermore, it follows from conditions (b), (d) and (e) that

$$(5.12)\qquad Z_0\text{ is holomorphic on }\boldsymbol{C}^+.$$

By (5.9), (5.11) and (5.12), we can get a holomorphic function h_0 on $\boldsymbol{C}^+$ such that

(5.13) $$h_0(\zeta)=\frac{\alpha}{\sqrt{2\pi}}\frac{1}{Z_0(\zeta)}$$

and

(5.14) $$\mathrm{Re}(h_0(\zeta))\geq 0.$$

Since $\mathrm{Re}(h_0)$ is non-negative and harmonic on $\boldsymbol{C}^+$, there exists a Borel measure $\Delta(d\lambda)$ on $\boldsymbol{R}$ such that for any $\zeta=\xi+i\eta\in\boldsymbol{C}^+$

(5.15) $$\mathrm{Re}(h_0(\xi+i\eta))=\frac{1}{2\pi}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\Delta(d\lambda).$$

Furthermore we see from condition (h) in $\mathscr{L}$, (5.9), (5.10) and (5.15) that for any $\eta>0$

(5.16) $$\Delta(\boldsymbol{R})=\int_{\boldsymbol{R}}\mathrm{Re}(h_0(\xi+i\eta))d\xi<\infty.$$

Therefore, we can define a continuous and non-negative definite function R on $\boldsymbol{R}$ by

(5.17) $$R(t)=\int_{\boldsymbol{R}}e^{-it\xi}\Delta(d\xi).$$

We shall show that $R\in\mathscr{R}$ and $\Phi(R)=(\alpha,\beta,\varkappa)$. By (4.60), conditions (c) and (e) in $\mathscr{L}$, we note that for any $\xi\in\boldsymbol{R}-\Lambda$

(5.18) $$\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{\varepsilon}{(\lambda-\xi)^2+\varepsilon^2}\varkappa(d\lambda)\ \text{exists}$$

and

(5.19) $$\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\left(\frac{\lambda-\xi}{(\lambda-\xi)^2+\varepsilon^2}-\frac{\lambda}{\lambda^2+\varepsilon^2}\right)\varkappa(d\lambda)\ \text{exists.}$$

It is clear that

(5.20) $$\lim_{\eta\to\infty}\eta[R](i\eta)=\frac{R(0)}{2\pi}.$$

We claim

(5.21) $$\lim_{\eta\to\infty}\eta\ h_0(i\eta)=\frac{\alpha}{\sqrt{2\pi}}.$$

By (4.57) and (5.18), we see that for any $\eta>0$

(5.22) $$\lim_{\varepsilon\downarrow 0}(-i(i\eta))K_\varepsilon(i\eta)$$

$$=\frac{1}{i}\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\left(\frac{\lambda}{\lambda^2+\eta^2}-\frac{\lambda}{\lambda^2+\varepsilon^2}\right)\kappa(d\lambda)+\int_{\boldsymbol{R}}\frac{\eta}{\lambda^2+\eta^2}\kappa(d\lambda)$$
$$-\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{\varepsilon}{\lambda^2+\varepsilon^2}\kappa(d\lambda).$$

By considering the following decomposition

$$\int_{\boldsymbol{R}}\left(\frac{\lambda}{\lambda^2+\eta^2}-\frac{\lambda}{\lambda^2+\varepsilon^2}\right)\kappa(d\lambda)$$
$$=\int_{\boldsymbol{R}}\left(\frac{\lambda}{\lambda^2+\eta^2}-\frac{\lambda}{\lambda^2+1}\right)\kappa(d\lambda)+\int_{\boldsymbol{R}}\left(\frac{\lambda}{\lambda^2+1}-\frac{\lambda}{\lambda^2+\varepsilon^2}\right)\kappa(d\lambda),$$

we see from condition (b) in $\mathscr{L}$ and (5.22) that

$$(5.23)\quad \lim_{\varepsilon\downarrow 0}(-i(i\eta))K_\varepsilon(i\eta)$$
$$=\int_{\boldsymbol{R}}\left(\frac{\lambda}{\lambda^2+\eta^2}-\frac{\lambda}{\lambda^2+1}\right)\kappa(d\lambda)+o(\eta)\ \text{ as } \eta\to\infty.$$

Since

$$\left|\frac{\lambda}{\lambda^2+\eta^2}-\frac{\lambda}{\lambda^2+1}\right|=\left|\eta\frac{\lambda\eta}{\lambda^2+\eta^2}\frac{1}{\lambda^2+1}\right|\leq\frac{\eta}{2}\frac{1}{\lambda^2+1},$$

we find from (5.23) that

$$(5.24)\quad \lim_{\varepsilon\downarrow 0}(-i(i\eta))K_\varepsilon(i\eta)=o(\eta)\ \text{ as } \eta\to\infty.$$

Therefore, (5.21) follows from (5.8), (5.13) and (5.24).

Next, by Lemma 4.28, (5.15) and (5.17), we have

$$(5.25)\quad \mathrm{Re}[R](\zeta)=\mathrm{Re}(h_0(\zeta))\qquad \text{for any } \zeta\in\boldsymbol{C}^+.$$

In particular, it follows from (5.20), (5.21) and (5.25) that

$$(5.26)\quad R(0)=\sqrt{2\pi}\,\alpha,$$

which implies that R satisfies condition (i) in $\mathscr{R}$. Furthermore we can see from (5.20), (5.21), (5.25) and (5.26) that

$$(5.27)\quad [R](\zeta)=h_0(\zeta)\ \text{ for any } \zeta\in\boldsymbol{C}^+,$$

which gives by condition (d) in $\mathscr{S}$, (5.8) and (5.13) that R satisfies condition (iii) in $\mathscr{R}$.

By taking the same consideration as Lemma 4.25, we note that it follows from conditions (b), (e) and (f) that for any $\xi\in\boldsymbol{R}-\Lambda$

$$(5.28)\quad \lim_{\eta\downarrow 0}(\lim_{\varepsilon\downarrow 0}(-i(\xi+i\eta))K_\varepsilon(\xi+i\eta))=\lim_{\varepsilon\downarrow 0}(-i\xi)K_\varepsilon(\xi).$$

Therefore, by condition (g) in $\mathscr{L}$, (5.8), (5.13), (5.27) and (5.28), we find that for any $\xi \in \boldsymbol{R}-\Lambda$

$$(5.29)\qquad \lim_{\eta \downarrow 0}[R](\xi+i\eta)=\frac{\alpha}{\sqrt{2\pi}}\frac{1}{\beta-i\xi+\lim_{\varepsilon\downarrow 0}(-i\xi)K_{\varepsilon}(\xi)},$$

which with (5.27) implies that R satisfies condition (iii) in $\mathscr{R}$ and relation (5.5). Thus we have completed the proof of Theorem 5.1. (Q. E. D.)

REMARK 5.2 By (5.16), we note that condition (h) in $\mathscr{L}$ can be replaced by

(h)′ $\int_{\boldsymbol{R}} M_{\eta}(\xi)d\xi$ is independent of $\eta>0$.

