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Expansive homeomorphisms of solenoidal groups

N. AOKI and K. MORIYASU
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§ 1. Introduction

Our investigation will be related to a conjugacy problem in a class of expansive homeomorphisms of solenoidal groups. Every compact connected finite-dimensional abelian group is called *solenoidal* (classical solenoid). Thus finite-dimensional tori are solenoidal.

Let X be a metric space with metric d. A homeomorphism $f: X\rightarrow X$ is called *expansive* if there is a constant $c>0$ such that $x\neq y$ implies $d(f^{n}(x), f^{n}(y))>c$ for some integer n. Such a constant c is an *expansive constant* of f. A continuous surjection $f: X\rightarrow X$ is positively expansive if there is a constant $c>0$ such that if $x\neq y$ then $d(f^{n}(x), f^{n}(y))>c$ for some non-negative integer n (here c is called an expansive constant of f). For f : $X\rightarrow X$ a continuous surjection, we let $\boldsymbol{X}_{f}=\{(x_{i}) : x_{i}\in X$ and $f(x_{i})=x_{i+1}$, $i\in \boldsymbol{Z}\}$. Then $f: X\rightarrow X$ is called *c-expansive* if there is a constant $c>0$ such that for (x_{i}), $(y_{i})\in\boldsymbol{X}_{f}$ if $d(x_{i}, y_{i})\leqq c$ for $i\in \boldsymbol{Z}$ then $(x_{i})=(y_{i})$. For compact spaces these notions are independent of a compatible metric used. c-expansiveness for continuous surjections is weaker than positive expansiveness. For homeomorphisms c-expansiveness implies expansiveness.

A sequence of points $\{x_{i}: a<i<b\}$ of a metric space X is called a *δ-pseudo orbit* of a continuous surjection f if $d(f(x_{i}), x_{i+1})\leqq\delta$ for $i\in(a, b-1)$. For $\varepsilon>0$ a δ-pseudo orbit $\{x_{i}\}$ is called to be *ε-traced* by a point $x\in X$ if $d(f^{i}(x), x_{i})\leq\varepsilon$ for $i\in(a, b)$. Here the symbols a and b are taken as $-\infty\leq a<b\leq\infty$ if f is bijective and as $0\leq a<b\leq\infty$ if f is not bijective. We call f to have the *pseudo orbit tracing property* (abbrev: POTP) if for every $\varepsilon>0$ there is $\delta>0$ such that each δ-pseudo orbit of f can be ε-traced by some point of X. For compact spaces this is independent of a compatible metric used.

We say a continuous covering map $f: X\rightarrow X$ is a *c-map* if f is c-expansive and satisfies *POTP*. A c-map $f: X\rightarrow X$ is called *special* if f satisfies the following: for every (x_{i}), $(y_{i})\in\boldsymbol{X}_{f}$ with $x_{0}=y_{0}$,

$$W^{u}((x_{i}))=W^{u}((y_{i}))$$

where $W^{u}((x_{i}))=\{z_{0}\in X$: there is $(z_{i})\in\boldsymbol{X}_{f}$ such that $d(x_{-i}, z_{-i})\rightarrow 0$ as i

$\to\infty\}$.

For a conjugacy problem, a problem of whether every Anosov diffeomorphism of an infra-nil-manifold is topologically conjugate to an algebraic hyperbolic map has been studied by Shub [12], Franks [6, 7], Manning [10] and etc. For continuous maps of finite-dimensional tori, the following was proved in [3, 4, 9]:

(i) every homeomorphism with both expansiveness and *POTP* is topologically conjugate to a hyperbolic toral automorphism ([9]),

(ii) every positively expansive map is topologically conjugate to an expanding toral endomorphism ([3]),

(iii) every special c-map, which is neither bijective nor positively expansive, is topologically conjugate to a hyperbolic toral endomorphism ([4]),

(iv) for every c-map which is not special, there is a finite cover and a c-map on which their inverse limit is topologically conjugate to a solenoidal automorphism ([4]).

For continuous surjection f of an n-solenoidal group, the following is known [2].

FACT 1.1. Assume that f is a local homeomorphism and $f(0)=0$ where 0 denotes the identity of X, then there is a totally disconnected subgroup F such that

(a) X/F is an n-torus,

and for each $\lambda>0$ there exist

(b) a continuous map $\varphi_f: F\times X\to X$ with diam $(\varphi_f(F\times X))<\lambda$ and

(c) a group endomorphism $\sigma_f: X\to X$ (if f is a homeomorphism then σ_f is a group automorphism)

such that

$$f(x+y)=f(x)+\varphi_f(y, x)+\sigma_f(y) \qquad (x\in X, y\in F).$$

FACT 1.2. Let d_n denote the euclidean metric of $\boldsymbol{R}^n$. Under the assumption of Fact 1.1, there exist a continuous homomorphism $\psi: \boldsymbol{R}^n\to X$ and a d_n-biuniformly continuous bijection $\bar{f}: \boldsymbol{R}^n\to\boldsymbol{R}^n$ such that the diagram

$$\begin{array}{ccc} \boldsymbol{R}^n & \xrightarrow{\ \bar{f}\ } & \boldsymbol{R}^n \\ \psi\downarrow & & \downarrow\psi \\ \psi(\boldsymbol{R}^n) & \xrightarrow[\ f\ ]{} & \psi(\boldsymbol{R}^n) \end{array} \qquad \text{commutes,}$$

and for $\lambda>0$ there are a subgroup C of finite index in $\boldsymbol{Z}^n$, a linear map $\bar{\sigma}_f$:

