Hokkaido Mathematical Journal Vol. 17 (1988), p. 83～92

Construction of a parametrix for a weakly hyperbolic differential operator

Fausto SEGALA

(Received April 7, 1987, Revised June 23, 1987)

1. Introduction

In this paper we construct a microlocal parametrix for the Cauchy problem:

$$(1.1)\qquad \begin{cases} (D_t^2-A(t, x, D_x))u(t, x)=0, & (t, x)\in[0, T]\times \boldsymbol{R}^N \\ u(0, x)=g_0(x) & , x\in \boldsymbol{R}^N \\ \partial u/\partial t(0, x)=g_1(x) & , x\in \boldsymbol{R}^N \end{cases}$$

for small $T>0$, where $A(t, x, D_x)$ is a second order differential operator whose principal symbol $A_2(t, x, \xi)$ satisfies the following conditions:

$$(1.2)\qquad A_2(t, x, \xi)\ \geq 0 \text{ for } t\geq 0$$

$$(1.3)\qquad \frac{\partial A_2}{\partial t}(0, x, \xi)\neq 0 \text{ where } A_2(0, x, \xi)=0,\ \xi\neq 0$$

Conditions (1. 2), (1. 3) imply that in a conic neighborhood of a point (x_0, ξ_0) for which $A_2(0, x_0, \xi_0)=0$ and for small $t\geq 0$, we can factor A_2 as

$$(1.4)\qquad A_2(t, x, \xi)=(t+B(x, \xi))C(t, x, \xi)$$

with B homogeneous of degree 0, $B\geq 0$ and C is homogeneous of degree 2 with $C>0$. We recall that a parametrix for (1. 1), under conditions (1. 2), (1. 3) is already known in some particular cases.

When $B\equiv 0$, i. e. $A_2(t, x, \xi)=tC(t, x, \xi)$ (TRICOMI operator), a parametrix for (1. 1) was given by IMAI [2], who treated also the case when B does not depend on x (cfr. IMAI [3]). Furthermore, if B does not depend on ξ, a parametrix for (1, 1) has been constructed in SEGALA [6] (see also YOSHIKAWA [8]). We point out that ESKIN [1] and MELROSE [4] treated the "diffractive" case, where A_2 is factored as in (1. 4) and B satisfies $\nabla_\xi B\neq 0$ where $B=0$.

Here we will treat the general case when conditions (1. 2), (1. 3) are satisfied. Precisely, for every point $(x_0, \xi_0)\in T^*\boldsymbol{R}^N\backslash\{0\}$ where $A_2(0, x_0, \xi_0)=0$, we will find a conic neighborhood Γ of (x_0, ξ_0) and for small $T>0$ construct a continuous operator

$$E: \mathscr{E}'_\Gamma(\boldsymbol{R}^N)\times\mathscr{E}'_\Gamma(\boldsymbol{R}^N)\to C^\infty([0, T], \mathscr{D}'(\boldsymbol{R}^N))$$

such that

$$(1.5)\qquad \begin{cases} PE(g_0, g_1)\in C^\infty([0, T]\times\boldsymbol{R}^N) \\ E(g_0, g_1)_{|t=0}-g_0 C^\infty(\boldsymbol{R}^N) \\ \dfrac{\partial E}{\partial t}(g_0, g_1)_{|t=0}-g_1 C^\infty(\boldsymbol{R}^N) \end{cases}$$

We use the notation $\mathscr{E}'_\Gamma=\{v\in\mathscr{E}'(\boldsymbol{R}^N)|\,WF(v)\subset\Gamma\}$.

2. Eikonal equation

We consider the eikonal equation associated with the operator D_t^2-A:

$$(2.1)\qquad \left(\frac{\partial\varphi}{\partial t}(t, x, \xi)\right)^2=A_2(t, x, \nabla_x\varphi(t, x, \xi)).$$

Let $(x_0, \xi_0))\in T^*\ \boldsymbol{R}^N\backslash\{0\}$ for which $B(x_0, \xi_0)=0$. We have the following non-trivial result proved by IMAI[3].

PROPOSITION 2.1.