REMARK 5.3 By (5.9), (5.10), (5.13) and (5.27), we note

$$(5.30)\qquad \mathrm{Re}([R](\xi+i\eta))=\frac{\alpha}{\sqrt{2\pi}}M_{\eta}(\xi) \text{ for any } \zeta=\xi+i\eta\in\boldsymbol{C}^{+}.$$

REMARK 5.4 The element of $\mathscr{R}$ can not always have its spectral density. Such an example is given as follows. We consider the non-negative definite function R defined by

$$(5.31)\qquad R(t)=\sigma_1 e^{-p_1|t|}+\sigma_2 e^{-ip_2 t} \qquad (t\in\boldsymbol{R}),$$

where $\sigma_1>0$, $\sigma_2>0$, $p_1>0$ and $p_2\in\boldsymbol{R}-\{0\}$. It can be seen that the spectral measure Δ of R is given by

$$(5.32)\qquad \Delta(d\xi)=\frac{1}{\pi}\frac{p_1\sigma_1}{p_1^2+\xi^2}d\xi+\sigma_2\delta_{\{p_2\}}(d\xi).$$

Furthermore we see that for any $\zeta=\xi+i\eta\in\boldsymbol{C}^{+}$

$$(5.33)\qquad [R](\zeta)=\frac{1}{2\pi}\left(\frac{\sigma_1}{p_1+\eta-i\xi}+\frac{\sigma_2}{\eta-i(\xi-p_2)}\right).$$

Therefore we find that R belongs to the set $\mathscr{R}$ by choosing a null set $\Lambda_R=\{p_2\}$. Then it can be seen from Theorem 4.2 that the second KMO-Langevin data (α,β,κ) associated with R is given by

$$(5.34)\qquad \begin{cases} \alpha = \dfrac{\sigma_1+\sigma_2}{\sqrt{2\pi}} \\ \beta = (\sigma_1+\sigma_2)\left(\dfrac{\sigma_1}{p_1}-i\dfrac{\sigma_2}{p_2}\right)^{-1} \\ \kappa(d\xi)=\dfrac{p_1\sigma_1(\sigma_1+\sigma_2)}{\pi}\dfrac{(\xi-p_2)^2}{\{(\sigma_1+\sigma_2)\xi-p_2\sigma_1\}^2+(p_1\sigma_2)^2}d\xi. \end{cases}$$

By taking account of Remark 5.4, we define a subclass $\mathscr{R}_0$ of $\mathscr{R}$

$$(5.36)\qquad \mathscr{R}_0 == \{R \in \mathscr{R};\ R \text{ has a spectral density}\}.$$

Then we shall show

THEOREM 5.2 $\Phi(\mathscr{R}_0) = \mathscr{L}_0$,
where

$$(5.37)\qquad \mathscr{L}_0 \equiv \{(\alpha, \beta, \kappa) \in \mathscr{L};$$

$$(\mathrm{i})\qquad \int_{\boldsymbol{R}} M_\eta(\xi)\,d\xi = \int_{\boldsymbol{R}} \lim_{\varepsilon \downarrow 0} M_\varepsilon(\xi)\,d\xi \qquad \textit{for any } \eta > 0\}.$$

PROOF Let R be any element of $\mathscr{R}_0$ and Δ' be the spectral density of R. We put $\Phi(R) = (\alpha, \beta, \kappa)$. By Lemma 4.28,

$$(5.37)\qquad \frac{1}{\pi}\int_{\boldsymbol{R}} \frac{\eta}{(\lambda - \xi)^2 + \eta^2}\Delta'(\lambda)\,d\lambda = 2\,\mathrm{Re}([R](\xi + i\eta)).$$

By letting η tend to zero, we have

$$(5.38)\qquad \Delta'(\xi) = 2\,\mathrm{Re}([R](\xi + i0)) \qquad a.\ e.\ \xi \in \boldsymbol{R} - \Lambda_R.$$

Therefore, we see from (5.37) and (5.38) that for any $\eta > 0$

$$(5.39)\qquad \int_{\boldsymbol{R}} \mathrm{Re}([R](\xi + i\eta))\,d\xi = \int_{\boldsymbol{R}} \mathrm{Re}([R](\xi + i0))\,d\xi,$$

which with (5.30) implies that (i) holds and so the triple (α, β, κ) belongs to $\mathscr{L}_0$.

Conversely, let (α, β, κ) be any element of $\mathscr{L}_0$. We put $\Phi^{-1}(\alpha, \beta, \kappa) = R$ and denote by Δ the spectral measure of R. We define a bounded Borel measure m on $\boldsymbol{R}$ by

$$(5.40)\qquad m(d\xi) = 2\,\mathrm{Re}([R](\xi + i0))\,d\xi.$$

By (5.30) and condition (i) in $\mathscr{L}_0$, we see that for any $\eta > 0$

$$(5.41)\qquad \int_{\boldsymbol{R}} \mathrm{Re}([R](\xi + i\eta))\,d\xi = \int_{\boldsymbol{R}} \lim_{\varepsilon \downarrow 0} \mathrm{Re}([R](\xi + i\varepsilon))\,d\xi.$$

Since $\mathrm{Re}([R](\xi + i\eta)) \geq 0$ for any $\zeta = \xi + i\eta \in \boldsymbol{C}^+$, we can apply Lebesgue's convergence theorem to obtain that for any bounded measurable function f on $\boldsymbol{R}$

$$(5.42)\qquad \lim_{\varepsilon \downarrow 0} \int_{\boldsymbol{R}} f(\xi)\mathrm{Re}([R](\xi + i\varepsilon))\,d\xi = \int_{\boldsymbol{R}} f(\xi)\mathrm{Re}([R](\xi + i0))\,d\xi.$$

In particular, we see from (5.40) and (5.42) that for any $\zeta=\xi+i\eta\in \boldsymbol{C}^{+}$

$$(5.43)\quad \frac{1}{\pi}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^{2}+\eta^{2}}m(d\lambda)=2\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^{2}+\eta^{2}}\mathrm{Re}([R](\lambda+i\varepsilon))d\lambda.$$

Furthermore, we can see from Lemma 4.29 that for any $\zeta=\xi+i\eta\in \boldsymbol{C}^{+}$

$$\begin{aligned}(5.44)\quad & 2\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^{2}+\eta^{2}}\mathrm{Re}([R](\lambda+i\varepsilon))d\lambda \\ &=\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\Big(\int_{\boldsymbol{R}}\frac{1}{\pi}\frac{\varepsilon}{(t-\lambda)^{2}+\varepsilon^{2}}\frac{\eta}{(\lambda-\xi)^{2}+\eta^{2}}d\lambda\Big)\Delta(dt) \\ &=\frac{1}{\pi}\int_{\boldsymbol{R}}\frac{\eta}{(t-\xi)^{2}+\eta^{2}}\Delta(dt).\end{aligned}$$

Therefore, by Lemma 4.27, (5.43) and (5.44), we see that $\Delta(d\lambda)=m(d\lambda)$, which with (5.40) implies that $R\in\mathscr{R}_{0}$. (Q.E.D.)

Finally, we shall give a characterization of the second KMO-Langevin data in a regular case introduced in §3. For that purpose we define a subclass $\mathscr{R}_{1}$ of $\mathscr{R}_{0}$ by

$$(5.45)\quad \mathscr{R}_{1}=\{R\in\mathscr{R};\ R \text{ has a spectral density with finite second moment}\}.$$

Then we shall show

THEOREM 5.3 $\Phi(\mathscr{R}_{1})=\mathscr{L}_{1}$,

where

(5.46) $\mathscr{L}_{1}\equiv\{(\alpha,\beta,\kappa)$;

(i) $\alpha>0$

(ii) $\beta\in\boldsymbol{C}-\{0\}$

(iii) κ *is a Borel measure on* $\boldsymbol{R}$ *such that*

(a) $\kappa(\{0\})=0$

(b)′ $\kappa(\boldsymbol{R})<\infty$

(c) $\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{\varepsilon}{\lambda^{2}+\varepsilon^{2}}\kappa(d\lambda)=\mathrm{Re}\beta$

(d) *there exist positive constants* c *and* m *such that*

$$\sup_{0<\varepsilon<1}|(-i\zeta K_{\varepsilon}(\zeta)|\le c(1+|\zeta|^{m})\ \textit{for any}\ \zeta\in\boldsymbol{C}^{+}$$

(e)′ *there exists a null set* $\Lambda=\Lambda_{(\alpha,\beta,\kappa)}$ *in* $\boldsymbol{R}-\{0\}$ *such that for any* $\xi\in\boldsymbol{R}-\Lambda$