$\boldsymbol{R}^n \to \boldsymbol{R}^n$ and a continuous map $\bar{\varphi}_f: C \times \boldsymbol{R}^n \to B(0, \lambda)$, where $B(0, \lambda) = \{v \in \boldsymbol{R}^n : d_n(0, v) < \lambda\}$ such that the diagram

$$\begin{array}{ccc} \boldsymbol{R}^n & \xrightarrow{\bar{\sigma}_f} & \boldsymbol{R}^n \\ \psi \downarrow & & \downarrow \psi \\ \psi(\boldsymbol{R}^n) & \xrightarrow[\sigma_f]{} & \psi(\boldsymbol{R}^n) \end{array} \quad \text{commutes and}$$

$$\bar{f}(v+l) = \bar{f}(v) + \bar{\varphi}_f(l, v) + \bar{\sigma}_f(l) \qquad (v \in \boldsymbol{R}^n,\ l \in C).$$

FACT 1.3. If in addition $\bar{f}: \boldsymbol{R}^n \to \boldsymbol{R}^n$ is positively expansive, then so is $\bar{\sigma}_f: \boldsymbol{R}^n \to \boldsymbol{R}^n$ and f is topologically conjugate to a solenoidal group endomorphism σ_f.

Thus it will be natural to ask whether Fact 1.3 holds for a homeomorphism which provides expansiveness and *POTP*. The purpose of this paper is to discuss this problem.

THEOREM 1. *Let $f: X \to X$ be a homeomorphism of an n-solenoidal group with $f(0)=0$, and σ_f, $\bar{\sigma}_f$ and $\bar{f}$ be as in Facts* 1.1 *and* 1.2. *Assume that f is expansive and has POTP. Then $\bar{f}: \boldsymbol{R}^n \to \boldsymbol{R}^n$ is expansive and has POTP with respect to the euclidean metric. If in addition $\bar{\sigma}_f: \boldsymbol{R}^n \to \boldsymbol{R}^n$ is hyperbolic and an expansive constant of $\bar{f}$ is arbitrary, then f is topologically conjugate to a solenoidal automorphism σ_f.*

In Theorem 1 it seems likely that f has a fixed point, and that two the assumptions of the second statement can be dropped. However the authors can not follow them. For one-dimensional case we have the following:

THEOREM 2. *Let X be a one-dimensional solenoidal group and $f: X \to X$ be an expansive homeomorphism with POTP. If f has a fixed point, then f is topologically conjugate to a solenoidal automorphism σ_f which is found as in Fact* 1.1.

§ 2. Topological properties derived from expansiveness and POTP.

Let Y be a metric space with metric d and $f: Y \to Y$ be a (bijective) homeomorphism. If $\varepsilon > 0$ and $x \in Y$, then a local stable set $W^s_\varepsilon(x, f, d)$ and a local unstable set $W^u_\varepsilon(x, f, d)$ are defined by

$$W^s_\varepsilon(x, f, d) = \{y \in Y : d(f^i(x), f^i(y)) \leqq \varepsilon,\ i \geq 0\},$$
$$W^u_\varepsilon(x, f, d) = \{y \in Y : d(f^{-i}(x), f^{-i}(y)) \leqq \varepsilon,\ i \geq 0\}.$$

The following is easily checked: if $f: Y \to Y$ has *POTP*, then for $\varepsilon_0 > 0$ there is $\delta_0 > 0$ such that $d(x, y) < \delta_0 (x, y \in Y)$ implies $W^s_{\varepsilon_0}(x, f, d) \cap W^u_{\varepsilon_0}(y, f, d) \neq \phi$ and if in addition f is expansive and $c > 0$ is an expansive constant, for $0<\varepsilon_0<c/2$ the set $W^s_{\varepsilon_0}(x, f, d)\cap W^u_{\varepsilon_0}(y, f, d)$ is a single point.

Therefore, for the case when $f: Y\to Y$ has expansiveness and *POTP*, a map $[\ ,\ ]: \Delta(\delta_0)\to Y$ is defined by

$$[x, y]=W^s_{\varepsilon_0}(x, f, d)\cap W^u_{\varepsilon_0}(y, f, d)$$

where $\Delta(\delta_0)=\{(x, y)\in Y\times Y: d(x, y)<\delta_0\}$.

Write $d=d_n$ for simplicity and assume that $f: \boldsymbol{R}^n\to\boldsymbol{R}^n$ has expansiveness and *POTP* under the metric $d=d_n$. Then the following lemmas are well known:

LEMMA 2.1. [8] *A map* $[\ ,\ ]: \Delta(\delta_0)\to\boldsymbol{R}^n$ *is continuous and for* $x, y, z\in\boldsymbol{R}^n$

$$[x, x]=x,\ [[x, y], z]=[x, z],\ [x, [y, z]]=[x, z] \text{ and}$$
$$f([x, y])=[f(x), f(y)],$$

where the two sides of these relations are defined.