There exist a conic neighborhood $[0, T]\times\Gamma$, $\Gamma\subset T^*\ \boldsymbol{R}^N\backslash\{0\}$, *of* $(0, x_0, \xi_0)$ *and two real functions* $\theta(t, x, \xi)$, $\rho(t, x, \xi)\in C^\infty(\boldsymbol{R}^{N+1}\times\boldsymbol{R}^N\backslash\{0\})$ *homogeneous of degree* 1 *and* 2/3 *respectively, such that*

$$(2.2)\qquad \rho(t, x, \xi)=0 \text{ on } t+B(x, \xi)=0$$

$$(2.3)\qquad \frac{\partial\rho}{\partial t}(0, x_0, \xi_0)>0$$

$$(2.4)\qquad \frac{\partial\theta}{\partial t}(t, x, \xi)=0 \text{ on } t+B(x, \xi)=0$$

$$(2.5)\qquad \det\ (\theta_{x\xi}(0, x_0, \xi_0))\neq 0.$$

Moreover, putting $\varphi_\pm=\theta\pm 2/3\ \rho^{3/2}$, *we have*

$$(2.6)\qquad \left(\frac{\partial\varphi_\pm}{\partial t}(t, x, \xi)\right)^2=A_2(t, x, \nabla_x\varphi_\pm(t, x, \xi))$$

for $(t, x, \xi)\in[0, T]\times\Gamma\cap\{(t, x, \xi)|\rho(t, x, \xi)\geq 0\}=X$.
From (2.4) *and* (2.6) *it follows that on a conic neighborhood of* (x_0, ξ_0), *we have*:

$$(2.7)\qquad B(x, \xi)=B(x, \nabla_x\tilde{\theta}(x, \xi))$$

where $\tilde{\theta}(x, \xi)=\theta(-B(x, \xi), x, \xi)$.

3. Transport equations

For simplicity, we assume in the sequel that $A(t, x, \xi)=A_2(t, x, \xi)=\sum_{p,q=1}^{N} a_{pq}(t, x)\xi_p\xi_q$. This is not a restriction, because we shall see that adding terms of degree 1 and 0 to A will not influence the construction. Our aim is to find two amplitudes $k_\pm$ in such a way that

$$(D_1^2-A)(e^{i\varphi_\pm}k_\pm)\equiv 0$$

on some conic neighborhood $[0, T']\times\Gamma'\subset[0, T]\times\Gamma$ of $(0, x_0, \xi_0)$.

From (2.6) it follows that on X (We omit $\pm$ and use the notation $g_t=\partial g/\partial t$, $g_q=\partial g/\partial x_q$);

$$(3.1)\qquad \begin{aligned}(D_t^2-A)(e^{i\varphi}k)=e^{i\varphi}\{&2\varphi_tD_tk-2\sum a_{pq}\varphi_pD_qk\\ &-i\varphi_{tt}k+i\sum a_{pq}\varphi_{pq}k+D_t^2k-\sum a_{pq}D_pD_qk\}.\end{aligned}$$

Our idea is to seek k as an asymptotic expansion $\sum_0^\infty k_{-\nu}$ with $k_{-\nu}$ homogeneous (in ξ) of degree-ν. Therefore, collecting terms according to their homogeneity, the relevant contribution to the right hand side of (3.1) is given by

$$e^{i\varphi}\{2\varphi_tD_tk-2\sum a_{pq}\varphi_pD_qk-i(\varphi_{tt}-\sum a_{pq}-\varphi_{pq})k\}$$

Since $\varphi_{tt}=\theta_{tt}+1/2\,\rho^{-1/2}\rho_t^2+\rho^{1/2}\rho_{tt}$, the coefficient $P\varphi$ of k has a singularity of the type $\rho^{-1/2}$. To overcome this obstacle we seek formally k of the form $\rho(t, x, \xi)^{-1/4}\sum_0^\infty k_{-\nu}(t, x, \xi)$.

with $k_{-\nu}(t, x, \xi)$ homogeneous (in ξ) of degree-ν. Consider the change of variable $t\to\rho(t, x, \xi)$ with its inverse $\rho\to t(\rho, x, \xi)$ and put

$$\tilde{k}_{-\nu}(\rho, x, \xi)=k_{-\nu}(t(\rho, x, \xi), x, \xi)$$

The transport equation for $\tilde{k}_0$ becomes:

$$\begin{aligned}&-ie^{i\varphi}\rho^{-/4}\{2(\sqrt{\rho}\,\rho_t^2+\theta_t\rho_t-\sum a_{pq}\theta_p\rho_q-\sqrt{\rho}\sum a_{pq}\rho_p\rho_q)\frac{\partial\tilde{k}_0}{\partial\rho}\\ &-2\sum(a_{pq}\theta_p+\sqrt{\rho}\,a_{pq}\rho_p)\frac{\partial\tilde{k}_0}{\partial x_q}\\ &+(-\frac{1}{2}\theta_t\rho_t\rho^{-1}+\theta_{tt}-\sqrt{\rho}\,\rho_{tt}-\sum a_{pq}\theta_{pq}-\sqrt{\rho}\sum a_{pq}\\ &+\frac{1}{2}\sum a_{pq}\theta_p\rho_q\rho^{-1})\tilde{k}_0\}=0.\end{aligned}$$

