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Solvability of convolution equations in K'_M

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Abstract. Net S be a convolution operator in the space K'_M, M is an increasing continuous function, of distributions in $\boldsymbol{R}^n$ growing not faster than $e^{M(kx)}$ for some positive integer k. We give necessary and sufficient condition on S to have $S*K'_M=K'_M$, then we use this condition to get necessary and sufficient conditions for solvability of systems of convolution equations in K'_M.

1. Introduction

S. Sznajder and Z. Zielezny [6] characterized convolution operators in the spaces K'_p, $p>1$, having solutions in K'_p. The space K'_P, $p>1$, is one of the spaces K'_M where $M(x)=|x|^p$, $p>1$. The spaces K'_M first appeared in the work of Gelfand and Shilov [3], where they used them to study the Cauchy problem. In [5], D. Pahk studied hypoelliptic convolution operators in the space K'_M after making a little modefication in the definition of the space K_M of test functions. Here I will follow Pahk and use the Paley-Wiener type theorem which he proved for the spaces. The results of this paper extend a result of Sznajder and Zielezny [6].

2. Terminology and preleminary results

Let $\mu(\xi)$, $0\leq\xi\leq\infty$, be a continuous, increasing function such that $\mu(0)=0$, $\mu(\infty)=\infty$. For $x\geq 0$ the function $M(x)$ is defined as

$$M(x)=\int_0^x \mu(\xi)\ d\xi.$$

It turns out the function $M(x)$ is continuous, increasing and convex with $M(0)=0$, $M(\infty)=\infty$, hence the function M is invertible. For negative x we define $M(x)$ to be $M(-x)$ and for $x=(x_1, \ldots, x_n)\in\boldsymbol{R}^n$, $n\geq 2$, we define $M(x_1, \ldots, x_n)$ to be $M(x_1)+\ldots+M(x_n)$. We notice that $M(x)$ grows faster than any linear function of $|x|$ as $|x|\rightarrow\infty$, hence $M^{-1}(x)\leq x$ whenever $x\geq 0$ and large enough, where M^{-1} is the inverse of M. Let M and Ω be functions corresponding to μ and ω respectively, as in the above definition, the functions M and Ω are said to be dual in the sense of Young if and only if $\mu\circ\omega=\omega\circ\mu=$ the identity function. As example of functions dual in the sense of

Young one has $M(x)=\frac{x^p}{p}$ and $\Omega(x)=\frac{x^q}{q}$ where $\frac{1}{p}+\frac{1}{q}=1$. We list some of the properties of the function M which are going to be used later in the proofs.

(1) $M(x)+M(y)\leq M(x+y)$ for all $x, y\geq 0$.

(2) $M(\frac{1}{k}x)\leq\frac{1}{k}M(x)$ for any $k>0$.

(3) $M(x+y)\leq M(2x)+M(2y)$ for all $x, y\geq 0$.

For a function M as above, we define the space K_M to be the space of all infinitely differentiable functions ψ on $\boldsymbol{R}^n$ so that

$$\nu_k(\psi)=\sup_{\substack{x\in\boldsymbol{R}^n\\|\alpha|\leq k}} e^{M(kx)}|\mathrm{D}^\alpha(\psi)(x)|<\infty,\ k=1, 2, 3, \dots,$$

where $D^\alpha=(-i\frac{\partial}{\partial x_1})^{\alpha_1}\dots(-i\frac{\partial}{\partial x_n})^{\alpha_n}$, $\alpha=(\alpha_1,\dots, \alpha_n)\in N^n$. The space K_M will be provided with the topology generated by the semi-norms ν_k: $k=1, 2, \dots$. It follows that K_M is a Frechet space with the embeddings $D\hookrightarrow K_M\hookrightarrow E$ being continuous, where D and E are Schwartz's spaces [5]. By K'_M we denote the space of all continuous linear functionals on K_M and we provide K'_M with the strong dual topology. It follows that $E'\subset K'_M\subset D'$. The following theorem characterizes the elements of K'_M.

THEOREM A. [5, Theorem 1.2.3]. *A distribution $T\in D'$ is in K'_M if and only if there exist positive integers m, k and a bounded continuous function $f(x)$ on $\boldsymbol{R}^n$ so that*

$$T=\frac{\partial^{mn}}{\partial x_1^m\dots\partial x_n^m}\left[e^{M(kx)}f(x)\right].$$

THEOREM B. [5, Theorem 1.3.2]. *For any distribution $S\in K'_M$ the following three conditions are equivalent:*

(i) *The distributions $S_1=e^{M(kx)}S$; $k=1, 2, 3, \dots$ are in S', the space of tempered distributions.*

(ii) *For every integer $k\geq 0$, there exists an integer $m\geq 0$ such that*

$$S=\sum_{|\alpha|\leq m} D^\alpha f_\alpha,$$

where the f_α s; $\alpha\in\boldsymbol{N}^n$ are continuous functions in $\boldsymbol{R}^n$ whose products with $e^{M(kx)}$ are bounded.