① $\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\boldsymbol{R}}\frac{\lambda-\xi}{(\lambda-\xi)^{2}+\varepsilon^{2}}\kappa(d\lambda)\equiv(H\kappa)(\xi)$ *exists*

② $\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\boldsymbol{R}}\frac{\varepsilon}{(\lambda-\xi)^{2}+\varepsilon^{2}}\kappa(d\lambda)\equiv(P\kappa)(\xi)$ *exists* (g)′ *for any* $\xi\in \boldsymbol{R}-\Lambda$

$$\pi(P\kappa)(\xi)+i(\mathrm{Im}\beta-\xi-\pi((H\kappa)(\xi)-(H\kappa)(0)))\neq 0$$

(i) $\int_{\boldsymbol{R}} M_\eta(\xi)d\xi=\int_{\boldsymbol{R}}\lim_{\varepsilon\downarrow 0} M_\varepsilon(\xi)d\xi$ *for any* $\eta>0$

(j) $\displaystyle\int_{\boldsymbol{R}}\frac{(|(P\kappa)(\xi)|^2+|(H\kappa)(\xi)|^2)(P\kappa)(\xi)}{|\pi(P\kappa)(\xi)|^2+|\mathrm{Im}\beta-\xi-\pi((H\kappa)(\xi)-(H\kappa)(0))|^2}d\xi<\infty\}$.

Proof Let R be any element of $\mathscr{R}_1$. Similarly as in the proof of Theorem 5.1, we take a stationary curve $X=(X(t)\,;\,t\in\boldsymbol{R})$ in a Hilbert space $\mathscr{H}$ and a one-parameter group of unitary operators $(U(t)\,;\,t\in\boldsymbol{R})$ such that $(X(s),\ X(t))_{\mathscr{H}}=R(t-s)$ and $U(t)X(s)=X(s+t)$ $(s,t\in\boldsymbol{R})$. We can see from the condition $\mathscr{R}\in\mathscr{L}_1$ that $X(0)$ belongs to $\mathscr{D}(L)$, where iL is an infinitesimal generator of $(U(t)\,;\,t\in\boldsymbol{R})$. Therefore we are in the same situation as § 2 and § 3. We put $\Phi(R)=(\alpha,\beta,\kappa)$. It then follows from Theorems 3.1 and 5.1 that the triple (α,β,κ) is equal to the second KMO-Langevin data in Definition 3.1. In particular, we find from (3.2) that $\kappa(\boldsymbol{R})<\infty$, that is, (b)′ holds.

Next we shall show (e)′. Since $\kappa(\boldsymbol{R})<\infty$, we note from (4.51) that for any $\zeta\in\boldsymbol{C}^+$

$$(-i\zeta)K_\varepsilon(\zeta)=-i\int_{\boldsymbol{R}}\frac{1}{\lambda-\zeta-i\varepsilon}\kappa(d\lambda)+i\int_{\boldsymbol{R}}\frac{1}{\lambda-i\varepsilon}\kappa(d\lambda). \tag{5.47}$$

By (3.4) and Lemma 3.2 (i), we see that

$$\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{\lambda}{\lambda^2+\varepsilon^2}\kappa(d\lambda)\text{exists.} \tag{5.48}$$

Therefore, it folows from condition (e) in $\mathscr{L}$, (5.47) and (5.48) that condition (e)′ in $\mathscr{L}_1$ holds. We note that (f) holds automatically under condition (b)′. Furthermore, we can see from (c), (e)′ and (5.44) that (g)′ in $\mathscr{L}_1$ is equivalent to (g) in $\mathscr{L}$. Since it follows from Lemma 4.28 that

$$\lim_{\varepsilon\downarrow 0} M_\varepsilon(\xi)=\frac{\sqrt{2\pi}}{\alpha}\mathrm{Re}([R](\xi+i0))\in L_1(\boldsymbol{R}_\xi), \tag{5.49}$$

we note that (h) in $\mathscr{L}$ is automatically included in (i) in $\mathscr{L}$. Finally, by Theorem 4.2 (iii) and (5.38), we have

$$\kappa(d\lambda)=\frac{R(0)}{(2\pi)^2}|\beta-i\lambda+\lim_{\varepsilon\downarrow 0}(-i\lambda)K_\varepsilon(\lambda)|^2\Delta'(\lambda)d\lambda. \tag{5.50}$$

Since $\kappa(\boldsymbol{R})<\infty$ and $\int_{\boldsymbol{R}}|\lambda|^2\Delta(d\lambda)<\infty$, we find from (5.50) that

$$(5.51)\qquad \int_{\boldsymbol{R}}|\lim_{\varepsilon\downarrow 0}(-i\lambda)K_{\varepsilon}(\lambda)|^2\Delta'(\lambda)d\lambda<\infty.$$

Therefore, by using (5.38) again, we can see from (5.47), (c) and (e)$'$ that (5.51) is equivalent to (j). Thus we have proved that $\Phi(R)=(\alpha,\beta,\varkappa)$ belongs to $\mathscr{L}_1$.

Conversely, let $(\alpha,\beta,\varkappa)$ be any element of $\mathscr{L}_1$ and put $R=\Phi^{-1}((\alpha,\beta,\varkappa))$. From the consideration above, Theorems 5.1 and 5.2 yield that R belongs to $\mathscr{R}_0$. Furhtermore, since (j) implies (5.51), we can see from (b)$'$ and (5.50) that $\int_{\boldsymbol{R}}|\lambda|^2\Delta'(\lambda)d\lambda<\infty$, which gives that R belongs to $\mathscr{R}_1$. (Q. E. D.)

§ 6. A Kubo noise

Let R be any fixed element of $\mathscr{R}_0$. In the sequel, we shall consider any stationary curve $A=(A(t)\,;\,t\in\boldsymbol{R})$ in a Hilbert space $\mathscr{H}$ such that

$$(6.1)\qquad (A(s),A(t))_{\mathscr{H}}=R(s-t)\qquad (s,t\in\boldsymbol{R})$$

and

$$(6.2)\qquad U(t)A(s)=A(t+s)\qquad (s,t\in\boldsymbol{R}),$$

where $(U(t)\,;\,t\in\boldsymbol{R})$ is a one-parameter group of unitary operators on $\mathscr{H}$.

By Stone's theorem, we have a resolution of identity $(E(\lambda)\,;\,\lambda\in\boldsymbol{R})$ such that

$$(6.3)\qquad U(t)=\int_{\boldsymbol{R}}e^{-it\lambda}dE(\lambda)\qquad (t\in\boldsymbol{R}).$$

Then it follows from (6.2) and (6.3) that

$$(6.4)\qquad A(t)=\int_{\boldsymbol{R}}e^{-it\lambda}dE(\lambda)A(0)\qquad (t\in\boldsymbol{R}).$$

Therefore, by defining a bounded Borel measure Δ_A on $\boldsymbol{R}$ by

$$(6.5)\qquad \Delta_A(d\lambda)=d(E(\lambda)A(0),A(0))_{\mathscr{H}},$$

we see that Δ_A is a spectral measure of R, that is,

$$(6.6)\qquad R(t)=\int_{\boldsymbol{R}}e^{-it\lambda}\,\Delta_A(d\lambda)\qquad (t\in\boldsymbol{R}).$$

Since $\|A(t)\|_{\mathscr{H}}\le\sqrt{R(0)}$ $(t\in\boldsymbol{R})$, we can define for each $\phi\in\mathscr{S}(\boldsymbol{R})$ an element $A(\phi)$ of $\mathscr{H}$ by

$$(6.7)\qquad A(\phi)=\int_{\boldsymbol{R}}\phi(t)A(t)dt.$$

By (6.4), we note that

$$(6.8)\qquad A(\phi)=\int_{\boldsymbol{R}}\hat{\phi}(\lambda)dE(\lambda)A(0)\quad(\phi\in\mathscr{S}(\boldsymbol{R})).$$