By the results of sect. 2 it follows that the coefficient of $\dfrac{\partial\tilde{k}_0}{\partial\rho}$ can be written as $2\sqrt{\rho}\quad a(\sqrt{\rho}, x, \xi)+<\tilde{\theta}_{x\xi}^{-1}(x, \xi)\nabla_{\xi}B(x, \xi), \quad \nabla_{x}B(x, \xi)>q(x, \xi)$, where $a(t, x, \xi)$ *is* C^{∞} and elliptic for $s\geq 0$ and for (x, ξ) in a conic neighborhood of (x_0, ξ_0)($<, >$ denotes the scalar product in $\boldsymbol{R}^N$). Moreover $a(\sqrt{\rho}(t, x, \xi), x, \xi)$ is homogeneous of degree 4/3. Finally $q(x, \xi)$ is homogeneous of degree 8/3 and elliptic in a conic neighborhood of (x_0, ξ_0).

The coefficient of $\dfrac{\partial\tilde{k}_0}{\partial x_q}$ is a C^{∞} function of $(\sqrt{\rho}, x, \xi)$ and the coefficient of $\tilde{k}_0$ can be written as a sum of the type $c(\sqrt{\rho}, x, \xi)+\rho^{-1}e(x, \xi)$, where c is a C^{∞} function and $e(x, \xi)=<\tilde{\theta}_{x\xi}^{-1}\nabla_{\xi}B(x, \xi), \nabla_{x}B(x, \xi)>p(x, \xi)$, with p elliptic of order 8/3. We rewrite the equation for $\tilde{k}_0$ as:

$$(3.2)\qquad (2\sqrt{\rho}\ a+b)\frac{\partial\tilde{k}_0}{\partial\rho}+<L, \frac{\partial\tilde{k}_0}{\partial x}>+(c+\rho^{-1}e)\tilde{k}_0=0.$$

We seek $\tilde{k}_0(\rho, x, \xi)$ as a sum $\sum_0^{\infty}\tilde{k}_0^j(\rho, x, \xi)$ with $\tilde{k}_0^j(\rho(t, x, \xi)x, \xi)$ homogeneous of degree 0.

Precisely we have the following equations (we add the initial condition for $\tilde{k}_0^0$)

$$(3.3)_0\qquad \begin{cases} 2\sqrt{\rho}\ a\dfrac{\partial\tilde{k}_0^0}{\partial\rho}+<L, \dfrac{\partial\tilde{k}_0^0}{\partial x}>+c\ \tilde{k}_0^0=0 \\ \tilde{k}_{0|\rho=0}^0=1 \end{cases}$$

$$(3.3)_j\qquad 2\sqrt{\rho}\ a\frac{\partial\tilde{k}_0^j}{\partial\rho}+<L, \frac{\partial\tilde{k}_0^j}{\partial x}>+c\ \tilde{k}_0^j$$
$$=-b\frac{\partial\tilde{k}_0^{j-1}}{\partial\rho}-\rho^{-1}e\ \tilde{k}_0^{j-1},\ j\geq 1.$$

By the change of variable $\rho=s^2$, and by writing $\hat{k}(s, x, \xi)=\tilde{k}(s, x, \xi)$, we obtain the following equations with new a, b, c, L, e

$$(3.4)_0\qquad \begin{cases} \dfrac{\partial\hat{k}_0}{\partial s}+<\dfrac{L}{a}, \dfrac{\partial\hat{k}_0^0}{\partial s}>+\dfrac{c}{a}\hat{k}_0^0=0 \\ \hat{k}_{0|s=0}^0=1 \end{cases}$$

$$(3.4)_j\qquad \frac{\partial\hat{k}_0}{\partial s}+<\frac{L}{a}, \frac{\partial\hat{k}_0^j}{\partial x}>+\frac{c}{a}\hat{k}_0^j$$
$$=-\frac{1}{a}\left[\frac{b}{2s}\ \frac{\partial\hat{k}_0^{j-1}}{\partial s}+\frac{e}{s^2}\hat{k}_0^{j-1}\right]=\hat{f}_0^0,\ j\geq 1.$$

The solution of $(3.4)_0$ is a C^{∞} function $\hat{k}_0^0(s, x, \xi)$ and $\hat{k}_0^0(\sqrt{\rho}(t, x, \xi), x, \xi)$ is homogeneous of degree 0.