(iii) *For every $\phi\in K_M$, the convolution $S*\phi$ is in K_M and the mapping $\phi\rightarrow S*\phi$ from K_M into itself is continuous, where $(S*\phi)(x)=\langle S_y, \phi(x-y)\rangle$.*

By $O'_c(K'_M : K'_M)$ we denote the space of all $S \in K'_M$ which satisfy any of the above conditions. Then it follows from (iii) *that $O'_c(K'_M : K'_M)$ is the space of convolution operators in K'_M. For $S \in O'_c(K'_M : K'_M)$ and $T \in K'_M$ we define the convolution $S*T$ by*

$$<S*T,\ \phi>=<T,\ \check{S}*\phi>,\quad \phi \in K_M,$$

where $<\check{S}, \psi>=<S, \check{\psi}>$ for all $\psi \in K_M$, $\check{\psi}(x)=\psi(-x)$.
For $\phi \in K_M$, the Fourier transform $\hat{\phi}$ is defined by

$$\hat{\phi}(\xi)=\int_{R^n} e^{-i<x,\xi>}\phi(x)dx,$$

as usual, and for $S \in O'_c(K'_M : K'_M)$ it follows that $S \in S'$ and its Fourier transform is well-defined by the formula

$$<\hat{S},\ \phi>=<S,\ \hat{\phi}>,\quad \phi \in K_M$$

Also, we define the Fourier transform $\hat{u}$ for any $u \in K'_M$ by the formula

$$<\hat{u},\ \hat{\phi}>=<u,\ \check{\phi}>,\quad \phi \in K'_M,$$

we denote by $\boldsymbol{K}'_M$ and $\boldsymbol{K}_M$ the Fourier transform of K'_M and K_M respectively. The elements of K_M and $O'_c(K'_M : K'_M)$ satisfy the following Paley-Wiener type theorem :

THEOREM C. [5, Theorem 1. 4. 1].
(a) *An entire function $F(\zeta)$ is a Fourier transform of a function ϕ in K_M if and only if, for every integer $N \geq 0$ and every $\varepsilon > 0$ there exists a constant C such that*

$$|F(\zeta) \leq C(1+|\zeta|)^{-N}\ e^{\Omega(\varepsilon\eta)},\ \textit{where}\ \zeta=\xi+i\eta \in \boldsymbol{C}^n,$$

and Ω is dual to M in the sense of Young.
(b) *An entire function $F(\zeta)$ is the Fourier transform of a distribution S in $O'_c(K'_M : K'_M)$ if and only if, for every $\varepsilon > 0$ there exist constants C, N such that*

$$(*)\qquad |F(\zeta)| \leq C(1+|\zeta|)^N e^{\Omega(\varepsilon\eta)},\ \textit{where}\ \zeta=\xi+i\eta \in \boldsymbol{C}^n,$$

and Ω is dual to M in the sense of Young.
From the Paley-Wiener theorem it follows that the differential operator of infinite order $exp(D)$ is an element of K'_M for all M.

We provide $\boldsymbol{K}_M$, the space of all $\hat{\phi}$ where $\phi \in K_M$, with the locally convex topology defined by the following family of semi-norms :

$$\omega_k(\hat{\phi})=\sup_{\zeta=\xi+i\eta\in C^n}(1+|\zeta|)^k e^{-\Omega(\frac{\eta}{k})}|\hat{\phi}(\zeta)|\,;\ k=1,2,3,\dots.$$

Since Ω grows faster than any linear function in $|\eta|$ whenever $|\eta|$ is large enough it follows that an equivalent family of semi-norms on $\boldsymbol{K}_M$ is

$$\beta_k(\hat{\phi})=\sup_{\zeta=\xi+i\eta\in C^n}(1+|\xi|)^k e^{-\Omega(\frac{\eta}{k})}|\hat{\phi}(\zeta)|\,;\ k=1,2,3,\dots.$$

It follows that the Fourier transform is a topological isomorphism of K_M onto $\boldsymbol{K}_M$.

Let m be a positive integer greater than 1, and by K_M^m, $K_M'^m$, $\boldsymbol{K}_M^m$, $O_c'(K_M'^m\,;K_M'^m)$ we denote the product of m-copies of K_M, K_M', $\boldsymbol{K}_M$ and $O_c'(K_M';K_M')$ respectively. We peovide $\boldsymbol{K}_M^m$ with the topology generated by the following family of semi-norms:

$$\nu_k(\hat{\phi})=\nu_k((\hat{\phi}_1,\hat{\phi}_2,\dots,\hat{\phi}_m)^t)=\sum_{j=1}^m\omega_k(\hat{\phi}_j)\,;\ k=1,2,3,\dots.$$

In the proofs we will use the following improved version of a lemma of Hörmander.

THEOREM D. [1, Corollary II.1.2]: *Let f, g be entire functions in $\boldsymbol{C}^n$ so that f/g is entire, then for every positive real number ρ and for every $\zeta\in\boldsymbol{C}^n$ one has*

$$\left|\frac{f(\zeta)}{g(\zeta)}\right|\le\sup_{\substack{|z|\le 3\rho\\ z\in C^n}}|f(z+\zeta)|\sup_{\substack{|z|\le 3\rho\\ z\in C^n}}|g(z+\zeta)|/[\sup_{\substack{|z|\le\rho\\ z\in C^n}}|g(z+\zeta)|]^2$$

3. The results

The first result give necessary and sufficient conditions for the solvability of convolution equations in K_M'. This generalizes the main theorem of [6]. A convolution operator which satisfies the conditions of the theorem is called invertible.