LEMMA 2.2. [8] *Under the above notations, there are* $0<\delta_1<\delta_0/3$ *and* $0<\rho<\delta_1$ *such that for* $x\in\boldsymbol{R}^n$ *letting*

$$W^\sigma_{\varepsilon_0,\delta_1}(x, f, d)=\{y\in W^\sigma_{\varepsilon_0}(x, f, d): d(x, y)<\delta_1\}\quad \sigma=s,\ u,$$
$$N(x)=[W^u_{\varepsilon_0,\delta_1}(x, f, d),\ W^s_{\varepsilon_0,\delta_1}(x, f, d)],$$

the following holds:

(a) $N(x)$ *is an open subset of* $\boldsymbol{R}^n$,

(b) $\operatorname{diam}(N(x))<2\delta_0/3$,

(c) $[\ ,\ ]: W^u_{\varepsilon_0,\delta_1}(x, f, d)\times W^s_{\varepsilon_0,\delta_1}(x, f, d)\to N(x)$ *is a homeomorphism,*

(d) $N(x)$ *contains the ball* $B_\rho(x)=\{y\in\boldsymbol{R}^n: d(x, y)<\rho\}$.

For $x\in\boldsymbol{R}^n$ we let

$D^s(x)$ is the connected component of x in $W^s_{\varepsilon_0,\delta_1}(x, f, d)$,

$D^u(x)$ is that of x in $W^u_{\varepsilon_0,\delta_1}(x, f, d)$,

and define

$$\bar{N}(x)=[D^u(x), D^s(x)].$$

LEMMA 2.3. *Under the above notations, for* $x\in\boldsymbol{R}^n$

(1) $\bar{N}(x)$ *is connected and open in* $\boldsymbol{R}^n$,

(2) $\operatorname{diam}(\bar{N}(x))<2\delta_0/3$,

(3) $[\ ,\ ]: D^u(x)\times D^s(x)\to\bar{N}(x)$ *is a homeomorphism,*

(4) $\bar{N}(x)\supset B_\rho(x)$.

The proof follows from Lemma 2.2.

LEMMA 2. 4. [11] *Let $c>0$ be an expansive constant of $f: \boldsymbol{R}^n \to \boldsymbol{R}^n$ and assume that*

$$(*)\begin{cases} \text{for } \varepsilon>0 \text{ there is } n_0>0 \text{ such that} \\ \max\{d(f^j(x), f^j(y)): -n_0 \leq j \leq n_0\}<c \\ \text{implies } d(x, y)<\varepsilon. \end{cases}$$

Then there exist a compatible metric $\bar{D}$ of $\boldsymbol{R}^n$, numbers $\varepsilon_1>0$, $a_1>0$ and $0<\lambda_1<1$ such that for $x \in \boldsymbol{R}^n$ and $i \geq 0$

$$\bar{D}(f^i(x), f^i(y)) \leq a_1 \lambda_1^i \bar{D}(x, y) \text{ for } y \in W_{\varepsilon_1}^s(x, f, \bar{D})$$
$$\bar{D}(f^{-i}(x), f^{-i}(y)) \leq a_1 \lambda_1^i \bar{D}(x, y) \text{ for } y \in W_{\varepsilon_1}^u(x, f, \bar{D})$$

Such a metric is called a hyperbolic metric.

Hereafter it will be assume that f has the condition $(*)$.

LEMMA 2. 5. [4] *Let $\varepsilon_1>0$ be as in Lemma 2. 4 and choose $\delta_1>0$ as in Lemma 2. 2 such that $\delta_1<\varepsilon_1$. Then we have*

(1) *if $f: \boldsymbol{R}^n \to \boldsymbol{R}^n$ is not positively expansive and $f^{-1}: \boldsymbol{R}^n \to \boldsymbol{R}^n$ has the same condition, then $D^\sigma(x) \neq (x)$ for $\sigma=s, u$,*

(2) *if $f: \boldsymbol{R}^n \to \boldsymbol{R}^n$ is positively expansive, then $D^s(x)=\{x\}$ and $D^u(x)=\bar{N}(x)$.*

To define generalized foliations, as a space we consider a locally connected metric space M. Let $\mathscr{F}$ be a family of subsets of M. We call $\mathscr{F}$ is a *generalized foliation* if the following holds:

(i) $\mathscr{F}$ is a decomposition of M,

(ii) each $L \in \mathscr{F}$ is arcwise connected,

(iii) if $x \in M$ then there exist non-trivial connected subsets D_x and K_x with $D_x \cap K_x=\{x\}$, a connected open neighborhood N_x of x in M, a homeomorphism $\varphi_x: D_x \times K_x \to N_x$ such that

(a) $\varphi_x(x, x)=x$,

(b) $\varphi_x(y, x)=y$ $(y \in D_x)$, $\varphi_x(x, z)=z$ $(z \in K_x)$,

(c) *for each $L \in \mathscr{F}$* there exists at most countable set $B \subset K_x$ such that $N_x \cap L=\varphi_x(D_x \times B)$.