To solve $(3.4)_j$, we write $\hat{k}_0^j=h_j\hat{k}_0^0$ and then we have

$$(3.5)\qquad \frac{\partial h_j}{\partial s}+<\frac{L}{a}, \frac{\partial h_j}{\partial x}>=\frac{\hat{f}_0^j}{\hat{k}_0^0}$$

Now consider the system

$$\begin{cases} \dot{x}(s)=L(s, x(s), \xi) \\ x(0)=x_0 \end{cases}$$

and let $x(s, x, \xi)$ be its solution. If $x_0(s, x, \xi)$ is the inverse map, we solve (3.5) by taking

$$(3.6)_0^j \qquad h_j(s, x, \xi)=\int_{|\xi|^{1/3}}^{s} (\hat{f}_0^j/\hat{k}_0^0)(z, x(z, x_0(s, x, \xi), \xi), \xi)\, dz.$$

The problem to sum $\sum_0^\infty \hat{k}_0^j$ will be examined later on. Now consider the transport equation for $\tilde{k}_{-\nu}$, $\nu\geq 1$. By (3.1) we have the equation

$$\begin{aligned}(3.7) \qquad &(2\sqrt{\rho}\ a+b)\frac{\partial\tilde{k}_{-\nu}}{\partial\rho}+<L, \frac{\partial\tilde{k}_{-\nu}}{\partial x}>+(c+\rho^{-1}e)\tilde{k}_{-\nu} \\
&=\frac{5}{16}\rho^{-2}(\rho_t^2-\sum a_{ij}\rho_i\rho_j)\tilde{k}_{-\nu+1} \\
&-\frac{1}{4}\rho^{-1}(2\rho_t^2\frac{\partial\tilde{k}_{-\nu+1}}{\partial\rho}+\rho_{tt}\tilde{k}_{-\nu+1}-2\sum a_{ij}\rho_i\rho_j\frac{\partial\tilde{k}_{-\nu+1}}{\partial\rho} \\
&-\sum a_{ij}\rho_{ij}\tilde{k}_{-\nu+1}-2\sum a_{ij}\rho_i\frac{\partial\tilde{k}_{-\nu+1}}{\partial x_j}) \\
&+\frac{\partial^2\tilde{k}_{-\nu+1}}{\partial\rho^2}\rho_t^2-\frac{\partial\tilde{k}_{-\nu+1}}{\partial\rho}\rho_{tt}-\sum a_{ij}\rho_i\rho_j\frac{\partial^2\tilde{k}_{-\nu+1}}{\partial\rho^2} \\
&-2\sum a_{ij}\rho_i\frac{\partial^2\tilde{k}_{-\nu+1}}{\partial\rho\partial x_j}\ \frac{\partial^2\tilde{k}_{-\nu+1}}{\partial\rho}-\sum a_{ij}-\sum a_{ij}\rho_{ij}\frac{\partial^2\tilde{k}_{-\nu+1}}{\partial x_j\partial x_j}=f_{-\nu}^0.\end{aligned}$$

Formally, we write $\tilde{k}_{-\nu}=\sum_0^\infty \tilde{k}_{-\nu}^j$ with $\tilde{k}_{-\nu}^j$ $(\rho((t, x, \xi), x, \xi)$ homogeneous of degree 0. Then

$$(3.8)_\nu^0 \qquad 2\sqrt{\rho}\ a\frac{\partial\tilde{k}_{-\nu}^0}{\partial\rho}+<L, \nabla_x\tilde{k}_{-\nu}^0>+c\ \tilde{k}_{-\nu}^0=f_{-\nu}^0.$$

$$\begin{aligned}(3.8)_\nu^j \qquad &2\sqrt{\rho}\ a\frac{\partial\tilde{k}_{-\nu}^j}{\partial\rho}+<L, \nabla_x\tilde{k}_{-\nu}^j>+c\ k_{-\nu}^j \\
&=-b\frac{\partial\tilde{k}_{-\nu}^{j-1}}{\partial\rho}-\rho^{-1}\ c\ \tilde{k}_{-\nu}^{j-1}\end{aligned}$$

We make the change of variable $\rho=s^2$ and so, $(3.8)_\nu$, $(3.9)_\nu$ assume the form

$$(3.9)_\nu^j \qquad \frac{\partial\hat{k}_{-\nu}^j}{\partial s}+<\frac{L}{a}, \nabla_x\hat{k}_{-\nu}^j>+\frac{C}{a}\hat{k}_{-\nu}^j=\hat{f}_{-\nu}^j$$

for $j=0, 1, 2, \ldots$. We seek a soluion of $(3.9)_\nu^j$ on the form $\hat{k}_{-\nu}^j=h_{-\nu}^j k_0^0$. Then, we can take

$$(3.10)^j_\nu \quad h^j_{-\nu}(s, x, \xi)=\int_{|\xi|^{1/3}}^{s}(f^j_{-\nu}/\hat{k}^0_0)(z, x(z, x_0(s, x, \xi), \xi), \xi)\, dz.$$