THEOREM 1. *Let $S\in O_c'(K_M':K_M')$, the following conditions are equivalent:*

(a) *There exist positive constants C, N and A so that*

(I) $$\sup_{\substack{|z|\le A\Omega^{-1}[\log(2+|\xi|)]\\ z\in C^n}}|\hat{S}(z+\xi)|\ge C(1+|\xi|)^{-N}\,;\ \xi\in\boldsymbol{R}^n,$$

where Ω^{-1} is the inverse of Ω, which is the dual to M in the sense of young.

(b) $S*K_M'=K_M'$.

THEOREM 2. *Let S be an invertible convolution operator of K_M', let v*

be another convolution operator of K'_M with $\hat{v}/\hat{S}$ an entire function, then there exists a $u \in O'_c(K'_M : K'_M)$ so that $S*u=v$. Moreover, if $v \in K_M$ then $u \in K_M$.

We use Theorem 1 to prove the main result of the paper, it provides necessary conditions and sufficient conditions for solvability of determined systems of convolution equations in K'_M.

One can represent such system of convolution equations in K'_M by an equation of the form $S*U=V$ where $S=(S_{ij}) \in O'_c(K'^m_M, K'^m_M)$, $U=(u_1, \ldots, u_m)$, $V=(v_1, \ldots, v_m) \in K'^m_M$. For S as above, S^t will denote the matrix $(\check{S}_{ji})$, we recall that det $(\hat{S}^t)$ is an entire function which satisfies the growth condition $(*)$.

THEOREM 3. *Let $S \in O'_c(K'^m_M ; K'^m_M)$, among the following properties of S, the implications $(a) \to (b) \to (c) \to (b)$ hold.*

(*a*) *det $(\hat{S}^t)$ satisfy the growth condtion (I).*

(*b*) *The map $S^t * \phi \to \phi$ from $S^t * K^m_M$ into K^m_M is continuous.*

(*c*) $S * K'^m_M = K'^m_M$.

4. Proofs of the results

PROOF OF THEOREM 1: The implication $(a) \Longrightarrow (b)$.

From the Hahn-Banach theorem and the continuity of the Fourier transform it suffices to prove that the map $T\hat{\phi} \to \hat{\phi}$ from $\boldsymbol{T K_M}$ into $\boldsymbol{K_M}$ is continuous; where $T=\check{S}$. Let k be any positive integer, we would like to show that there exist positive numbers l, c so that $\omega_k(\hat{\phi}) \le c\omega_l(T\hat{\phi})$, $\hat{\phi} \in \boldsymbol{K_M}$. Let $\zeta=(\zeta_1, \ldots, \zeta_n)$ be any given point of $\boldsymbol{C}^n$, $\zeta_j=\xi_j+i\eta_j$, by applying Theorem D to the function $\hat{\phi}=T\hat{\phi}/T$ with $\rho=A\Omega^{-1}[\log(2+|\xi|)]+|\eta|$, one has

$$|\hat{\phi}(\zeta)| \le \sup_{\substack{|z|\le 3\rho \\ z\in \boldsymbol{C}^n}} |(T\hat{\phi})(z+\zeta)| \cdot \sup_{\substack{|z|\le 3\rho \\ z\in \boldsymbol{D}^n}} |T(z+\zeta)| / [\sup_{\substack{|z|\le \rho \\ z\in \boldsymbol{D}^n}} |T(z+\zeta)|]^2. \tag{4}$$

From condition (I) it follows that

$$\sup_{|z|\le\rho} |T(z+\zeta)| \ge \sup_{|z|\le A\Omega^{-1}[\log(2+|\xi|)]} |T(z+\zeta)| \ge c_1(1+|\xi|)^{-N} \ge c_1(1+|\zeta|)^{-N},$$

hence (4) becomes

$$|\hat{\phi}(\zeta)| \le c_1^{-2}(1+|\zeta|)^{2N} \sup_{|z|\le 3\rho} |(T\hat{\phi})(z+\zeta)| \sup_{|z|\le 3\rho} |T(z+\zeta)|. \tag{5}$$

From the definition of semi-norms on $\boldsymbol{K_M}$ it follows that for every positive integer l and $z=x+iy \in \boldsymbol{C}^n$; $|z| \le 3\rho$ one has

$$\begin{aligned} |(T\hat{\phi})(z+\zeta)| &\le (1+|z+\zeta|)^{-l} e^{\Omega\left(\frac{y+\eta}{l}\right)} \omega_l(T\hat{\phi}); \\ &\le (1+|z|)^l (1+|\zeta|)^{-l} e^{\Omega\left(\frac{y+\eta}{l}\right)} \omega_l(T\hat{\phi}). \end{aligned} \tag{6}$$

We will determine l later. By (6) and the Paley-Wiener theorem applied to T, inequality (5) gives

$$(7)\quad |\hat{\phi}(\zeta)\leq c_2(1+|\zeta|)^{2N-l}\omega_l(T\hat{\phi})\sup_{|z|\leq 3\rho}(1+|z+\zeta|)^{N_2}e^{\Omega\left(\frac{y+\eta}{80k2^{n+1}}\right)}\sup_{|z|\leq 3\rho}(1+|z|)^l e^{\Omega\left(\frac{y+\eta}{l}\right)},$$

where c_2, N_2 are positive constants depending on k only and n is the dimension. From the properties of Ω one has for any $y=Imz$, $y=(y_1,\dots,y_n)$, $|z|\leq 3\rho$