Let $\mathscr{F}$ be a generalized foliation of M. For fixed $L \in \mathscr{F}$, let $\mathscr{O}_L$ be a family of subsets of L such that for any $D \in \mathscr{O}_L$ there is an open subset O of M such that D is a connected component in $O \cap L$. Then the topology generated by $\mathscr{O}_L$ is called a leaf topology of L. The leaf topology has the following properties;

(1) arcwise connected,

(2) locally connected,

(3) a countable base.

For $x\in\boldsymbol{R}^n$ define a stable set $W^s(x, f)$ and an unstable set $W^u(x, f)$ by

$$W^s(x, f)=\{y\in\boldsymbol{R}^n : d(f^i(x), f^i(y))\rightarrow 0 \text{ as } i\rightarrow\infty\},$$
$$W^u(x, f)=\{y\in\boldsymbol{R}^n : d(f^{-i}(x), f^{-i}(y))\rightarrow 0 \text{ as } i\rightarrow\infty\},$$

Then we have the following:

LEMMA 2.6. [4] *If $f:\boldsymbol{R}^n\rightarrow\boldsymbol{R}^n$ and $f^{-1}:\boldsymbol{R}^n\rightarrow\boldsymbol{R}^n$ are not positively expansive and f has the condition* (*) *of Lemma 2, 4, then $\mathscr{F}^s=\{W^s(x, f): x\in\boldsymbol{R}^n\}$ and $\mathscr{F}^u=\{W^u(x, f): x\in\boldsymbol{R}^n\}$ are generalized foliations.*

§ 3. The structure of solenoidal groups.

The results of solenoidal groups that suffice for our needs are preparated in this section and no new results are discussed here.

Let X be an n-solenoidal group and G be the dual group of X. The rank of G is n (rank$(G)=n$) and G is torsion free. Thus there is a finite sequence $\Theta=\{e_1, e_2,\cdots, e_n\}$ such that rank$(G/\text{gp}\Theta)=0$. Here gpΘ is the smallest subgroup generated by Θ. This implies that each $0\neq\xi\in G$ is expressed as $a\xi=a_1e_1+\cdots+a_ne_n$ for some integers $a\neq 0$ and $(a_1\cdots, a_n)\neq(0,\cdots,0)$. Since $(a_1/a,\cdots, a_n/a)$ is uniquely determined, an isomorphism $\varphi: G\rightarrow\boldsymbol{R}^n$ is defined by $\varphi(\xi)=(a_1/a,\cdots, a_n/a)$. Identify ξ with $(a_1/a,\cdots, a_n/a)$. Then Θ is the canonical basis of $\boldsymbol{Z}^n$ so that gp$\Theta=\boldsymbol{Z}^n\subset G\subset\boldsymbol{Q}^n\subset\boldsymbol{R}^n$. For $t=(t_1,\cdots, t_n)\in\boldsymbol{R}^n$ define a map $\psi:\boldsymbol{R}^n\rightarrow X$ by $\psi(t)\xi=t_1a_1/a+\cdots+t_na_n/a$ (addition mod 1) for $\xi=(a_1/a,\cdots, a_n/a)\in G$. Then $\psi(t)\in X$ and $\psi:\boldsymbol{R}^n\rightarrow X$ is continuous. But $\psi^{-1}:\psi(\boldsymbol{R}^n)\rightarrow\boldsymbol{R}^n$ is not continuous even if ψ is one to one.

FACT 3.1. [2] $\psi(\boldsymbol{R}^n)$ is dense in X. If X is a torus then $\psi(\boldsymbol{R}^n)=X$.

FACT 3.2. [2] Let $F=\{x\in X: \xi(x)=0 \text{ for } \xi\in\text{gp}\Theta\}$. Then the following holds:

(i) F is totally disconnected and $\psi^{-1}\{\psi(\boldsymbol{R}^n)\cap\text{F}\}=\boldsymbol{Z}^n$,

(ii) $X=\psi(\boldsymbol{R}^n)+F$,

(iii) there is a small closed neighborhood U of 0 in $\boldsymbol{R}^n$ such that $\psi(U)\cap F=\{0\}$, $\psi(U)+F$ is a closed neighborhood of 0 in X and the direct product $U\times F$ is homeomorphic to $\psi(U)+F$, in which case we write $\Psi(U)+F=\psi(U)\oplus F$.

As an easy corollary of Fact 3.2 we have the following:

FACT 3.3. Let U be an open neighborhood of 0 in $\boldsymbol{R}^n$. If diam(U) is small enough, then

(i) $\psi: U\rightarrow\psi(U)$ is bijective,

(ii) $\psi(U)\times\psi(\boldsymbol{Z}^n)$ is open in $\psi(\boldsymbol{R}^n)$.

FACT 3.4. [2] Let U be as in Fact 3.3. Assume that $\psi:\boldsymbol{R}^n\rightarrow\psi(\boldsymbol{R}^n)$

is bijective. If $F(\delta)$ is an open subgroup of F contained in an open set with radius δ, then

(i) $C(\delta)=\psi^{-1}\{F(\delta)\cap\psi(\boldsymbol{R}^n)\}$ is a subgroup of $\boldsymbol{Z}^n$,

(ii) $\boldsymbol{Z}^n/C(\delta)$ is finite (i.e. $C(\delta)$ is a subgroup of finite index in $\boldsymbol{Z}^n$).