To give a meaning to the sum $\sum_\nu \sum_j \hat{k}^j_{-\nu}$, we introduce a class of symbols. We say that a function $p(s, x, \xi)\in C^\infty([1, \infty]\times \boldsymbol{R}^N\times \boldsymbol{R}^N)$ is a symbol of class $S^{M,Q}(M,\ Q\in \boldsymbol{R})$ if

$$|D^j_s D^\alpha_\xi D^\beta_x p(s, x, \xi)|\le Cs^{M-j}|\xi|^{Q-2|\alpha|/3+|\xi|/3}$$

for $s\ge 1,\ x\in \boldsymbol{R}^N,\ |\xi|\ge 1$.
Then, we can write

$$(3.11) \quad \hat{k}^0_0(s, x, \xi)=1+s\ m^0_0(s, x, \xi)$$

with $m_0\in S^{1,-1/3}$. This follows from the fact that m_0 is C^∞ for $s\ge 0$ and m_0 $(\sqrt{\rho}(t, x, \xi), x, \xi)$ is homogeneous of degree $-1/3$.
Now observe that for $t\ge 0$,

$$(3.12) \quad \begin{aligned}&|<\tilde{\theta}^{-1}_{x\xi}\nabla_\xi B(x, \xi),\ \nabla_x B(x, \xi)>|q(x, \xi)\\ &\le CB(x, \xi)|\xi|^{5/3}\le C(t+B(x, \xi))|\xi|^{2/3}|\xi|\le C'\rho|\xi|.\end{aligned}$$

From (3. 11) and (3. 12), by using an inductive argument as in [5], we conclude that for $j\ge 1$:

$$(3.13)^j_0 \qquad \hat{k}^j_0\in S^{j,-j/3}$$

We observe that for $s|\xi|^{-1/3}\le 1/2$, $S^{j+1,-(j+1)/3}\subset S^{j,-j/3}$. Then it is possible to define $\hat{k}'_0\in S^{1,-1/3}$ in such a way that $\hat{k}'_0-\sum_1^N \hat{k}^j_0\in S^{N+1,-(N+1)/3}$ for every $N\in \boldsymbol{N}$.
Similarly, for $\nu\ge 1$ we can write (see [5])

$$(3.14) \quad \hat{k}_{-\nu}=(-1)^\nu\frac{\prod_1^\nu(6m\text{-}5)\prod_1^\nu(6m\text{-}1)}{\nu\,!\,48^\nu}(s^{-3\nu}+\hat{k}'_{-\nu})$$

with $k'_{-\nu}\in S^{1,-1/3}$. We introduce the notation $f^\#(s, x, \xi)=f(t(s^2, x, \xi), x, \xi)$. By $p(t, x, \xi)\cong\sum p_j(t, x, \xi)$, we mean that $p^\#\cong\sum p^\#_j$. We denote by $\alpha^\pm_{-\nu}$ the coefficient of the asymptotic expansion of the Airy functions $A_\pm(z)=2\pi e^{\pm i2\pi 3}$ $Ai(e^{\pm i2\pi 3}(-z))$(see WASOW [7]).
Summing up, we have the following result

PROPOSITION 3. 1.
There exist $\mu^\pm_{-\nu}\in S^{0,-1/3}$ *such that* $k_\pm$ *defined by*

$$(3.15) \quad \begin{aligned}&k_\pm(t, x, \xi)\cong\rho(t, x, \xi)^{-1/4}\times\\ &\sum_0^\infty \rho(t, x, \xi)^{-3\nu/3}(\alpha^\pm_{-\nu}+\sqrt{\rho}\,\mu^\mp_{-\nu}(\sqrt{\rho(t, x, \xi)}, x, \xi)),\end{aligned}$$

satisfy :

(3.16) $\quad (D_t^2-A)(e^{i\varphi_\pm}k_\pm)=e^{i\varphi_\pm}(|\xi|^{4/3}h_\pm+n_\pm)$

With $h_\pm^\#\in S^{-\infty,0}$ *and* $n_\pm^\#\in\bigcap_0^\infty S^{j,-j/3}$ *for* (x,ξ) microlocally near to (x_0,ξ_0) and $s|\xi|^{-1/3}\in[0,T]$.
From (3.15) *it follows that*

(3.17) $\quad (e^{i\varphi_\pm}k_\pm/A_\pm-1)^\#\in S^{1,-1/3}\oplus S^{-\infty,0}.$

Since $\varphi_\pm(t,x,\xi)=\tilde\theta(x,\xi)+\rho(t,x,\xi)^2\delta(t,x,\xi)\pm2\rho(t,x,\xi)^{3/2}/3$ *with* $\tilde\theta$ *homogeneous of degree* 1 *and* δ *homogeneous of degree*-1/3, *we can write (from* (3.16)) :