$$\begin{aligned}\Omega\left(\frac{y+\eta}{80k\ 2^{n+1}}\right)&=\sum_{j=1}^{n}\Omega\left(\frac{y_j+\eta_j}{80k\ 2^{n+1}}\right)\leq\sum_{j=1}^{n}\Omega\left(\frac{3A\Omega^{-1}[\log(2+|\xi|)]+3|\eta|}{40k\ 2^{n+1}}\right)\\&\quad+\Omega\left(\frac{|\eta_j|}{40k\ 2^{n+1}}\right)\\&\leq n\Omega\left(\frac{3|\eta|}{20k\ 2^{n+1}}\right)+\frac{3A}{40k}\log(2+|\xi|)+\Omega\left(\frac{\eta}{40k}\right)\\&\leq n\Omega\left(\frac{3\eta}{40k}\right)+\Omega\left(\frac{\eta}{40k}\right)+\frac{3A}{40k}\log(2+|\zeta|),\end{aligned}$$

hence

$$\sup_{|z|\leq 3\rho}e^{\Omega\left(\frac{y+\eta}{80k\ 2^{n+1}}\right)}\leq(2+|\zeta|)^{\frac{3A}{40k}}e^{\Omega\left(\frac{3n+1}{40k}\eta\right)}.$$

And similarly

$$\sup_{|z|\leq 3\rho}e^{\Omega\left(\frac{y+\eta}{l}\right)}\leq(2+|\zeta|)^{\frac{6A}{l}}e^{\Omega\left(\frac{3n2^{n+2}+2}{l}\eta\right)}.$$

On the other hand, since $\Omega^{-1}(x)\leq x$ for large x and $\Omega(\eta)$ grows faster than any linear function of $|\eta|$ for large $|\eta|$ it follows that

$$\begin{aligned}\sup_{|z|\leq 3\rho}(1+|z|)^l&\leq C_l e^{\frac{|z|}{3}}\leq C_l e A^{\Omega^{-1}[\log(2+|\xi|)]+|\eta|}\\&\leq C_l(2+|\xi|)^A e^{|\eta|}\leq C_l^1 2^A(1+|\zeta|)^A e^{\Omega\left(\frac{\eta}{20k}\right)},\end{aligned}$$

where C_l, C_l^1 are constants which depend on k and l only. Thus (7) becomes

$$(8)\qquad \begin{aligned}|\hat{\phi}(\zeta)|&\leq 2^{2A}C_3C_l^1\omega_l(T\hat{\phi})(1+|\zeta|)^{2N+N_2+\frac{3A}{40k}+\frac{6A}{l}+2A-l}\times\\&\times e^{\Omega\left(\frac{3n+1}{40k}+\frac{2+3n2^{n+2}}{l}\right)\eta+2\Omega\left(\frac{\eta}{20k}\right)},\end{aligned}$$

where C_3 is a constant which depends on k only. Hence

$$(9)\qquad \begin{aligned}(1+|\zeta|)^k e^{-\Omega\left(\frac{\eta}{k}\right)}|\hat{\phi}(\zeta)|&\leq 2^{2A}C_3C_l^1\omega_l(T\hat{\phi})(1+|\zeta|)^{[-l+k+2N+N_2+\frac{3A}{40k}+\frac{6A}{l}+2A]}\times\\&\times e^{\Omega\left[\left(\frac{3n+1}{40k}+\frac{2+2n2^{n+2}}{l}+\frac{1}{10k}\right)\eta\right]-\Omega\left(\frac{\eta}{k}\right)}.\end{aligned}$$

By taking $l=\max\{2+k+2N+N_2+\frac{3A}{40k}+8A,\ 13\ 2^{n+2}k\}$ it follows from (9) that

$$\omega_k(\hat{\phi})\leq C\omega_l(T\hat{\phi}),$$

where C is a constant which depends on k only.

The implication (b)⇒(a).

Here I am going to use a technique which was introduced by Ehrenpreise [2] and was used by many other authors. First we remark that condition (I) is implied by the condition: there exist positive constants N, A, L so that

$$\text{(II)} \qquad \sup_{|z|\leq A\Omega^{-1}(\log(2+|\xi|))}|\hat{S}(z+\xi)|\geq(1+|\xi|)^{-N},\ \xi\in\boldsymbol{R}^n,\ |\xi|\geq L.$$

The proof will be by contradiction. If (II) does not hold, then for all $j\in\boldsymbol{N}$, there exists $\xi_j\in\boldsymbol{R}^n$ so that $|\xi_j|>e^j$ and

$$\text{(10)} \qquad \sup_{|z|\leq A_j\alpha_j}|\hat{S}(z+\xi_j)|<(1+|\xi_j|)^{-j},$$

where $A_j=e^{2j}$ and $\alpha_j=\Omega^{-1}(\log(2+|\xi_j|))$. For each $j\in\boldsymbol{N}$ we define k_j to be the greatest integer equal or less than $\log\alpha_j+1$. Let $\phi\in D$ so that supp $\phi\subset B(0,1)$, $\phi\geq0$ and $\hat{\phi}(0)=1$. For each $j\in\boldsymbol{N}$ we define the function ϕ_j by $\phi_j(\xi)=\alpha_j\phi(\alpha_j\xi)$, thus supp $\phi_j\subset B(0,\frac{1}{\alpha_j})$, we also define the function ψ_j^1 by

$$\psi_j^1(\xi)=e^{i<\xi_j,\xi>}(\phi_j*\phi_j*\ldots*\phi_j)(\xi),$$

where the convolution product is being taken k_j-times. Hence, supp $\psi_j^1\subset B(0,\frac{k_j}{\alpha_j})\subset B(0,1)$. Define the function ψ_j as the convolution $\psi_j^1*\psi_j^1$, thus supp $\psi_j\subset B(0,2)$.