It is easy to see that

Proposition 6.1

(i) *There exists a positive constant* c_3 *such that*

$$\|A(\phi)\|_{\mathscr{H}}\leq c_3\sup_{\lambda\in\boldsymbol{R}}|\hat{\phi}(\lambda)|\quad(\phi\in\mathscr{S}(\boldsymbol{R})).$$

(ii) *For any* $\phi_j\in\mathscr{S}(\boldsymbol{R})$ *and* $a_j\in\boldsymbol{C}$ $(j=1,2)$

$$A(a_1\phi_1+a_2\phi_2)=a_1A(\phi_1)+a_2A(\phi_2).$$

(iii) *For any* $\phi,\psi\in\mathscr{S}(\boldsymbol{R})$

$$(A(\phi),A(\psi))_{\mathscr{H}}=\int_{\boldsymbol{R}}\hat{\phi}(\lambda)\overline{\hat{\psi}(\lambda)}\,\Delta_A(d\lambda).$$

By virtue of conditions (ii) and (iii) in $\mathscr{R}$, we can define for each element $\phi\in\mathscr{S}(\boldsymbol{R})$ an element $I(\phi)$ of $\mathscr{H}$ by

$$(6.9)\qquad I(\phi)=\frac{1}{\sqrt{2\pi}}\int_{\boldsymbol{R}}\hat{\phi}(\lambda)([R](\lambda+i0))^{-1}\,dE(\lambda)A(0).$$

Since $R\in\mathscr{R}_0$, we find that $\Delta_A(\Lambda_R)=0$ and so (6.9) is well-defined. It then is easy to see that

Proposition 6.2

(i) *There exist positive constants* c_4 *and* m_3 *such that*

$$\|I(\phi)\|_{\mathscr{H}}\leq c_4\sup_{\lambda\in\boldsymbol{R}}\{(1+|\lambda|^{m_3})|\hat{\phi}(\lambda)|\}\quad(\phi\in\mathscr{S}(\boldsymbol{R})).$$

(ii) *For any* $\phi_j\in\mathscr{S}(\boldsymbol{R})$ *and* $a_j\in\boldsymbol{C}(j=1,2)$,

$$I(a_1\phi_1+a_2\phi_2)=a_1I(\phi_1)+a_2I(\phi_2).$$

(iii) *For any* $\phi,\psi\in\mathscr{S}(\boldsymbol{R})$,

$$(I(\phi),I(\psi))_{\mathscr{H}}=\int_{\boldsymbol{R}}\hat{\phi}(\lambda)\overline{\hat{\psi}(\lambda)}\,\Delta_I(d\lambda).$$

Here Δ_I *is a Borel measure on* $\boldsymbol{R}$ *such that*

$$(6.10)\qquad \Delta_I(d\lambda)=\frac{1}{2\pi}|[R](\lambda+i0)|^{-2}\Delta_A(d\lambda).$$

Following [7], by Propositions 6.1 and 6.2, we can give

Definition 6.1 (i) We call $(A(\phi);\phi\in\mathscr{S}(\boldsymbol{R}))$ an $\mathscr{H}$-valued stationary tempered distribution corresponding to the stationary curve $A=(A(t);t\in\boldsymbol{R})$.

(ii) We call $I=(I(\phi)\,;\,\phi\in\mathscr{S}(\boldsymbol{R}))$ an $\mathscr{H}$-valued stationary tempered distribution associated with the stationary curve $A=(A(t)\,;\,t\in\boldsymbol{R})$ and then Δ_I its spectral measure.

Concerning a relation between two stationary tempered distributions above, we shall show

THEOREM 6.1 (i) *We suppose that there exist positive constants c_5 and m_4 such that*

$$(6.11)\qquad |[R](\zeta)|\leq c_5(1+|\zeta|^{m_4}) \qquad (\zeta\in\boldsymbol{C}^+).$$

Then we have

$$(6.12)\qquad A(\phi)=\frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\downarrow 0}\int_0^\infty e^{-\varepsilon t}R(t)I(\phi(\cdot+t))dt \qquad (\phi\in\mathscr{S}(\boldsymbol{R}))$$

(ii) *If $R\in L^1(\boldsymbol{R})$, then*

$$(6.13)\qquad A(\phi)=\frac{1}{\sqrt{2\pi}}\int_0^\infty R(t)I(\phi(\cdot+t))dt \qquad (\phi\in\mathscr{S}(\boldsymbol{R})).$$

PROOF Since

$$(6.14)\qquad [R](\cdot+i\varepsilon)=(\chi_{(0,\infty)}e^{-\varepsilon|\cdot|}R)^{\sim}\quad(\varepsilon>0),$$

it follows from (6.9) and (6.14) that for any $\varepsilon>0$

$$(6.15)\qquad \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\varepsilon t}R(t)I(\phi(\cdot+t))dt=\int_{\boldsymbol{R}}\hat{\phi}(\lambda)\frac{[R](\lambda+i\varepsilon)}{[R](\lambda+i0)}dE(\lambda)A(0).$$

By condition (iv) in $\mathscr{R}$ and further assumption (6.11), there exist positive constants c_6 and m_5 such that for any $\varepsilon>0$

$$(6.16)\qquad \left|\frac{[R](\lambda+i\varepsilon)}{[R](\lambda+i0)}\right|\leq c_6\,(1+|\lambda|^{m_5}) \qquad (\lambda\in\boldsymbol{R}).$$

Therefore, (6.12) holds from (6.15) and (6.16). Next we suppose that $R\in L^1$. Then, since

$$(6.17)\qquad [R](\cdot+i0)=(\chi_{(0,\infty)}R)^{\sim},$$

we see that (6.15) holds for $\varepsilon=0$, that is, (6.13) holds. (Q.E.D.)

REMARK 6.1 If $R\in L^1$, then we find that the spectral measure Δ of R is given by $2\,\mathrm{Re}([R](\lambda+i0))d\lambda$ and so R belongs to the set $\mathscr{R}_0$.

By Theorem 6.1, we find that there exists a linear black box such that stationary tempered distributions $(I(\phi)\,;\,\phi\in\mathscr{S}(\boldsymbol{R}))$, $(A(\phi)\,;\,\phi\in\mathscr{S}(\boldsymbol{R}))$ and a function $\chi_{(0,\infty)}R$ can be regarded as an input, an output and a response function, respectively. In Kubo's linear response theory ([8],[9],

[10] and [11]), it seems to the author that it is fundamental to construct such a linear black box. For that reason, we shall give

DEFINITION 6.2 The stationary tempered distribution $I=(I(\phi)\,;\,\phi\in\mathscr{S}(\boldsymbol{R}))$ is said to be a Kubo noise associated with the stationary curve $A=(A(t)\,;\,t\in\boldsymbol{R})$.

Next we shall represent the spectral measure Δ_I of the Kubo noise I in terms of the second KMO-Langevin data $(\alpha,\beta,\kappa)(=\Phi(R))$. By Theorem 5.1 and Proposition 6.2 (iii), we have

THEOREM 6.2

$$(6.18)\qquad \Delta_I(d\xi)=\left|\frac{1}{\alpha}(\beta-i\xi+\lim_{\varepsilon\downarrow 0}(-i\xi)K_\varepsilon(\xi))\right|^2\Delta_A(d\xi).$$

Here

$$(6.19)\qquad K_\varepsilon(\xi)=\int_{\boldsymbol{R}}\frac{1}{(\lambda-\xi-i\varepsilon)(\lambda-i\varepsilon)}\kappa(d\lambda)\qquad (\xi\in\boldsymbol{R}).$$

Furthermore we shall show

THEOREM 6.3

(i) $\Delta_I(d\xi)=\sqrt{\dfrac{2}{\pi}}\dfrac{1}{\alpha}\mathrm{Re}(\beta-i\xi+\lim\limits_{\varepsilon\downarrow 0}(-i\xi)K_\varepsilon(\xi))d\xi$

(ii) $\Delta_I(d\xi)=\dfrac{\sqrt{2\pi}}{\alpha}(P\kappa)(\xi)d\xi.$

Here

$$(6.20)\qquad (P\kappa)(\xi)=\lim_{\eta\downarrow 0}\int_{\boldsymbol{R}}\frac{1}{\pi}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\kappa(d\lambda)\qquad (\xi\in\boldsymbol{R}-\Lambda_R).$$

PROOF By (5.38) and (6.10), we see that

$$(6.21)\qquad \Delta_I(d\xi)=\frac{1}{\pi}\mathrm{Re}(([R](\xi+i0))^{-1})d\xi$$

and so (i) follows from Theorem 5.1 and (6.21). By noting (5.8) and (5.9), we find that (ii) follows from (i). (Q.E.D.)