(3.18) $\quad (D_t^2-A)(e^{i\varphi_\pm}k_\pm)=|\xi|^{4/3}e^{i\tilde\theta_\pm}h_\pm+e^{i\varphi_\pm}n_\pm$

for some new $h_\pm^\#\in S^{-\infty,0}$.
Now set $\zeta(t,x,\xi)=(t+B(x,\xi))A_t'(-B(x,\xi))^{1/3}$ and introduce a cut-off function $\chi(\zeta)$ which equals 0 for $\zeta\le1$ and equals 1 for $\zeta\ge M>>1$. Then, by defining $B_\pm=e^{i(\theta-\tilde\theta\pm2\rho^{3/2}/3)}\,k_\pm$, we obtain

$$
\begin{aligned}
(3.19)\quad &(D_t^2-A)(e^{i\varphi_\pm}(\chi k_\pm))=(D_t^2-A)(\chi e^{i\tilde\theta}e^{i\theta-\tilde\theta\pm2\rho^{3/2}/3)}k_\pm)\\
&=e^{i\theta}A_t'(-B(x,\xi),x,\xi)^{1/3}(-\chi''B_\pm-2\chi'\frac{\partial B_\pm}{\partial\zeta}+\Sigma^\pm)\\
&+e^{i\tilde\theta}|\xi|^{4/3}\chi h_\pm+e^{i\varphi_\pm}\chi n_\pm,\ \text{with } \Sigma^\pm\in S^{0,-1/3}
\end{aligned}
$$

Eq. (3.19) is a consequence of (3.18) and of the fact that if $g(s)\in S^{M,Q}$, and $q(s)$ has compact support, then $(qg)(s)\in S^{-\infty,Q}$.
Finally,we try to construct two amplitudes $a_\pm(t,x,\xi)\in S^0_{2/3,2/3,1/3}$ (in the sense that $|D_t^jD_\xi^\alpha D_x^\beta a|\le C(1+|\xi|)^{2j/3-2|\alpha|/3+|\beta|/3}$) such that

(3.20) $\quad (D_t^2-A)(e^{i\varphi_\pm}(\chi k_\pm)+e^{i\tilde\theta}a_\pm)\in S^\infty_{2/3,2/3,1/3}$

We seek $\tilde a_\pm$ as an asymptotic development $\sum_0^\infty a_\nu^\pm$ with $a_\nu^\pm\in S^{\nu/3}_{2/3,2/3,1/3}$. By putting $a_\pm(\zeta(t,x,\xi),x,\xi)=a_\pm(t,x,\xi)$, we have

$$
\begin{aligned}
&(D_t^2-A)(e^{i\tilde\theta}a_\pm)\\
&=e^{i\tilde\theta}A_t'(-B(x,\xi),x,\xi)^{2/3}\frac{\partial^2\tilde a_\pm}{\partial\zeta^2}\\
&-e^{i\theta}\{A(t,x,\nabla_x\tilde\theta(x,\xi))\tilde a_\pm-<\nabla_xA(t,x,\nabla_x\tilde\theta(x,\xi)),\\
&\frac{\partial\tilde a_\pm}{\partial\zeta}\nabla_x\zeta(t,x,\xi)>-<\nabla_xA(t,x,\nabla_x\tilde\theta(x,\xi)),\ \nabla_x\tilde a>+\dots\}
\end{aligned}
$$

Observe that $A(-B(x,\xi),x,\nabla_x\tilde\theta(x,\xi))=0$ and hence we can write (at least formally), $A(t,x,\xi)=\sum_1^\infty(t+B(x,\xi))^iA^{(j)}(-B(x,\xi),x,\xi)/j!$. This implies that the transport equations for $\tilde a_\pm$ are of the following type:

$$\frac{\partial^2 \tilde{a}_\pm}{\partial \zeta^2}+\zeta\tilde{a}_\pm+A'_t(-B(x,\xi),x,\xi)^{-2/3}L\tilde{a}_\pm=F_\pm+G_\pm$$

where $F_\pm=\chi'' B_\pm+2\chi'\frac{\partial B_\pm}{\partial\zeta}+\Sigma^\pm+\chi h_\pm$, $G_\pm=\chi|\xi|^{-4/3}e^{i[\theta-\tilde{\theta}+\frac{2}{3}\rho^{2/3}]}n_\pm$ and $L=[B'_\xi(x,\nabla_x\tilde{\theta}(x,\xi))C(x,\xi)\nabla_x B(x,\xi)+\zeta\nabla_x \mathrm{C}(x,\xi)\nabla_x \mathrm{B}(x,\xi)]\frac{\partial}{\partial\rho}+\dots$. Then $a^0_\pm$ will be a solution of

$$(3.21)_0 \quad \frac{\partial^2 \tilde{a}^0_\pm}{\partial\zeta^2}+\zeta\tilde{a}^0_\pm=F_\pm+G_\pm$$

and $a^\nu_\pm$, for $\nu\geq 1$, will be a solution of

$$(3.21)_\nu \quad \frac{\partial^2 \tilde{a}^\nu_\pm}{\partial\zeta^2}+\zeta\tilde{a}^\nu_\pm=L\tilde{a}^{\nu-1}_\pm$$