From the above definitions it follows that

$$\hat{\psi}_j^1(z+\xi_j)=\int e^{-i<z,\xi>}(\phi_j*\ldots*\phi_j)(\xi)d\xi=(\widehat{\phi_j*\ldots*\phi_j})(z)\,;$$
$$=(\hat{\phi}_j(z))^{k_j},$$

and

$$\hat{\phi}_j(z)=\int e^{-i<\frac{z}{\alpha_j},\xi>}\phi(\xi)d\xi=\hat{\phi}\Big(\frac{z}{\alpha_j}\Big).$$

Thus

$$\text{(11)} \qquad \hat{\psi}_j^1(z+\xi_j)=\hat{\phi}\Big(\frac{z}{\alpha_j}\Big)^{k_j}\text{and } \psi_j^1(\xi_j)=(\hat{\phi}(0))^{k_j}=1.$$

Since $S*K'_M=K'_M$ it follows that $S*E=\delta$ for some $E\in K'_M$, and since $\hat{E}$ is a continuous linear functional on $\boldsymbol{K}_M$ there exist positive integers A, k so that

$$(12)\qquad |\psi_j(\xi)|=|<\delta, \tau_\xi\check{\psi}_j>|=<S*E, \tau_\xi\check{\psi}_j>|=|<\hat{E}, \tau_{-\xi}(S*\hat{\psi}_j)>|$$
$$\le A\omega_k(\widehat{\tau_{-\xi}(S*\psi_j)})\le A\sup_{z\in C^n}(1+|z|)^k e^{-\Omega(\frac{y}{k})}|\hat{S}(z)|\cdot|\hat{\psi}_j(z)|$$
$$\le A\sup_{\substack{|z|\le A_j\alpha_j\\ z\in C^n}}(1+|z+\xi_j|)^k e^{-\Omega(\frac{y}{k})}|\hat{S}(z+\xi_j)|\cdot|\hat{\psi}_j(z+\xi_j)|+$$
$$+A\sup_{|z|>A_j\alpha_j}(1+|z+\xi_j|)^k e^{-\Omega(\frac{y}{k})}|\hat{S}(z+\xi_j)|\cdot|\hat{\psi}_j(z+\xi_j)|.$$

Now we estimate each of the terms on the right hand side of (12). For the first term we apply the Paley-Wiener theorem to ψ_j as element of D, use (10) and the fact that Ω grows faster than any linear function of $|y|$ as $|y|$ gets large, one has for j large enough

$$(13)\qquad A\sup_{|z|\le A_j\alpha_j}(1+|z+\xi_j|)^k e^{-\Omega(\frac{y}{k})}|\hat{S}(z+\xi_j)||\hat{\psi}_j(z+\xi_j)|$$
$$\le AC_k\sup_{|z|\le A_j\alpha_j}(1+|z+\xi_j|)^{k-2k}e^{-\Omega(\frac{y}{k})+\Omega(\frac{y}{k})}\sup_{|z|\le A_j\alpha_j}|\hat{S}(z+\xi_j)|$$
$$\le AC_k(1+|\xi_j|)^{-j}<\frac{1}{2}e^{-j},$$

where C_k is constant which depends on k only. Next, we estimate the second term. By the Paley-Wiener theorem applied to ϕ_j, ψ_j^1 as elements of D and S as element of $O'_c(K'_M: K'_M)$ and the definition of k_j, α_j it follows from (11) that

$$(14)\qquad A\sup_{|z|>A_j\alpha_j}(1+|z+\xi_j|)^k e^{-\Omega(\frac{y}{k})}|\hat{S}(z+\xi_j)|\cdot|\hat{\psi}_j(z+\xi_j)$$
$$\le AC_k^1\sup_{|z|>A_j\alpha_j}(1+|z+\xi_j|)^{k+N}e^{-\Omega(\frac{y}{k})+\Omega(\frac{y}{6k})}|\hat{\psi}_j^1(z+\xi_j)|\cdot|\hat{\psi}_j^1(z+\xi_j)|$$
$$\le AC_k^2\sup_{|z|>A_j\alpha_j}(1+|z+\xi_j|)^{k+N-(2k+N)}e^{-\Omega(\frac{y}{k})+2\Omega(\frac{y}{6k})}|\hat{\psi}_j^1(z+\xi_j)|$$
$$\le \mathrm{A}C_k^2\sup_{|z|>A_j\alpha_j}e^{-\Omega(\frac{y}{k})+2\Omega(\frac{y}{6k})}(C_1(1+|\frac{z}{\alpha_j}|)^{-1}e^{\frac{|y|}{\alpha_j}})^{k_j}$$
$$\le AC_k^3\sup_{|z|>A_j\alpha_j}e^{-\Omega(\frac{y}{k})+3\Omega(\frac{y}{6k})}C_1^{k_j}(1+|\frac{z}{\alpha_j}|)^{-k_j}$$
$$\le \mathrm{A}C_k^3C_1^{k_j}\sup_{|z|>A_j\alpha_j}(1+|\frac{z}{\alpha_j}|)^{-k_j},$$

where C_k^1, C_k^2 and C_k^3 are constants which depend on k only. For any $z\in \boldsymbol{C}^n$, $|z|>A_j\alpha_j$ one has $1+\left|\frac{z}{\alpha_j}\right|>1+A_j$, hence $\sup_{|z|>A_j\alpha_j}(1+\left|\frac{z}{\alpha_j}\right|)^{-k_j}\le(1+A_j)^{-k_j}$, and (14) gives

