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Ergodic H^1 Is Not A Dual Space

Raul E. CURTO*, Paul S. MUHLY* and Takahiko NAKAZI†
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§1. Introduction

Suppose that X is a standard Borel measure space with probability measure m and that $\{T_t\}_{t\in\boldsymbol{R}}$ is an ergodic, measurable action of $\boldsymbol{R}$ on X preserving m. Via composition, $\{T_t\}_{t\in\boldsymbol{R}}$ acts on functions over X. $(T_t f)(x)=f(T_t x)$, and when restricted to $L^p(X)$, $p<\infty$, $\{T_t\}_{t\in\boldsymbol{R}}$ is strongly continuous; on $L^\infty(X)$, $\{T_t\}_{t\in\boldsymbol{R}}$ is only weak-* continuous. Ergodic H^∞, $H^\infty(X)$, is defined to be the subspace of $f\in L^\infty(X)$ such that, for almost all x, the function of t, $f(T_t x)$, lies in $H^\infty(\boldsymbol{R})$; i.e. this function admits an extension to a bounded analytic function in the upper half-plane. For p in the range $0<p<\infty$, ergodic H^p, $H^p(X)$, is defined to be the closure of $H^\infty(X)$ in $L^p(X)$. As is shown in [10], when $1\leq p$, $H^p(X)$ is the subspace of all $f\in L^p(X)$ such that, with the exception of a null set of x, the function of t, $f(T_t x)$, when divided by $t+i$ lies in the usual Hardy space $H^p(\boldsymbol{R})$ associated with the upper half-plane. Our objective is to prove the theorem that is our title.

THEOREM. *If $\{T_t\}_{t\in\boldsymbol{R}}$ is not periodic, then $H^1(X)$ is not a dual space.*

This result was conjectured by the second author in [11] and it was noted there that the theorem is true if $\{T_t\}_{t\in\boldsymbol{R}}$ has pure point spectrum. In this case X is a quotient of the Bohr groups and harmonic analysis on X is the key to the proof (see [8]). In our more general setting this tool is not at our disposal. Of course. if $\{T_t\}_{t\in\boldsymbol{R}}$ is periodic, then $H^1(X)$ is (isometrically isomorphic to) the classical Hardy space, $H^1(\boldsymbol{T})$, on the circle $\boldsymbol{T}$, and this space is well-known to be a dual space by the F. and M. Riesz theorem. Namely, $H^1(\boldsymbol{T})$ is the dual of the quotient space $C(\boldsymbol{T})/A_0(\boldsymbol{T})$, where $A_0(\boldsymbol{T})$ is the space of (boundary values of) functions which are continuous on the closed unit disc, analytic on the interior, and vanish at the origin. The proof is simple. The dual of $C(\boldsymbol{T})$ is the space of measures on

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$\boldsymbol{T}$ and the F. and M. Riesz theorem asserts that a measure μ on $\boldsymbol{T}$ annihilates $A_0(\boldsymbol{T})$ if and only if it is absolutely continuous and its derivative lies in $H^1(\boldsymbol{T})$. In [6], Forelli proved a generalization of the F. and M. Riesz theorem to the context of flows. To state his theorem, suppose that X is a locally compact Hausdorff space and that $\{T_t\}_{t\in\boldsymbol{R}}$ acts continuously on X. Define $A(X)$ to be those functions $f\in C_0(X)$ such that for each $x\in X$, the function of t, $f(T_tx)$, lies in $H^\infty(\boldsymbol{R})$. Forelli showed that each Borel measure μ on X that annihilates $A(X)$ must be quasi-invariant in the sense that the class of null sets of $|\mu|$ is invariant under $\{T_t\}_{t\in\boldsymbol{R}}$. If m is an invariant ergodic probability measure on X, as we are assuming, and if $f\in H^1(X)$ (with $\int f dm=0$). then the measure fdm annihilates $A(X)$. On the other hand, it is easy to construct measures on X that annihilate $A(X)$ and are singular with respect to m (assuming $X\neq\boldsymbol{T}$). Thus Forelli's theorem does not provide a complete generalization of the F. and M. Riesz theorem. Even though the annhilator of $A(X)$ is quite a bit bigger than $H^1(X)$, one can still ask if a qualitative vestige of the F. and M. Riesz theorem remains : Is $H^1(X)$ a dual space? Our theorem shows that even this fails.

We assume that our measure space is a standard Borel space for two reasons. First, it is essential that $L^1(X)$ be separable—as will be apparent shortly. Second, we require techniques from ergodic theory that are true only provisionally in nonstandard spaces. It would be interesting to know if our theorem is true in the nonseparable context.

§ 2., The Proof

The proof of our theorem makes essential use of the techniques of[11] and the following lemma.