FACT 3.5. [2] The following holds:

(i) $\psi(\boldsymbol{Z}^n)$ is closed under the relative topology of $\psi(R^n)$,

(ii) $\psi(\boldsymbol{Z}^n)$ is dense in F,

(iii) $X=\psi(\boldsymbol{R}^n)+F_0$ for each open subgroup F_0 of F.

FACT 3.6. [2] If X contains no torus subgroup, then $\psi:\boldsymbol{R}^n\rightarrow\psi(\boldsymbol{R}^n)$ is bijective.

FACT 3.7. [1] If V is the maximal torus subgroup of X, then there exists a solenoidal group S without tori such that X splits into a direct sum $X=S\oplus V$.

Let $s=\dim(S)$, then S is expressed as $S=\psi_1(\boldsymbol{R}^s)+F$ by Fact 3.2, and by Fact 3.6, $\psi_1:\boldsymbol{R}^s\rightarrow\psi_1(\boldsymbol{R}^s)$ is bijective. Thus a homomorphism $\psi:\boldsymbol{R}^s\times V\rightarrow\psi_1(\boldsymbol{R}^s)\oplus V$ is defined by

$$\psi(v, x)=\psi_1(v)+x \quad ((v, x)\in\boldsymbol{R}^s\times V).$$

Obviously ψ is bijective and continuous.

Let d denote the translation invariant metric for X and d_s denote the euclidean metric for $\boldsymbol{R}^s$. For $\delta>0$ put

$U(0, \delta)=\{v\in\boldsymbol{R}^s: d_s(v, 0)\leq\delta\}$,

$V(0, \delta)=\{x\in V: d(x, 0)\leq\delta\}$,

$F(0, \delta)=\{x\in F: d(x, 0)\leq\delta\}$.

Since F is totally disconnected and $F(0, \delta)$ is symmetric, $F(0, \delta)$ contains an open subgroup of F. Therefore, to avoid complication we promise that every closed neighborhood $F(0, \delta)$ of 0 itself is an open subgroup of F.

By Fact 3.2 we can find $\alpha_0>0$ such that $\psi_1(U(0, \alpha_0))\cap F(0, \alpha_0)=\{0\}$ and $W(0, \alpha_0)=\psi_1(U(0, \alpha_0))\oplus F(0, \alpha_0)\oplus V(0, \alpha_0)$ is an open neighborhood of 0 in X.

Hereafter we fix the number $\alpha_0>0$ and define a function κ by

$$\kappa(x)=\max\{d_s(v_S, 0),\ d(v_V, 0),\ d(v_F, 0)\}$$

for $x=\psi_1(v_S)+v_F+v_V\in W(0, \alpha_0)$. For $x, y\in X$ put

$$d_0(x, y)=\begin{cases}\kappa(x-y) & \text{for for } x-y\in W(0, \alpha_0)\\ \alpha_0 & \text{otherwise}\end{cases} \qquad (3.1).$$

Then d_0 is uniformly equivalent to the original metric d for X and a translation invariant metric for X. If in particular $v, v' \in U(0, \alpha_0)$ then we have

$$d_0(\psi_1(v), \psi_1(v')) = d_s(v, v').$$

For the direct product space $\boldsymbol{R}^s \times V$ we define a metric d_1 by

$$d_1((v, x), (v', x')) = \max\{d_s(v, v'), d(x, x')\} \tag{3.2}$$

for $(v, x), (v', x') \in \boldsymbol{R}^s \times V$. If $d_1((v, x), (v', x')) < \alpha_0$ then we have

$$d_0(\psi(v, x), \psi(v', x')) = d_1((v, x), (v', x')). \tag{3.3}$$

For simplicity we write

$$\begin{aligned} &K = \psi(\boldsymbol{R}^s \times V) = \psi_1(\boldsymbol{R}^s) \oplus V \\ &K(0, \alpha) = \psi_1(U(0, \alpha)) \oplus V(0, \alpha) \quad (0 < \alpha \leq \alpha_0). \end{aligned} \tag{3.4}$$

If in particular $x, x' \in K(0, \alpha_0)$ then we have

$$d_0(x, x') = d_1(\psi^{-1}(x), \psi^{-1}(x')). \tag{3.5}$$

Let $f: X \to X$ be a (bijective) homeomorphism. Then for $0 < \varepsilon < \alpha_0$ there is $\delta > 0$ such that

$$f(K(0, \delta) + x) \subset K(0, \varepsilon) + f(x) \quad (x \in X). \tag{3.6}$$

FACT 3.8. [1] If $f(0) \in K$ then $f(K) = K$.

FACT 3.9. [1] For $0 < \lambda_0 < \alpha_0/3$ there exist $\delta_f > 0$, a continuous injective homomorphism $\sigma_f: F(0, \delta_f) \to F(0, \lambda_0)$ and a continuous map $\varphi_f: F(0, \delta_f) \times X \to K(0, \lambda_0)$ such that

$$f(x+y) = f(x) + \varphi_f(y, x) + \sigma_f(y)$$

for all $x \in X$ and all $y \in F(0, \delta_f)$.