§ 7. A second KMO-Langevin equation

Under the same situation as § 6, we shall derive an equation describing the time evolution of $A=(A(t)\,;\,t\in\boldsymbol{R})$.

As a refinement of Lemma 4.21, we shall show

LEMMA 7.1 *For each* $\varepsilon>0$

(i) $K_\varepsilon \in C^\infty(\boldsymbol{R})$

(ii) $|\frac{\partial^m}{\partial\xi^m}K_\varepsilon(\xi)|\le c_{m,\varepsilon}(1+|\xi|)$ $(m\in\boldsymbol{N}^*=\boldsymbol{N}\cup\{0\},\ \xi\in\boldsymbol{R})$,

where $c_{m,\varepsilon}=m!\frac{(\varepsilon+1)}{\varepsilon^{m+1}}\int_{\boldsymbol{R}}\frac{1}{\lambda^2+\varepsilon^2}\varkappa(d\lambda)$.

PROOF Since

$$(7.1)\qquad |\frac{1}{(\lambda-\xi-i\varepsilon)(\lambda-i\varepsilon)}|=|\frac{\lambda-i\varepsilon}{\lambda-\xi-i\varepsilon}|\frac{1}{\lambda^2+\varepsilon^2}\le$$

$$\le\frac{|\lambda-\xi-i\varepsilon|+|\xi|}{|\lambda-\xi-i\varepsilon|}\frac{1}{\lambda^2+\varepsilon^2}\le(1+\frac{|\xi|}{\varepsilon})\frac{1}{\lambda^2+\varepsilon^2},$$

the estimate (ii) for $m=0$ holds. Let m be any element of $\boldsymbol{N}$. Then we note that

$$\frac{\partial^m}{\partial\xi^m}\left(\frac{1}{(\lambda-\xi-i\varepsilon)(\lambda-i\varepsilon)}\right)=\frac{m!}{(\lambda-\xi-i\varepsilon)^{m+1}(\lambda-i\varepsilon)}$$

and so similarly as (7.1)

$$|\frac{\partial^m}{\partial\xi^m}\left(\frac{1}{(\lambda-\xi-i\varepsilon)(\lambda-i\varepsilon)}\right)|\le m!\left(\frac{1}{\varepsilon^m}+\frac{|\xi|}{\varepsilon^{m+1}}\right)\frac{1}{\lambda^2+\varepsilon^2}.$$

Therefore, we can exchange the order of integration and differentiation to obtain Lemma 7.1. (Q. E. D.)

By virtue of Lemma 7.1, we can define for each $\varepsilon>0$ an $\mathscr{H}$-valued tempered distribution $\widehat{K_\varepsilon\tilde{A}}$ by

$$(7.2)\qquad \widehat{K_\varepsilon\tilde{A}}(\phi)\equiv(K_\varepsilon\tilde{A})(\hat{\phi})\equiv\tilde{A}(K_\varepsilon\hat{\phi}).$$

By (6.8) and (7.2), we have

LEMMA 7.2 *For each* $\varepsilon>0$ *and any* $\phi\in\mathscr{S}(\boldsymbol{R})$

$$\widehat{K_\varepsilon\tilde{A}}(\phi)=\int_{\boldsymbol{R}}K_\varepsilon(\xi)(-i\xi)\hat{\phi}(\xi)dE(\xi)A(0).$$

On the other hand, by using Lemma 7.1 again, we can define for each $\varepsilon>0$ a tempered distribution $\gamma_\varepsilon\in\mathscr{S}'(\boldsymbol{R})$ by

$$(7.3)\qquad \gamma_\varepsilon=\frac{1}{2\pi}\hat{K}_\varepsilon.$$

We note that for each $\varepsilon>0$, any $\phi\in\mathscr{S}(\boldsymbol{R})$ and any $t\in\boldsymbol{R}$

$$(7.4)\qquad (\gamma_\varepsilon * \check{\phi})(t)\equiv\gamma_\varepsilon(\check{\phi}(t-\bullet)) = \frac{1}{2\pi}(K_\varepsilon\hat{\phi})\hat{\ }(t).$$

Since we see from Lemma 7.1 that $K_\varepsilon\hat{\phi}\in\mathscr{S}(\boldsymbol{R})$, it follows from (7.4) that for each $\varepsilon>0$ and any $\phi\in\mathscr{S}(\boldsymbol{R})$

$$(7.5)\qquad \gamma_\varepsilon * \check{\phi}\in\mathscr{S}(\boldsymbol{R}).$$

Therefore we can define for each $\varepsilon>0$ an $\mathscr{H}$-valued tempered distribution $\dot{A}*\gamma_\varepsilon$ by

$$(7.6)\qquad (\dot{A}*\gamma_\varepsilon)(\phi)\equiv(\dot{A}*(\gamma_\varepsilon*\check{\phi}))(0) = \dot{A}((\gamma_\varepsilon*\check{\phi})\check{\ }).$$

By (6.8), (7.4) and (7.6), we have

LEMMA 7.3 *For each $\varepsilon>0$ and any $\phi\in\mathscr{S}(\boldsymbol{R})$*

$$(\dot{A}*\gamma_\varepsilon)(\phi)=\int_{\boldsymbol{R}}K_\varepsilon(\xi)(-i\xi)\hat{\phi}(\xi)dE(\xi)A(0)=(A*\dot{\gamma}_\varepsilon)(\phi).$$

By taking account of Lemma 7.2, (7.3) and Lemma 7.3, it is reasonable to give

DEFINITION 7.1 We define for each $\varepsilon>0$ a stationary tempered distribution $\dot{\gamma}_\varepsilon*A$ by

$$(7.7)\qquad \dot{\gamma}_\varepsilon*A\equiv\widehat{K_\varepsilon\tilde{\dot{A}}}=\dot{A}*\gamma_\varepsilon=A*\dot{\gamma}_\varepsilon.$$

Now we shall show

THEOREM 7.1 *As $\mathscr{H}$-valued tempered distributions,*

$$(7.8)\qquad \dot{A}=-\beta A-\lim_{\varepsilon\downarrow 0}\dot{\gamma}_\varepsilon*A+\alpha I.$$

PROOF Let ϕ be any fixed element of $\mathscr{S}(\boldsymbol{R})$. By (6.8), we have

$$(7.9)\qquad \dot{A}(\phi)=\int_{\boldsymbol{R}}(-i\xi)\hat{\phi}(\xi)dE(\xi)A(0) = \int_{\boldsymbol{R}}(-i\xi)[R](\xi+i0)\hat{\phi}(\xi)([R](\xi+i0))^{-1}dE(\xi)A(0).$$

Furthermore it follows from (5.5) in Theorem 5.1 that for any $\xi\in\boldsymbol{R}-\Lambda_R$

$$(7.10)\qquad (-i\xi)[R](\xi+i0)=\frac{\alpha}{\sqrt{2\pi}}-\beta[\mathrm{R}](\xi+i0) -\lim_{\varepsilon\downarrow 0}((-i\xi)K_\varepsilon(\xi))[R](\xi+i0).$$