LEMMA. *Let E be a seqarable Banach space that is the dual of some other Banach space. Then there is no Banach space isomorphism from $L^1([0,1])$ onto a closed subspace of E.*

This result is due to Gelfand [7], originally ; for more recent proofs see [2], [5], and [12]. Note that since two non-atomic, standard Borel, probability measure spaces are measure theoretically isomorphic, their corresponding L^1-spaces are isometrically isomorphic. Therefore, in the lemma, we may replace $L^1([0,1])$ by L^1 of any other non-atomic standard Borel probability space. Our objective is to show that $H^1(X)$ contains a copy of such an L^1.

To this end, we need Ambrose's theorem about flows built under functions [1] and we need some calculations from [11]. Ambrose's theorem says that we may assume, without loss of generality, that our flow has the special form that we now describe. Let μ be a probability measure on a standard Borel space Ω and let τ be an ergodic, invertible, measurable transformation on Ω preserving μ. Suppose, too, that F is a real-valued measurable function on Ω satisfying $0<c\leq F(\omega)\leq C$ for certain constants c and C, and all $\omega\in\Omega$, and satisfying $\int_\Omega F d\mu=1$. We extend F to a function ϕ on $\boldsymbol{Z}\times\Omega$ by the formula

$$\phi(n,\omega)=\begin{cases}\sum_{k=0}^{n-1} F(\tau^k\omega), & n>0\\ 0, & n=0\\ -\phi(-n,\tau^n\omega), & n<0\end{cases}$$

Then ϕ satisfies the equation $\phi(n+m,\omega)=\phi(n,\omega)+\phi(m,\tau^n\omega)$ for all m, $n\in\boldsymbol{Z}$ and $\omega\in\Omega$. Let $\tilde{X}=\Omega\times\boldsymbol{R}$ and let $\tilde{m}=\mu\times\lambda$, where λ is Lebesgue measure on R. Also, let X be the region under the graph of F, i. e., $X=\{(\omega,r)\in\tilde{X}\,|\,0\leq r<F(\omega)\}$, and let m be the restriction of $\tilde{m}$ to X. Then since $\int_\Omega F d\mu=1$, m is a probability measure on X. On $\tilde{X}$, let $\{S_t\}_{t\in\boldsymbol{R}}$ denote the group of $\tilde{m}$-preserving transformations defined by the formula $S_t(\omega,r)=(\omega,r+t)$, and let σ be the measurable, invertible $\tilde{m}$-preserving transformation given by the formula

$$\sigma(\omega,r)=(\tau\omega,r+F(\omega)).$$

Then σ commutes with $\{S_t\}_{t\in\boldsymbol{R}}$, and $\sigma^n(\omega,r)=(\tau^n\omega,r+\phi(n,\omega))$, $n\in\boldsymbol{Z}$, $(\omega,r)\in\tilde{X}$. Observe that $\tilde{X}$ is the <u>disjoint</u> union $\bigcup_{n\in\boldsymbol{Z}}\sigma^n(X)$. We define Π mapping $\tilde{X}$ onto X by the formula $\Pi(\omega,r)=(\tau^n\omega,r-\phi(n,\omega))$, if $\phi(n,\omega)\leq r<\phi(n+1,\omega)$, and we define $\{T_t\}_{t\in\boldsymbol{R}}$ on X by the formula

$$T_t(\omega,r)=(\tau^n\omega,(r+t-\phi(n,\omega))),$$

if $\phi(n,\omega)\leq r+t<\phi(n+1,\omega)$. It is easily checked that $\{T_t\}_{t\in\boldsymbol{R}}$ is a 1-parameter group of m-preserving transformations on X that satisfies $\Pi S_t=T_t\Pi$ for all $t\in\boldsymbol{R}$. It is also easily checked that $\{T_t\}_{t\in\boldsymbol{R}}$ is ergodic (cf. [1] or [11]).

For f $\in L^1(\tilde{X},\tilde{m})$ and $(\omega,r)\in X$, set $(\boldsymbol{P}f)(\omega,r)=\sum_{n\in\boldsymbol{Z}}f\circ\sigma^n(\omega,r)$. Then for m-almost all $(\omega,r)\in X$, this series converges and defines an element in $L^1(X)$. The map $\boldsymbol{P}$ is a norm-one linear map from $L^1(\tilde{X})$ onto $L^1(X)$ satisfying $\boldsymbol{P}S_t=T_t\boldsymbol{P}$ for all $t\in\boldsymbol{R}$. (See Lemma 1 of [11] for a proof of these assertions.)