Since $C(\varepsilon) = \psi_1^{-1}(F(0, \varepsilon) \cap \psi_1(\boldsymbol{R}^s))$ is of finite index by Fact 3.4, we have $\sigma_f(\psi_1(C(\delta_f)) \subset \psi_1(C(\lambda_0))$ whenever $f(0) \in K$. For, since $f(K) = K$ by Fact 3.8 and $\psi_1(C(\delta_f)) \subset \psi_1(C(\lambda_0)) \subset K$, for $y \in \psi_1(C(\delta_f))$ we have

$$f(y) = f(0+y) = f(0) + \varphi_f(y, 0) + \sigma_f(y)$$

and so $\sigma_f(y) = f(y) - \varphi_f(y, 0) - f(0) \in K$. Since $\sigma_f(y) \in F(0, \lambda_0)$, consequently $\sigma_f(y) \in F(0, \lambda_0) \cap K = F(0, \lambda_0) \cap \{\psi_1(\boldsymbol{R}^s) \oplus V\} = \psi_1(C(\lambda_0))$ because $F(0, \lambda_0) \cap V = \{0\}$.

In the rest of this section, we prepare some lemmas under the assumption $f(0) \in K$.

Since $\psi: \boldsymbol{R}^s \times V \to \psi_1(\boldsymbol{R}^s) \oplus V$ is bijective, we have the commutative diagram

$$\begin{array}{ccc} \boldsymbol{R}^s\times V & \xrightarrow{\tilde f} & \boldsymbol{R}^s\times V \\ \psi\downarrow & & \downarrow\psi \\ \psi_1(\boldsymbol{R}^s)\oplus V & \xrightarrow[f]{} & \psi_1(\boldsymbol{R}^s)\oplus V. \end{array}$$

Then the following is easily checked.

FACT 3.10. $\tilde f$ is d_1-biuniformly continuous.

FACT 3.11. Let $\tilde\sigma_f: C(\delta_f)\to C(\lambda_0)$ be defined by $\tilde\sigma_f=\psi_1^{-1}\circ\sigma_f\circ\psi_1$. Then $\tilde\sigma_f$ is a homomorphism and $\tilde\sigma_f(C(\delta_f))$ is of finite index.

Obviously $\tilde\sigma_f$ is a homomorphism and $\tilde\sigma_f(C(\delta_f))$ is a subgroup of $\boldsymbol{Z}^s$. To see that $\tilde\sigma_f(C(\delta_f))$ is of finite index, we use (3.6), then there is $\delta>0$ such that $f(K(0,\delta))\subset K(0,\lambda_0))+f(0)$. Let $\delta_f>0$ and $\delta_1>0$ be numbers such that $0<\delta_f<\delta$ and $W(0,\delta_1)+f(0)\subset f(W(0,\delta_f))$. Then we have

$$\begin{aligned}
&\psi_1(U(0,\delta_1))+F(0,\delta_1))+f(0)\\
&\subset f(\psi_1(U(0,\delta_f))+F(0,\delta_f))\\
&\subset f(\psi_1(U(0,\delta_f))+\varphi_f(F(0,\delta_f),\ \psi_1(U(0,\delta_f)))+\sigma_f(F(0,\delta_f))\\
&\subset \psi_1(U(0,\lambda_0))+f(0)+\psi_1(U(0,\lambda_0))+\sigma_f(F(0,\delta_f))\\
&=\psi_1(U(0,2\lambda_0))+\sigma_f(F(0,\delta_f))+f(0)\\
&\subset \psi_1(U(0,2\lambda_0))+F(0,\lambda_0)+f(0)
\end{aligned}$$

from which

$$F(0,\delta_1)\subset\sigma_f(F(0,\delta_f))\subset F(0,\lambda_0).$$

Since $C(\delta_1)$ and $C(\lambda_0)$ are of finite index, so is $\tilde\sigma_f(C(\delta_f))$.

By Fact 3.11 the extension of $\tilde\sigma_f$ is an automorphism of $\boldsymbol{R}^s$ which is denoted by the same symbol.

Define $\tilde\varphi_f: (C(\delta_f))\times(\boldsymbol{R}^s\times V)\to U(0,\lambda_0))\times V(0,\lambda_0)$ by

$$\tilde\varphi_f(l,v)=\psi^{-1}\circ\varphi_f(\psi(l),\ \psi(v))\quad((l,v)\in C(\delta_f)\times(\boldsymbol{R}^s\times V)).$$

Then $\tilde\varphi_f$ is continuous and by Fact 3.9 we have

$$\tilde f(v+l)=\tilde f(v)+\tilde\varphi_f(l,v)+\tilde\sigma_f(l)\quad(v\in\boldsymbol{R}^s\times V,\ l\in C(\delta_f)).$$

Let $t=\dim(V)$ and $\pi_1: \boldsymbol{R}^t\to V$ be the natural projection. Then a covering projection $\pi: \boldsymbol{R}^s\times\boldsymbol{R}^t\to\boldsymbol{R}^s\times V$ is defined by $\pi(p,q)=(p,\pi_1(q))$ for $p\in\boldsymbol{R}^s$ and $q\in\boldsymbol{R}^t$, and by [5] we can find a translation invariant complete metric $\bar d$ for $\boldsymbol{R}^s\times\boldsymbol{R}^t$ satisfying the conditions for some $\alpha_1>0$