By substituting (7.10) into (7.9) and then noting (6.8) and (6.9), we have

$$(7.11)\qquad \dot{A}(\phi)=\frac{\alpha}{\sqrt{2\pi}}I(\phi)-\beta A(\phi)-\int_{\boldsymbol{R}}\lim_{\varepsilon\downarrow 0}(-i\xi)K_\varepsilon(\xi))\hat{\phi}(\xi)dE(\xi)A(0).$$

On the other hand, by condition (d) in $\mathscr{L}$, we can apply Lebesgue's convergence theorem to obtain

$$(7.12)\qquad \int_{\boldsymbol{R}}\lim_{\varepsilon\downarrow 0}(-i\xi)K_\varepsilon(\xi))\hat{\phi}(\xi)dE(\xi)A(0) =s-\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}(-i\xi)K_\varepsilon(\xi)\hat{\phi}(\xi)dE(\xi)A(0).$$

Therefore, we see from Lemma 7.2, (7.7), (7.11) and (7.12) that Theorem 7.11 holds. (Q.E.D.)

We call equation (7.8) a **second KMO-Langevin equation.**

§ 8. Fluctuation-Dissipation Theorem

According to the so-called fluctuation-dissipation theorem in statistical physics, we know ([6]) that in a physical linear system taking a reciprocal action with a microscopic and kinetic quantity which is itself doing a thermal motion, there exists a relation between the system function and the spectral density of the physical system.

Let R be any given element of $\mathscr{R}_0$ and then $A=(A(t)\,;\,t\in\boldsymbol{R})$ be any stationary curve in a Hilbert space $\mathscr{H}$ such that

$$(8.1)\qquad (A(s),\ A(t))_{\mathscr{H}}=R(s-t)\qquad (s,\ t\in\boldsymbol{R}).$$

We regard the stationary curve A as a **physical system** stated above.

Let $I=(I(\phi)\,;\,\phi\in\mathscr{S}(\boldsymbol{R}))$ be the Kubo noise associated with A defined by (6.9). We have proved in Theorem 6.1 that if $R\in L^1(\boldsymbol{R})$, then

$$(8.2)\qquad A(\phi)=\frac{1}{\sqrt{2\pi}}\int_0^\infty R(t)I(\phi(\cdot+t))dt\qquad (\phi\in\mathscr{S}(\boldsymbol{R})).$$

Therefore we find that this linear relation (8.2) gives a **physical linear system** where I and A can be regarded as an **input** and an **output** of the system with a function $\chi_{(0,\infty)}R$ as its **response function**, respectively. We note that the term $\chi_{(0,\infty)}$ in the response function $\chi_{(0,\infty)}R$ comes from a causality.

Furthermore we have proved in Theorem 7.1 that the time evolution of A is governed by the following **second KMO-Langevin equation**:

$$(8.3) \qquad \dot{A} = -\beta A - \lim_{\varepsilon \downarrow 0} \dot{\gamma}_\varepsilon * A + \alpha I.$$

Here the triple (α, β, κ) is the **second KMO-Langevin data** associated with R and then $\gamma_\varepsilon = \frac{1}{2\pi}\hat{K}_\varepsilon (\in \mathscr{S}(\boldsymbol{R})')$ and $K_\varepsilon(\xi) = \int_{\boldsymbol{R}} \frac{1}{(\lambda - \xi - i\varepsilon)(\lambda - i\varepsilon)} \kappa(d\lambda)$. When we regard a first term with a second term and then a third term in the right-hand side of the second KMO-Langevin equation (8.3) as a systematic part and a fluctuating part, respectively, we see that a **complex mobility** (resp. a complex admittance in general) μ of velocity (resp. current in general) in a stationary state described by the second KMO-Langevin equation (8.3) is given by

$$(8.4) \qquad \mu(\zeta) = \frac{1}{\beta - i\zeta + \lim_{\varepsilon \downarrow 0}(-i\zeta)K_\varepsilon(\zeta)} \qquad (\zeta \in \boldsymbol{C}^+).$$

It then follows from (3.3), Theorems 4.1 (i) and 5.1 that

THEOREM 8.1

$$(8.5) \qquad \mu(\zeta) = \frac{1}{R(0)} \int_0^\infty e^{i\zeta t} R(t)\, dt \qquad (\zeta \in \boldsymbol{C}^+).$$

This is the **first fluctuation-dissipation theorem** in Kubo's linear response theory ([8], [9], [10] and [11]). Furthermore it follows from (3.7) and Theorem 4.1 (ii) that

THEOREM 8.2

$$(8.6) \qquad D = \frac{R(0)}{\beta},$$

where

$$(8.7) \qquad D = \lim_{\eta \downarrow 0} \int_0^\infty e^{-\eta t} R(t)\, dt.$$

This is the **Einstein relation** which implies that the **fluctuation power** D of A is in a reciprocal proportion to the first **complex friction coefficient** β in the second KMO-Langevin equation (8.3).

REMARK 8.1 The Einstein relation (8.6) follows immediately from the first fluctuation-dissipation theorem (8.5), since

(8.8) $\beta=\lim_{\eta\downarrow 0}\mu(i\eta)$.

REMARK 8.2 We note that such an Einstein relation as (8.6) does not hold for Mori's memory kernel equation (2.13).

Furthermore, as a relation between the complex mobility μ and the spectral density Δ'_A of R, we see from (8.5)(c.f. (5.40)) that

THEOREM 8.3

(8.9) $$\mathrm{Re}(\mu(\xi+i0))=\frac{\pi}{R(0)}\Delta'_A(\xi) \qquad (a.\ e.\ \xi\in\boldsymbol{R}).$$

This is equivalent to the first fluctuation-dissipation theorem (8.5) as was proved in Theorem 5.1.

Next we define a stationary tempered distribution W by

(8.10) $W=\alpha I$,

which is regarded as a fluctuation part in the second KMO-Langevin equation (8.3). Corresponding to formula (8.9), we see from Theorems 4.1 (i), 6.3 (i), (8.4) and (8.10) that

THEOREM 8.4

(8.11) $$\mathrm{Re}\left(\frac{1}{\mu(\xi+i0)}\right)=\frac{\pi}{R(0)}\Delta'_W(\xi) \qquad (a.\ e.\ \xi\in\boldsymbol{R}),$$

where Δ'_W *is the spectral density of* W.

REMARK 8.3 It follows from Theorem 6.3 (ii) that

(8.12) $$\mathrm{Re}\left(\frac{1}{\mu(\xi+i0)}\right)=\lim_{\eta\downarrow 0}\int_{\boldsymbol{R}}\frac{\eta}{(\lambda-\xi)^2+\eta^2}\kappa(d\lambda).$$

This formula (8.11) is the **second fluctuation-dissipation theorem** in Kubo's linear response theory ([8], [9], [10] and [11]).

§ 9. Examples

In this final section, we shall consider two examples: one is Ornstein-Uhlenbeck's Brownian motion and the other is Mori's generalized Brownian motion.

EXAMPLE 9.1 Let $X=(X(t)\,;\,t\in\boldsymbol{R})$ be an Ornstein-Uhlenbeck's Brownian motion governed by the following $[\alpha_0,\beta_0,0]$-Langevin equation

(9.1) $$dX(t)=-\beta_0X(t)\,dt+\alpha_0dB(t) \qquad (t\in\boldsymbol{R}).$$

Here $\alpha_0>0$, $\beta_0>0$ and $(B(t)\,;\,t\in\boldsymbol{R})$ is a one-dimensional Brownian motion.