Fix $h \in H^1(\boldsymbol{R})$ and consider the map Φ_h from $L^1(\Omega)$ into $L^1(X)$ defined by the formula $\Phi_h(\mathrm{f}) = \boldsymbol{P}(fh)$ where $fh(\omega, r) = f(\omega)h(r)$. Then Φ_h is a bounded linear map from $L^1(\Omega)$ into $L^1(X)$ with norm dominated by $\|h\|_1$. Moreover, since $h \in H^1(\boldsymbol{R})$, the range of Φ_h is contained in $H^1(X)$. This is easy to see directly, or appeal can be made to Lemma 6 of [11]. So, to complete the proof of our theorem, all we need to do, by the lemma, is to show that we can choose h so that Φ_h is bounded below. To this end, choose ε such that $0 < \varepsilon < \dfrac{c}{c+1}$ where, (recall $c > 0$ is a lower bound for F), and then choose

$h \in H^1(\boldsymbol{R})$ such that $|1 - |h(r)|| < \varepsilon$, $r \in [0, c]$, and $\int_{\boldsymbol{R}/[0,c]} |h(r)| \mathrm{d}r < \varepsilon$.

Note that such a choice is easily made.

Assertion. *Φ_h is bounded below by the postive constant $c(1-\varepsilon)-\varepsilon$.*

To prove this, first note that the adjoint of $\boldsymbol{P}$, $\boldsymbol{P}^*$. maps $L^\infty(X)$ isometrically into $L^\infty(X)$ by the formula $\boldsymbol{P}^* g = \tilde{g}$ where $\tilde{g}$ is the automorphic extension of g to $\tilde{X}$; i. e., $\tilde{g}$ is the unique function on $\tilde{X}$ such that $\tilde{g} \circ \sigma = \tilde{g}$ and such that $\tilde{g}|_X = g$. So, for $f \in L^1(\Omega)$ we have

$$\begin{aligned}
\|\Phi_h(f)\|_1 &= \sup_{\|g\|_\infty \le 1} \int_X \boldsymbol{P}(fh) g dm \Big| \\
&= \sup_{\|g\|_\infty \le 1} \left| \int_{\tilde{X}} fh\tilde{g} d\tilde{m} \right| \\
&\ge \sup_{\|g\|_\infty \le 1} \left\{ \left| \int_\Omega \int_0^c f(\omega) h(r) \tilde{g}(\omega, r) d\lambda(r) d\mu(\omega) \right.\right. \\
&\qquad \left.\left. - \left| \int_\Omega \int_{\boldsymbol{R} \setminus [0,c]} f(\omega) h(r) \tilde{g}(\omega, r) d\lambda(r) d\mu(\omega) \right| \right\} \\
&\ge \sup_{\|g\|_\infty \le 1} \left\{ \left| \int_\Omega \int_0^c f(\omega) h(r) g(\omega, r) d\lambda(r) d\mu(\omega) \right| - \varepsilon \|f\|_1 \right\}.
\end{aligned}$$

It should be remarked that $\|g\|_\infty$ is the essential supremum taken over all of X, but any L^∞ function on $\Omega \times [0, c]$ can be extended to X without increasing its essential sup-norm. Consequently, the last term in our inequeality equals

$$\begin{aligned}
&\int_\Omega |f(\omega)| d\mu(\omega) \int_0^c |h(r)| d\lambda(r) - \varepsilon \|f\|_1 \\
&\ge c(1-\varepsilon)\|f\|_1 - \varepsilon \|f\|_1 = (c(1-\varepsilon)-\varepsilon)\|f\|_1,
\end{aligned}$$

which verifies the assertion and completes the proof of our theorem.

§ 3. Concluding Remarks

The space $H^{\infty}(X)$ is a weak-* Dirichlet algebra on X with respect to m (see [9]). It makes sense to ask the following question: If $H^1(m)$ is the abstract H^1-space associated to a weak-* Dirichlet algebra on a probability measure space, when is H^1(m) a dual space? We suspect that this is so only when $H^1(m)$ is the classical Hardy space $H^1(\boldsymbol{T})$. The problem seems to be particularly difficult for maximal weak-* Dirichlet algebras for in most, if not all, non maximal weak-* Dirichlet algebras it is possible to show directly that $H^1(m)$ contains a copy of $L^1([0,1])$.

Let G be a compact abelian group, let Λ be a subset of $\hat{G}$, and let L^1_Λ be the set of functions in $L^1(G)$ with the property that their Fourier transforms are supported in Λ. Among other things, Lust was concerned in [8] with the question: when is L^1_Λ a dual space? In view of our result, it is natural to ask: If L^1_Λ is not a dual space, must L^1_Λ contain a copy of $L^1([0,1])$?

Finally, we note that it is possible to transfer the real-variable approach to H^1 and to define an ergodic H^1 â la Stein and Weiss when $\boldsymbol{R}^n$ acts on a measure space (cf. [3, 4]). It this space a dual space? Does it contain a copy of L^1?

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The University of Iowa
The University of Iowa
and
Hokkaido University