$C(1+|\zeta|)^k e^{\Omega(\varepsilon\eta)}$; where Ω is a function dual to M in the sense of Young. Let $\zeta\in \boldsymbol{C}^n$ be given, by applying Hormander's lemma with $\rho=A\Omega^{-1}[\log(2+|\xi|)]+|\eta|$, where A is the constant of condition (I), one has from the invertibility of S

$$(18)\qquad |\hat{u}(\zeta)\leq C_1^2(1+|\zeta|)^{2N_1}\sup_{\substack{|z|\leq 3\rho\\ z\in \boldsymbol{C}^n}}\hat{v}(z+\zeta)\sup_{\substack{|z|\leq 3\rho\\ z\in \boldsymbol{C}^n}}|\hat{S}(z+\zeta)|.$$

where C_1, N_1 are the constants of condition (I) above. By applying the Paley-Wiener theorem to S and v it follows that there exist constants C_2, K_2, C_3 and K_3 which depend on ε and the dimension n so that

$$(19)\qquad \sup_{\substack{|z|\leq 3\rho\\ z\in \boldsymbol{C}^n}}|\hat{v}(z+\zeta)|\leq C_2(1+|\zeta|)^{k_2}\, e^{\Omega(\frac{\varepsilon}{3}\eta)},$$

and

$$(20)\qquad \sup_{\substack{|z|\leq 3\rho\\ z\in \boldsymbol{C}^n}}|\hat{S}(z+\zeta|\leq C_3(1+|\zeta|)^{k_3}\, e^{\Omega(\frac{\varepsilon}{3}\eta)}.$$

Using (19) and (20) to estimate the right hand side of (18) one gets

$$|\hat{u}(\zeta)|\leq C(1+|\zeta|)^k\, e^{\Omega(\varepsilon\eta)},$$

where $C=C_1^2C_2C_3$ and $k=2N_1+k_2+k_3$ are constants which depend on ε only.

Next, we consider the case when $v\in K_M$. To show that $u\in K_M$ we prove that for any $\varepsilon>0$ and k any positive integer there exists a C which depends on k and ε only so that $|\hat{u}(\zeta)|\leq C(1+|\zeta|)^{-k}\, e^{\Omega(\varepsilon\eta)}$. Let N_1 and k_3 be the constants of (18) and (20) respectively. Let $k_2=\max\{2N_1+k_3+13A+k+1, \frac{24}{\varepsilon}+1\}$. By the Paley-Wiener theorem applied to v it follows that there exists a constant C_2^1 which depends on k_2 only so that (with ρ as above)

$$(21)\qquad \sup_{\substack{|z|\leq 3\rho\\ z\in \boldsymbol{C}^n}}|\hat{v}(z+\zeta)|\leq C_2^1\sup_{\substack{|z|\leq 3\rho\\ z\in \boldsymbol{C}^n}}(1+|z+\zeta|)^{-k_2}\, e^{\Omega(\frac{y+\eta}{2^n k_2})}$$
$$\leq C_2^2((1+|\zeta|)^{13A-k_2}e^{\Omega((\frac{\varepsilon}{10}+\frac{12}{k_2})\eta)},$$

where C_2^2 in a constant which depends on k_2 (and A) only. Using (20) and (21) to estimate the right hand side of (18) one gets by the choice of k_2

$$|\hat{u}(\zeta)|\leq C_1^2C_2^2C_3(1+|\zeta|)^{2N_1+k_3+13A-k_2}\, e^{\Omega(\frac{\varepsilon}{3}\eta)+\Omega((\frac{\varepsilon}{10}+\frac{12}{k_2})\eta)}$$
$$\leq C(1+|\zeta|)^{-k}\, e^{\Omega(\varepsilon\eta)},$$

where $C=C_1^2C_2^2C_3$ is a constant which depends on k and ε only.

(15) $$A \sup_{|z|>A_j\alpha_j}(1+|z+\xi_j|)^k e^{-\Omega(\frac{y}{k})}|\hat{S}(z+\xi_j)||\hat{\psi}_j(z+\xi_j)|$$
$$\leq AC_k^3\left(\frac{1+A_j}{C_1}\right)^{-k_j}<\frac{1}{2}e^{-j},$$

whenever j is large enough. From (12), (13) and (15) one gets

(16) $|\psi_j(\xi)|\leq e^{-j}$ whenever j is large enough.

From the definition of ϕ, ϕ_j, ψ_j^1, ψ_j and inequality (16) it follows that

(17) $$1=|\hat{\phi}(0)|^{2k_j}=|\hat{\psi}_j^1(\xi_j)|\cdot|\hat{\psi}_j^1(\xi_j)|=|\hat{\psi}_j(\xi_j)|\leq\int|\psi_j(\xi)|d\xi$$
$$\leq e^{-j}\int_{B(0,2)}d\xi\leq 4^n e^{-j}.$$

As j goes to infinity the left hand side of (17) remains one but the right hand side converges to zero, the contradiction proves the implication. This completes the proof of the theorem.