We take as $\tilde{a}^0_\pm$, te following solutions of $(3.21)_0$:

$$\begin{aligned}(3.22)_0 \quad \tilde{a}^0_\pm(\zeta,x,\xi)=&\frac{A_+(\zeta)}{\gamma}[\int_\zeta^\infty A_-(z)F_\pm(z,x,\xi)dz\\ &-\int_0^\zeta A_-(z)G_\pm(z,x,\xi)dz]\\ &-\frac{A_-(\zeta)}{\gamma}[\int_\zeta^\infty A_+(z)F_\pm(z,x,\xi)dz-\int_0^\zeta A_+(z)G_\pm(z,x,\xi)dz]\end{aligned}$$

where $\gamma=A_+(\zeta)A'_-(\zeta)-A'_+(\zeta)A_-(\zeta)$ (which is a constant$\neq 0$).
It is not difficult to see that

$$(3.23) \quad \tilde{a}^0_\pm(\zeta,x,\xi)=b^0_\pm(\zeta,x,\xi)+c^0_\pm(\zeta,x,\xi)|\xi|^{-2/3}$$

with $b^0_\pm\in S^{-\infty,0}$ and $c^0_\pm\in\bigcap_0^\infty S^{j,-j/3}$.

Moreover we take

$$\begin{aligned}\tilde{a}^1_\pm(\zeta,x,\xi)=&\frac{A_+(\zeta)}{\gamma}[\int_\zeta^\infty A_-(t)Lb^0_\pm(z,x,\xi)dz\\ &-\int_0^\zeta A_-(z)L(c^0_\pm(z,x,\xi)|\xi|^{-2/3}dz\\ &-\frac{A_+(\zeta)}{\gamma}[\int_\zeta^\infty A_+(z)Lb^0_\pm(z,x,\xi)dz-\int_0^\zeta A_+(z)L(c^0_\pm(z,x,\xi)|\xi|^{-2/3}dz\end{aligned}$$

and $\tilde{a}^1_\pm$ can be written on the form $b^1_\pm+c^1_\pm|\xi|^{-2/3}$ with $b^1_\pm\in S^{-\infty,-1/3}$, $c^1_\pm\in\bigcap_0^\infty S^{j,-j/3}$. Now, the choice of $a^\nu_\pm$ for $\nu>1$ is obvious. From (3.23) it follows that $a^0_\pm\in S^0_{2/3,2/3,1/3}$ and by $(3.22)_0$ we obtain that the principal contribution to $a^0_\pm$ is given by

$$\frac{A_+(\zeta)}{\gamma}\int_\zeta^\infty A_-(\chi''B_\pm+2\chi'\frac{\partial B_\pm}{\partial\zeta})\,dz$$
$$+\frac{A_-(\zeta)}{\gamma}\int_\zeta^\infty A_-(\chi''B_\pm+2\chi'\frac{\partial B_\pm}{\partial\zeta})\,dz.$$

Taking into account (3.17), we conclude that the principal part of $a_\pm^0$ is given by

$$\frac{A_+(\zeta)}{\gamma}\int_\zeta^\infty A_-(\chi''A_\pm+2\chi'\frac{\partial A_\pm}{\partial\zeta})\,dz$$
$$+\frac{A_-(\zeta)}{\gamma}\int_\zeta^\infty A_-(\chi''A_\pm+2\chi'\frac{\partial A_\pm}{\partial\zeta})\,dz.$$

From the identity

$$\int_\zeta^\infty A_\pm(Y''+\rho Y)\,d\zeta=\int_\zeta^\infty(A_\pm Y''-A_\pm''Y)\,d\zeta=[A_\pm Y'-A_\pm'Y]_\zeta^\infty$$

we obain that the principal contribution to a_+^0 is given by

$$\frac{A_+}{\gamma}[A_-\chi A_+'+A_-\chi'A_+-\chi A_+A_-']_\zeta^\infty+$$
$$\frac{A_-}{\gamma}[A_+\chi'A_++A_+\chi A_+'-\chi A_+A_+']_\zeta^\infty=\frac{A_+}{\gamma}[\chi\gamma]_\zeta^\infty=(1-\chi)A_+$$

and therefore

$$(3.24)\qquad a_0^\pm(t,x,\xi)-(1-\chi(\zeta(t,x,\xi))A_\pm(\zeta(t,x,\xi))\in S^{-2/3}_{2/3,2/3,1/3}$$