(i) $\bar d(\bar p,\bar q)\le\alpha_1$ ($\bar p,\bar q\in\boldsymbol{R}^s\times\boldsymbol{R}^t$) implies $d_1(\pi(\bar p),\ \pi(\bar q))=\bar d(\bar p,\bar q)$,

(ii) for $\bar p\in\boldsymbol{R}^s\times\boldsymbol{R}^t$ *and* $\bar q\in\boldsymbol{R}^s\times V$ with $d_1(\pi(\bar p),q)\le\alpha_1$ there is a unique $\bar{q}\in\pi^{-1}(q)$ such that $d_1(\pi(\bar{p}),\ q)=\bar{d}(\bar{p},\ \bar{q})$, (3.8)

(iii) all the covering transformations are $\bar{d}$-isometries.

Note that $\bar{d}$ is uniformly equivalent to d_n. Let $\bar{p}\in\boldsymbol{R}^s\times\boldsymbol{R}^t$ satisfy $\bar{d}(\bar{0},\bar{p})=d_1(0,\tilde{f}(0))$ where $\bar{0}$ denotes the zero point in $\boldsymbol{R}^s\times\boldsymbol{R}^t$. Then there is a lift $\bar{f}$: $\boldsymbol{R}^s\times\boldsymbol{R}^t\rightarrow\boldsymbol{R}^s\times\boldsymbol{R}^t$ of $\tilde{f}$ such that $\bar{f}(\bar{0})=\bar{p}$, $\bar{f}$ is bijective and $\bar{d}$-biuniformly continuous and

$$\begin{array}{ccc} \boldsymbol{R}^s\times\boldsymbol{R}^t & \xrightarrow{\ \bar{f}\ } & \boldsymbol{R}^s\times\boldsymbol{R}^t \\ \pi\downarrow & & \downarrow\pi \\ \boldsymbol{R}^s\times V & \xrightarrow[\ \tilde{f}\ ]{} & \boldsymbol{R}^s\times V \end{array} \qquad \text{commutes.}$$

Note that the group of all the covering transformations of $\boldsymbol{R}^s\times\boldsymbol{R}^t$ is isomorphic to $\boldsymbol{Z}^t$ because π is the identity on $\boldsymbol{R}^s\times\{0\}$. For the simplicity we identify such two groups.

Then there is a group automorphism σ'_f: $\boldsymbol{Z}^t\rightarrow\boldsymbol{Z}^t$ such that

$$\bar{f}(v+z)=(0,\ \sigma'_f(z))+\bar{f}(v)\quad(v\in\boldsymbol{R}^s\times\boldsymbol{R}^t,\ z\in\boldsymbol{Z}^t).$$

Let $\tilde{\sigma}'_f$ be the extension of σ'_f and define an automorphism $\bar{\sigma}_f$: $\boldsymbol{R}^s\times\boldsymbol{R}^t\rightarrow\boldsymbol{R}^s\times\boldsymbol{R}^t$ by

$$\bar{\sigma}_f(v_s,\ v_t)=(\tilde{\sigma}_f(v_s),\ \tilde{\sigma}'_f(v_t))\quad(v_s\in\boldsymbol{R}^s,\ v_t\in\boldsymbol{R}^t).$$

We define a continuous map $\bar{\varphi}_f$: $(C(\delta_f)\times\boldsymbol{Z}^t)\times(\boldsymbol{R}^s\times\boldsymbol{R}^t)\rightarrow D(\bar{0},\lambda_0)$ by

$$\bar{\varphi}_f((l_s,\ l_t),\ v)=\tilde{\varphi}_f(l_s,\ \pi(v))\qquad(l_s\in C(\delta_f),\ l_t\in\boldsymbol{Z}^t,\ v\in\boldsymbol{R}^s\times\boldsymbol{R}^t),$$

where $D(\bar{0},\lambda_0)=\{v\in\boldsymbol{R}^s\times\boldsymbol{R}^t:\ \bar{d}(\bar{0},v)<\lambda_0\}$. Then we have

$$\bar{f}(v+l)=\bar{f}(v)+\bar{\varphi}_f(l,\ v)+\bar{\sigma}_f(l)\qquad(l\in C(\delta_f)\times\boldsymbol{Z}^t,\ v\in\boldsymbol{R}^s\times\boldsymbol{R}^t).$$

Then the following holds.

FACT 3.12. [1] $\bar{\sigma}$: $\boldsymbol{R}^s\times\boldsymbol{R}^t\rightarrow\boldsymbol{R}^s\times\boldsymbol{R}^t$ induces a solenoidal automorphism.

§ 4. Expansiveness and POTP on solenoidal groups.

As before let X be an n-solenoidal group and f : $X\rightarrow X$ be a (bijective) homeomorphism. If f is expansive and has *POTP*, then the following propositions are established.