If $S\in O'_c(K'_M, K'_M)$ satisfies any of the above condtions we say that the entire function $\hat{S}$ is slowly increasing.

REMARK From the above proof it follows that an entire function which satisfies the growth condition $(*)$ is slowly increasing if and only if the map $f\hat{\phi}\to\hat{\phi}$ from f $\boldsymbol{K}_M$ into $\boldsymbol{K}_M$ is continuous.

EXAMPLES:
(i) Let $\hat{S}(\zeta)=e^{i\zeta}$, then $S\in O'_c(K'_M: K'_M)$ for $M(x)=e^x-x-1$, [5]. For any $\xi\in\boldsymbol{R}^n$, $|\xi|\geq e^8$ one has with $\rho=\Omega^{-1}[\log(2+|\xi|)]=(\log(2+|\xi|)+1)\times(\log[\log(2+|\xi|)+1])-\log(2+|\xi|)$,

$$\sup_{\substack{|z|\leq\rho\\ z\in\boldsymbol{C}^n}}|\hat{S}(z+\xi)|=\sup_{\substack{|z|\leq\rho\\ z\in\boldsymbol{C}^n}} e^{-y}\geq 1\geq(1+|\xi|)^{-1}.$$

Thus S is invertible in K'_M.

(ii) Let $\hat{S}(\zeta)=1+e^{-\zeta^2}$, then $S\in O'_c(K'_M: K'_M)$ for $M(x)=|x|^{\frac{4}{3}}$, [5]. For any $\xi\in\boldsymbol{R}^n$ one has

$$\sup_{\substack{|z|\leq 5\Omega^{-1}[\log(2+|\xi|)]\\ z\in\boldsymbol{C}^n}}|\hat{S}(z+\xi)|=\sup_{\substack{|z|\leq 5[\log(2+|\xi|)]^{1/4}\\ z\in\boldsymbol{C}^n}}(1+e^{-(z+\xi)^2})\geq 1\geq(1+|\xi|)^{-1}$$

Thus S is invertible in K'_M.

PROOF OF THEOREM 2: Since S is invertible there exists a $u\in K'_M$ so that $S*u=v$. To show that $u\in O'_c(K'_M: K'_M)$ it suffices to show that for any $\varepsilon>0$ there exist positive constants k, C which depend on ε so that $|\hat{u}(\zeta)|\leq$

PROOF OF THEOREM 3: The implication (a)⇒(b). From the continuity of the Fourier transform it suffices to show that the map $\hat{\psi}=\hat{S}^t\hat{\phi}\to\hat{\phi}$ from $S^t\boldsymbol{K}_M^m$ into $\boldsymbol{K}_M^m$ is continuous. Since $\boldsymbol{K}_M$ and $\boldsymbol{K}_M^m$ are metrizable continuity is equivalent to sequential continuity. Suppose $(\hat{\psi}_j)$ converges to 0 in $\boldsymbol{K}_M^m$ as $j\to\infty$. Then, by Cramer's rule, we have

$$\text{(22)}\qquad \hat{\phi}_{lj}=\frac{h_{lj}}{\det(\hat{S}^t)}\ ;\ l=1,2,\ldots,m,$$

where $\hat{\phi}_{lj}$ is the l-th component of $\hat{\phi}_j$ and h_{lj} is the determinant of the matrix which one gets by replacing the l-th column of $\hat{S}^t$ by $\hat{\psi}_j$. By definition of the topology of $\boldsymbol{K}_M^m$, to prove that $(\hat{\phi}_j)$ converges to 0 it suffices to show that $\hat{\phi}_{lj}$ converges to 0 for each $l=1,2,\ldots,m$. Let k be any given positive integer, we prove that $\rho_k(\hat{\phi}_{lj})$ converges to zero as j goes to infinity, for $l=1,2,\ldots,m$.

Since $\hat{\psi}_j$ converges to 0 in $\boldsymbol{K}_M^m$ and $S^t\in O'_c(K'^m_M:K'^m_M)$ it follows from the Paley-Wiener theorem and the definition of the topology of $\boldsymbol{K}_M$ that (h_{lj}) converges to 0 in $\boldsymbol{K}_M$ as j goes to infinity, for each $l=1,2,\ldots,m$. Let $\zeta=\xi+i\eta$ be any point in $\boldsymbol{C}^n$, for each $l=1,2,\ldots,m$, applying Theorem D to $\hat{\phi}_{lj}$ in (22) with $\rho=A\Omega^{-1}[\log(2+|\xi|)]+|\eta|$ where A is the constant of condition (I), which $\det(\hat{S}^t)$ satisfy, it follows that

$$\text{(23)}\qquad |\hat{\phi}_{lj}(\zeta)|\leq\sup_{\substack{|z|\leq3\rho\\ z\in\boldsymbol{C}^n}} h_{lj}(z+\zeta)|\sup_{\substack{|z|\leq3\rho\\ z\in\boldsymbol{C}^n}}|\det(\hat{S}^t)(z+\zeta)|/[\sup_{\substack{|z|\leq\rho\\ z\in\boldsymbol{C}^n}}|\det(\hat{S}^t)(z+\zeta)]^2.$$