Then we find that the covariance function $R\equiv R_{\alpha_0,\beta_0}$ is given by

$$(9.2)\qquad R(t)=\frac{\alpha_0^2}{2\beta_0}e^{-\beta_0|t|}\qquad\qquad (t\in\boldsymbol{R}).$$

In particular we have

$$(9.3)\qquad R(0)=\frac{\alpha_0^2}{2\beta_0}.$$

Furthermore we see from (9.2) and (9.3) that

$$(9.4)\qquad [R](\zeta)=\frac{R(0)}{2\pi}\frac{1}{\beta_0-i\zeta}\qquad\qquad (\zeta\in\boldsymbol{C}^+)$$

and so R belongs to $\mathscr{R}_0$.

Let $(\alpha,\beta,\varkappa)$ be the second KMO-Langevin data associated with R. Then it follows from Theorem 4.2, (9.2) and (9.3) that

$$(9.5)\qquad \alpha=\frac{\alpha_0^2}{2\sqrt{2\pi}\beta_0}\left(=\frac{R(0)}{\sqrt{2\pi}}\right)$$

$$(9.6)\qquad \beta=\beta_0$$

and

$$(9.7)\qquad \varkappa(d\xi)=\frac{\beta_0}{\pi}d\xi.$$

By a residue theorem and (9.7), we see that for any $\varepsilon>0$ and any $\zeta\in\boldsymbol{C}^+$ with $0<\operatorname{Im}\zeta<\varepsilon$

$$\begin{aligned}(9.8)\qquad K_\varepsilon(\zeta)&=2\beta_0 i\frac{1}{2\pi i}\int_{\boldsymbol{R}}\frac{1}{z-\zeta-i\varepsilon}\,\frac{1}{z-i\varepsilon}\,dz\\ &=2\beta_0 i\left(\frac{1}{(\zeta+i\varepsilon)-i\varepsilon}+\frac{1}{(i\varepsilon)-\zeta-i\varepsilon}\right)\\ &=0\end{aligned}$$

and so

$$(9.9)\qquad K_\varepsilon(\xi)=0 \text{ for any } \varepsilon>0 \text{ and any } \xi\in\boldsymbol{R}.$$

In particular, by (7.3) and (9.9), we have

$$(9.10)\qquad \gamma_\varepsilon=0 \text{ for any } \varepsilon>0.$$

Next we shall calculate the Kubo noise I. By (6.8), (6.9), (9.4) and (9.5), we see that for any $\phi\in\mathscr{S}(\boldsymbol{R})$

(9.11)
$$\begin{aligned} & I(\phi) \\ &= \frac{\sqrt{2\pi}}{R(0)}\left(\beta_0 \int_{\boldsymbol{R}} \hat{\phi}(\xi) dE(\xi) A(0) - \int_{\boldsymbol{R}} \hat{\phi}(\xi)(i\xi) dE(\xi) A(0)\right) \\ &= \frac{\sqrt{2\pi}}{R(0)}(\beta_0 A(\phi) + A(\dot{\phi})) \\ &= \frac{\sqrt{2\pi}}{R(0)} \alpha_0 \dot{B}(\phi) \end{aligned}$$

and so by (9.5)

(9.12) $$\alpha I(\phi) = \alpha_0 \dot{B}(\phi).$$

Therefore, we find that the second KMO-Langevin equation (8.3) for X is equal to $[\alpha_0, \beta_0, 0]$-Langevin equation (9.1).

Example 9.2 Let us consider a stationary curve $A = (A(t)\,;\, t \in \boldsymbol{R})$ in a Hilbert space $\mathscr{H}$ treated in § 2 and § 3. Here we suppose that the covariance function R of A belongs to $\mathscr{R}_1$. Then, by Theorems 2.2 and 7.1, we have two kinds of equations describing the time evolution of A:

Mori's memory kernel equation

(9.13) $$\dot{A}(t) = i\omega A(t) - \int_0^t \phi_M(t-s) A(s) ds + I_M(t) \quad (t \in \boldsymbol{R})$$

and

A second KMO-Langevin equation

(9.14) $$\dot{A}(\phi) = -\beta A(\phi) - \lim_{\varepsilon \downarrow 0} (\gamma_\varepsilon * A)(\phi) + \alpha I_K(\phi) \quad (\phi \in \mathscr{S}(\boldsymbol{R})).$$

Here ω, ϕ_M and I_M are the frequency, memory kernel and Mori noise, respectively, and then (α, β, κ) is the second KMO-Langevin data associated with R and I_K is the Kubo noise.

Proposition 9.1

(i) $\omega = -(\mathrm{Im}\,\beta + \pi(H\kappa)(0))$

(ii) $\phi_M(t) = \int_{\boldsymbol{R}} e^{-it\lambda} \kappa(d\lambda) \qquad (t \in \boldsymbol{R})$

(iii) $\chi_{(0,\infty)}(t) \int_t^\infty e^{-\varepsilon s} \phi_M(s) ds = -\gamma_\varepsilon(t) \quad (t \in \boldsymbol{R})$

(iv) $\kappa(d\lambda) = (P\kappa)(\lambda) d\lambda.$

Proof By Lemma 3.3 and (3.10), we have

(9.15) $$-i\omega + 2\pi[\phi_M](0 + i0) = \beta.$$

On the other hand, it follows from (3.4), (c) and (e)' in $\mathscr{L}_1$ that

$$(9.16)\qquad 2\pi[\phi_M](0+i0)=\pi((P\kappa)(0)-i(H\kappa)(0))$$
$$=\mathrm{Re}\,\beta-i\pi(H\kappa)(0).$$

Therefore, (i) follows from (9.15) and (9.16) (ii) and (iii) are the very (3.2) and (3.8), respectively. (iv) follows from Lemma 4.26 and (e)′ in $\mathscr{L}_1$. (Q. E. D.)

Next we shall show

PROPOSITION 9.2

(i) $\Delta_{I_M}(d\lambda)=R(0)\kappa(d\lambda)$

(ii) $\Delta_{I_K}(d\lambda)=\dfrac{2\pi}{R(0)}\kappa(d\lambda),$

where Δ_{I_M} and Δ_{I_K} are spectral measures of the Mori noise I_M and the Kubo noise I_K, respectively.

PROOF (i) follows from (2.10) and Proposition 9.1 (ii). (ii) follows from Theorems 4.2 (i) and 6.2 (ii) (Q. E. D.)

Finally we shall give a relation between the Mori noise I_M and the Kubo noise I_K. For that purpose, we regard the Mori noise I_M as an $\mathscr{H}$-valued stationary tempered distribution $(I_M(\phi)\,;\ \phi\in\mathscr{S}(\boldsymbol{R}))$:

$$(9.17)\qquad I_M(\phi)=\int_{\boldsymbol{R}}\phi(t)I_M(t)\,dt.$$

Then we shall show

PROPOSITION 9.3 *For any $\phi\in\mathscr{S}(\boldsymbol{R})$*

$$(9.18)\qquad \alpha I_K(\phi)-I_M(\phi)$$
$$=\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{1}{i}\Big(\int_{\boldsymbol{R}}\frac{\hat{\phi}(\lambda)}{\lambda-\xi-i\varepsilon}\kappa(d\lambda)\Big)dE(\xi)A(0)$$
$$=\frac{1}{i}\int_{\boldsymbol{R}}\Big(\int_{\boldsymbol{R}}\frac{\hat{\phi}(\lambda)-\hat{\phi}(\xi)}{\lambda-\xi}\kappa(d\lambda)\Big)dE(\xi)A(0)$$
$$+\pi\int_{\boldsymbol{R}}((P\kappa)(\xi)-(H\kappa)(\xi))\hat{\phi}(\xi)\,dE(\xi)A(0),$$

where $(E(\xi)\,;\ \xi\in\boldsymbol{R})$ is the spectral resolution of A in (6.4).