Since $\rho-\zeta\in S^{2,-2/3}$ we conclude that

$$(3.25)\qquad a_0^\pm(t,x,\xi)-(1-\chi(\rho(t,x,\xi))A_\pm(\rho(t,x,\xi))\in S^{-2/3}_{2/3,2/3,1/3}$$

4. Construction of the microlocal parametrix

We define the operators

$$E_\pm(t)\,f(x)=\iint e^{i[\theta(t,x,\xi)-\theta(0,y,\xi)]}\frac{a_\pm(t,x,\xi)}{A_\pm(\rho(0,x,\xi))}f(y)\,dy\,d\xi$$
$$+\iint e^{i[\theta(t,x,\xi)-\theta(0,y,\xi)\pm\frac{2}{3}\rho(t,x,\xi)^{2/3}]}\frac{\chi(\rho(t,x,\xi))k_\pm(t,x,\xi)}{A_\pm(\rho(0,x,\xi))}f(y)\,dy\,d\xi$$

Clearly $E_\pm(0)$ is a pseudodifferential operator and (microlocally) $PE_\pm\equiv0$. From (3.17) and (3.25) it follows that the principal contribution to $E_\pm(t)$ is given by

$$E_\pm^\#(t)\,f(x)=\iint e^{i[\theta(t,x,\xi)-\theta(0,y,\xi)]}\frac{A_\pm(\rho(x,\xi))}{A_\pm(\rho(0,x,\xi))}f(y)\,dy\,d\xi$$

and the matrix

$$\begin{bmatrix} E_+^{\#}(0) & E_-^{\#}(0) \\ D_tE_+^{\#}(0) & D_tE_-^{\#}(0) \end{bmatrix}$$

is microlocally invertible. A detailed calculation in a similar case is given for example in [6].

We can write, for $\rho\geq 0$, $A_\pm(\rho)=e^{\pm i2\mu(\rho)\rho^{3/2}/3}\ F_\pm(\rho)$, where $\mu(\rho)=0$ for $\rho\leq 1$, $\mu(\rho)=1$, for $\rho\geq 2$ and $F_\pm(\rho)\in C^\infty[0,\infty]$, $F_\pm(\rho)\sim\rho^{-1/4}\sum_0^{+\infty} c_\pm^{-\nu}\rho^{-3\nu/2}$. So, we obtain

$$E_\pm^{\#}(t)\ f(x)=\iint e^{i\Psi_\pm(t,x,y,\xi)}h_\pm(t,x,y,\xi)f(y)\ dy\ d\xi$$

where

$$\begin{aligned}\Psi_\pm(t,x,y,\xi)&=\theta(t,x,\xi)-\theta(0,x,\xi)\\ &\pm(2/3)\mu(\rho(t,x,\xi))\rho(t,x,\xi)^{3/2}\mp(2/3)\mu(\rho(0,x,\xi))\rho(0,x,\xi)^{3/2}\end{aligned}$$

and $h_\pm$ is a pseudo-differential symbol of order 0. Then, we can describe the wave front set of $E_\pm^{\#}(t)f$, for fixed $t>0$, by following MELROSE [4]. We denote by $\gamma_\pm(t,y,\eta)$, the null-bicharacteristics of $\tau\pm(t+B(x,\xi))^{1/2}C(t,x,\xi)^{1/2}$ passing through (y,η). Therefore we have that, for fixed $t>0$,

$$WFE_\pm^{\#}(t)f=\{\gamma_\pm(t,y,\eta)\,|\,(y,\eta)\in WFf\}.$$

References

[1] G. ESKIN, A parametrix for mixed problem for strictly hyperbolic equation of arbitrary order. Comm. In PDE, (16) (1976), 521-560.

[2] M. IMAI, Cauchy problem for operators of Tricomi's type. Hokkaido Math. J. 8 (1979), 126-143.

[3] M. IMAI, On a parametrix for a weakly hyperbolic operator. Hokkaido Math. J. 9 (1980), 190-216.

[4] R. B. MELROSE, Local Fourier-Airy integral operators. Duke Math. J. 42 (4), 583-604.

[5] F. SEGALA, Parametrices for a clafss of differential operators with multiple characteristics. Ann. Math. Pura e Appl. (to appear).

[6] F. SEGALA, A simple construction of a parametrix for a regular hyperbolic operator (to appear).

[7] W. WASOW, A symptotic expansions for ordinary differential equatins. Interscience 1965.

[8] A. YOSHIKAWA, A parametrix for a completely regularly hyperbolic operator. Hokkaido Math. J. 10 (1981), 723-742.

Dipartimento di Matematica
Università di Bologna