Next we estimate each of the terms on the right hand side of (23). Since $\det(\hat{S}^t)$ is slowly increasing it follows that

$$\text{(24)}\qquad [\sup_{|z|\leq\rho}|\det(\hat{S}^t)(z+\zeta)|]^{-2}\leq C_1(1+|\zeta|)^{N_1}.$$

for some positive constants C_1, N_1.
By applying the Paley-Wiener theorem to the entries of $\hat{S}^t$ and using the properties of Ω it follows that there exist positive constants C_2, N_2 which depend on k only, such that

$$\text{(25)}\qquad \sup_{|z|\leq3\rho}|\det(\hat{S}^t)(z+\zeta)|\leq C_2(1+|\zeta|)^{N_2}\ e^{\Omega\left(\frac{\eta}{10k}\right)}.$$

Using (24) and (25) to estimate the right hand side of (23) one gets

$$\text{(26)}\qquad |\hat{\phi}_{lj}(\zeta)|\leq C_1C_2(1+|\zeta|)^{N_1+N_2}\ e^{\Omega\left(\frac{\eta}{10k}\right)}\sup_{|z|\leq3\rho}|h_{lj}(z+\zeta)|.$$

Since (h_{lj}) converges to 0 in $\boldsymbol{K}_M$ as j goes to infinity, it follows that for any $s\in\boldsymbol{N}$ one has

$$(27)\qquad \rho_s(h_{lj})=\sup_{z=x+iy\in C^n}(1+|z+\zeta|)^s\ e^{-\Omega\left(\frac{y+\eta}{s}\right)}|h_{lj}(z+\zeta)|\to 0 \text{ as } j\to\infty.$$

The number s will be fixed later. One also has for any $z\in \boldsymbol{C}^n$

$$|h_{lj}(z+\zeta)|\leq(1+|z+\zeta|)^{-s}\ e^{\Omega\left(\frac{\eta+y}{s}\right)}\rho_s(h_{lj})\leq(1+|\zeta|)^{-s}(1+|z|)^s\ e^{\Omega\left(\frac{\eta+y}{s}\right)}\rho_s(h_{lj}),$$

hence

$$(28)\qquad \sup_{\substack{|z|\leq 3\rho\\ z\in C^n}}|h_{lj}(z+\zeta)|\leq\rho_s(h_{lj})(1+|\zeta|)^{-s}\sup_{|z|\leq 3\rho}(1+|z|)^s\sup_{|z|\leq 3\rho}e^{\Omega\left(\frac{\eta+y}{s}\right)}$$
$$\leq C_s\rho_s(h_{lj})(1+|\zeta|)^{-s}2^A(1+|\zeta|)^A e^{\Omega\left(\frac{\eta}{s}\right)}2^{\frac{12A}{s}}(1+|\zeta|)^{\frac{12A}{s}}e^{\Omega\left(\frac{14}{s}\eta\right)}$$
$$\leq C_s 2^{13A}\rho_s(h_{lj})(1+|\zeta|)^{13A-s}\ e^{\Omega\left(\frac{15}{s}\eta\right)},$$

where C_s is a constant which depends on s only.
Using (28) to estimate the right hand side of (26) one gets

$$(29)\qquad |\hat{\phi}_{lj}(\zeta)|\leq C_1C_2C_s(1+|\zeta|)^{N_1+N_2+13A-s}\ e^{\Omega\left(\left(\frac{15}{s}+\frac{1}{10k}\right)\eta\right)}\ \rho_s(h_{lj}),$$

hence

$$(30)\qquad (1+|\zeta|)^k\ e^{-\Omega\left(\frac{\eta}{k}\right)}|\hat{\phi}_{lj}(\zeta)|\leq C_1C_2C_3(1+|\zeta|)^{k+N_1+N_2+13A-s}$$
$$e^{\Omega\left(\left(\frac{15}{s}+\frac{1}{10k}\right)\eta\right)-\Omega\left(\frac{\eta}{k}\right)}\times\rho_s(h_{lj}).$$

Let $s=\max\{k+N_1+N_2+13A+1, 30k\}$, thus s depends on k only and the same is true for the constant $C=C_1C_2C_s$. Thus one has

$$(31)\qquad \rho_k(\hat{\phi}_{lj})\leq C\rho_s(h_{lj}).$$

From (27) and (31) it follows that for every $l=1,\dots, m$, the sequence $(\hat{\phi}_{lj})$ converges to 0 in $\boldsymbol{K}_M$. This proves the implication.
The implication (b)$\Rightarrow$(c).
This follows immediately from the Hahn-Banach theorem.
The implication (c)$\Rightarrow$(b).
Since K_M^m is metrizable it suffices to show that the map takes bounded sets into bounded sets. By Mackey's theorem the bounded sets are the same in the strong and the weak topologies of K_M^m, hence we prove that the image of every weakly bounded subset of $S^t*K_M^m$ is weakly bounded in K_M^m. Let B be weakly bounded in $S^t*K_M^m$ and let $v\in K_M'^m$, since $S*K_M'^m=K_M'^m$ there exist a $u\in K_M'^m$ and a constant C which depends on u only such that

$$|<v, \phi>|=|<S*u, \phi>|=|<u, S^t*\phi>|\leq C$$

whenever $S^t*\phi\in B$. This proves the implication.

Closing remarks: (1) I beleive that the implication (b)$\Rightarrow$(a) of Theorem 3 is correct but I don't have a complete proof for it.
(2) The question of finding necessary and sufficient conditions for solvability of underdetermined systems of convolution equations in K'_M is an interesting one, and still open.

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