PROOF By (6.4), (9.13) and Lemma 9.1 (ii), we see that

$$(9.19)\qquad I_M(t)=\int_{\boldsymbol{R}}\Big\{(-i\xi-i\omega)e^{-it\xi}+\int_{\boldsymbol{R}}\frac{1-e^{i(\xi-\lambda)t}}{i(\lambda-\xi)}\kappa(d\lambda)\Big\}dE(\xi)A(0)$$

and so

$$(9.20)\qquad I_M(\phi)$$

$$= \int_{\boldsymbol{R}} (-i\xi - i\omega)\hat{\phi}(\xi) dE(\xi)A(0) +$$

$$+ \int_{\boldsymbol{R}} \Big(\int_{\boldsymbol{R}} e^{-it\xi} \Big(\int_{\boldsymbol{R}} \frac{1-e^{i(\xi-\lambda)t}}{i(\lambda-\xi)} \varkappa(d\lambda)\Big)\phi(t)dt\Big) dE(\xi)A(0).$$

On the other hand, by (7.7) and (9.14), we have

$$(9.21) \qquad \alpha I_K(\phi) = \int_{\boldsymbol{R}} \{\beta - i\xi + \lim_{\varepsilon \downarrow 0}(-i\xi)K_\varepsilon(\xi)\}\hat{\phi}(\xi)dE(\xi)A(0).$$

As we have seen in the proof of Theorem 5.3, we note that

$$(9.22) \qquad \lim_{\varepsilon \downarrow 0}(-i\xi)K_\varepsilon(\xi) = \pi((P\varkappa)(\xi) - (P\varkappa)(0)) - i\pi((H\varkappa)(\xi) - (H\varkappa)(0))$$
$$= \pi((P\varkappa)(\xi) - i(H\varkappa)(\xi)) - \mathrm{Re}\,\beta + i\pi(H\varkappa)(0)$$

and so by Lemma 9.1 (i)

$$(9.23) \qquad \beta - i\xi + \lim_{\varepsilon \downarrow 0}(-i\xi)K_\varepsilon(\xi)$$
$$= -i\xi - i\omega + \pi((P\varkappa)(\xi) - i(H\varkappa)(\xi)).$$

Therefore, it follows from (9.20), (9.21) and (9.23) that

$$(9.24) \qquad \alpha I_K(\phi) - I_M(\phi)$$
$$= \int_{\boldsymbol{R}} \{\pi((P\varkappa)(\xi) - i(H\varkappa)(\xi))\hat{\phi}(\xi)$$
$$- \int_{\boldsymbol{R}} \Big(\int_{\boldsymbol{R}} e^{-it\xi} \Big(\int_{\boldsymbol{R}} \frac{1-e^{i(\xi-\lambda)t}}{i(\lambda-\xi)} \varkappa(d\lambda)\Big)\phi(t)dt\Big)\} dE(\xi)A(0).$$

By noting (b)′ and (e)′ in $\mathscr{L}_1$, we see that for almost all $\xi \in \boldsymbol{R}$ and any $t \in \boldsymbol{R}$

$$(9.25) \qquad \int_{\boldsymbol{R}} \frac{1-e^{i(\xi-\lambda)t}}{i(\lambda-\xi)} \varkappa(d\lambda) = \frac{1}{i}\lim_{\varepsilon \downarrow 0}\int_{\boldsymbol{R}} \frac{1-e^{i(\xi-\lambda)t}}{\lambda-\xi-i\varepsilon}\varkappa(d\lambda)$$
$$= \frac{1}{i}\lim_{\varepsilon \downarrow 0}\int_{\boldsymbol{R}} \frac{1}{\lambda-\xi-i\varepsilon}\varkappa(d\lambda) - \frac{1}{i}\lim_{\varepsilon \downarrow 0}\int_{\boldsymbol{R}} \frac{e^{i(\xi-\lambda)t}}{\lambda-\xi-i\varepsilon}\varkappa(d\lambda)$$
$$= \pi((P\varkappa)(\xi) - i(H\varkappa)(\xi)) - \frac{1}{i}\lim_{\varepsilon \downarrow 0}\int_{\boldsymbol{R}} \frac{e^{i(\xi-\lambda)t}}{\lambda-\xi-i\varepsilon}\varkappa(d\lambda).$$

Substituting (9.25) into (9.24), we have

$$(9.26) \qquad \alpha I_K(\phi) - I_M(\phi)$$
$$= \frac{1}{i}\int_{\boldsymbol{R}} \Big(\int_{\boldsymbol{R}} e^{-it\xi} \Big(\lim_{\varepsilon \downarrow 0}\int_{\boldsymbol{R}} \frac{e^{i(\xi-\lambda)t}}{\lambda-\xi-i\varepsilon}\varkappa(d\lambda)\Big)\phi(t)dt\Big) dE(\xi)A(0).$$

By noting that $\left| \int_{\boldsymbol{R}} \frac{1-e^{i(\xi-\lambda)t}}{\lambda-\xi-i\varepsilon}\varkappa(d\lambda)\right| \leq |t|\varkappa(\boldsymbol{R})$, we see from (d) in $\mathscr{L}_1$ and

(9.25) that there exist positive constants c_7 and m_6 such that for any $\varepsilon>0$ and any $\xi\in\boldsymbol{R}$

$$(9.27)\qquad \left|\int_{\boldsymbol{R}}\frac{e^{i(\xi-\lambda)t}}{\lambda-\xi-i\varepsilon}\varkappa(d\lambda)\right|\leq c_7(1+|t|+|\xi|^{m_6}).$$

Therefore, by Lebesgue's convergence theorem, we see that for any $\xi\in\boldsymbol{R}$

$$(9.28)\qquad \begin{aligned}&\int_{\boldsymbol{R}}e^{-it\xi}\Big(\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{e^{i(\xi-\lambda)t}}{\lambda-\xi-i\varepsilon}\varkappa(d\lambda)\Big)\phi(t)\,dt\\ &=\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}e^{-it\xi}\Big(\int_{\boldsymbol{R}}\frac{e^{i(\xi-\lambda)t}}{\lambda-\xi-i\varepsilon}\varkappa(d\lambda)\Big)\phi(t)\,dt\\ &=\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{\hat{\phi}(\lambda)}{\lambda-\xi-i\varepsilon}\varkappa(d\lambda).\end{aligned}$$

Furthermore, we divide

$$(9.29)\qquad \int_{\boldsymbol{R}}\frac{\hat{\phi}(\lambda)}{\lambda-\xi-i\varepsilon}\varkappa(d\lambda)=\int_{\boldsymbol{R}}\frac{\hat{\phi}(\lambda)-\hat{\phi}(\xi)}{\lambda-\xi-i\varepsilon}\varkappa(d\lambda)+\hat{\phi}(\xi)\int_{\boldsymbol{R}}\frac{1}{\lambda-\xi-i\varepsilon}\varkappa(d\lambda).$$

By using (9.27) for $t=0$ and (e)$'$ in $\mathscr{L}_1$, we find that

$$(9.30)\qquad \left|\int_{\boldsymbol{R}}\frac{\hat{\phi}(\lambda)}{\lambda-\xi-i\varepsilon}\varkappa(d\lambda)\right|\leq\|\hat{\phi}'\|_\infty\varkappa(\boldsymbol{R})+c_7(1+|\xi|^{m_6})|\hat{\phi}(\xi)|$$

$$(9.31)\qquad \lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}}\frac{\hat{\phi}(\lambda)}{\lambda-\xi-i\varepsilon}\varkappa(d\lambda)=\int_{\boldsymbol{R}}\frac{\hat{\phi}(\lambda)-\hat{\phi}(\xi)}{\lambda-\xi}\varkappa(d\lambda)+\pi((H\varkappa)(\xi)+i(P\varkappa)(\xi))\hat{\phi}(\xi).$$

Thus, by (9.30) and (9.31), we can apply Lebesgue's convergence theorem to (9.26) and (9.28) to obtain (9.18). (Q. E. D.)

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