PROPOSITION 4.1. *Let K be as in* (3.4). *If $f(K)=K$, then a lift $\bar{f}$: $\boldsymbol{R}^n\rightarrow\boldsymbol{R}^n$ of f is expansive and has POTP (under the euclidean metric).*

PROPOSITION 4.2. [4] *There exist a continuous function $\bar{D}^\sigma$: $\cup\{W^\sigma$*

$(x, \bar{f})\times W^{\sigma}(x, \bar{f}): x\in \boldsymbol{R}^{n}\}\rightarrow \boldsymbol{R}$ $(\sigma=s, u)$ *and constants* $a_2>0, 0<\lambda_2<1$ *such that for* $i\geq 0$

$$\bar{D}^{s}(\bar{f}^{i}(x), \bar{f}^{i}(y))\leq a_2\lambda_2^{i}\bar{D}^{s}(x, y) \qquad (y\in W^{s}(x, \bar{f}),$$
$$\bar{D}^{u}(\bar{f}^{-i}(x), \bar{f}^{-i}(y))\leq a_2\lambda_2^{i}\bar{D}^{u}(x, y) \qquad (y\in W^{u}(x, \bar{f}),$$

and for any $\varepsilon_2>0$ *there exists* $\delta_2>0$ *such that if* $\bar{D}^{\sigma}(x, y)\leq\delta_2$ $(y\in W^{\sigma}(x, \bar{f}))$ *then* $d_n(x, y)\leq\varepsilon_2$ $(\sigma=s, u)$.

Proposition 4.1 shows the first statement of our Theorem 1. For the second statement Proposition 4.2 is prepared. For the proof of Proposition 4.1 we need the following five lemmas.

LEMMA 4.3. *Let* $\tilde{f}: \boldsymbol{R}^{s}\times V\rightarrow \boldsymbol{R}^{s}\times V$ *be as in Fact* 3.10. *Then* $\tilde{f}$ *is expansive.*

Let d_0 and d_1 be metrics for X and $\boldsymbol{R}^{s}\times V$ respectively defined by (3.1) and (3.2). Let $0<c<\alpha_0$ be an expansive constant of f under d_0. To see expansiveness of $\tilde{f}$, if $d_1(\tilde{f}^{i}(v), \tilde{f}^{i}(v'))\leq c$ for $i\in \boldsymbol{Z}$ $(v, v'\in \boldsymbol{R}^{s}\times V)$. Then by (3.3), for $i\in \boldsymbol{Z}$

$$d_1(\tilde{f}^{i}(v), \tilde{f}^{i}(v'))=d_0(\psi\circ\tilde{f}^{i}(v), \psi\circ\tilde{f}^{i}(v'))$$
$$=d_0(f^{i}\circ\psi(v), f^{i}\circ\psi(v'))\leq c,$$

from which $\psi(v)=\psi(v')$ and so $v=v'$ (since ψ is bijective).

LEMMA 4.4. $\tilde{f}$ *has a hyperbolic metric* $\tilde{D}$ *which is uniformly equivalent to* d_1.

By (3.3) it is easily checked that $\tilde{f}$ has the condition (*) of Lemma 2.4.

LEMMA 4.5. *Under the assumption of Proposition* 4.1, *there exists* $\alpha_2>0$ *such that for* $0<\varepsilon<\alpha_2$

$$\psi(W_{\varepsilon}^{\sigma}(v, \tilde{f}, d_1))=W_{\varepsilon}^{\sigma}(\psi(v), f, d_0)\cap\{K(0, \alpha_0/2)+\psi(v)\}$$

for $v\in \boldsymbol{R}^{s}\times V$ *and* $\sigma=s, u$.

For the proof use (3.6). Then we can find $\alpha_2>0$ such that $f(K(0, \alpha_2)+x)\subset K(0, \alpha_0/2)+f(x)$ and $f^{-1}(K(0, \alpha_2)+x)\subset K(0, \alpha_0/2)+f^{-1}(x)$ for $x\in X$. (3.3) ensures that for $0<\varepsilon<\alpha_2$,

$$\psi(W_{\varepsilon}^{\sigma}(v, \tilde{f}, d_1))\subset L_{v}^{\sigma} \quad (v\in \boldsymbol{R}^{s}\times v, \ \sigma=s, u)$$

where $L_{v}^{\sigma}=W_{\varepsilon}^{\sigma}(\psi(v), f, d_0)\cap\{K(0, \alpha_0/2)+\psi(v)\}$. If $y\in L_{v}^{s}$, then $d_0(f^{i}(y), f^{i}\circ\psi(v))\leq\varepsilon$ for $i\geq 0$ and $y\in K(0, \alpha_0/2)+\psi(v)$. This implies $y\in K(0, \varepsilon)+\psi(v)$. Since $\varepsilon<\alpha_2$, we have $f(y)\in K(0, \alpha_0/2)+f\circ\psi(v)$ and hence $f(y)\in$

$K(0, \varepsilon)+f\circ\psi(v)$ since $d_0(f(y), f\circ\psi(v))\leq\varepsilon$. Repeating in this process, we have $f^i(y)\in K(0, \varepsilon)+f^i\circ\psi(v)$ for $i\geq 0$ and by (3.5), $d_1(\psi^{-1}\circ f^i(y), \psi^{-1}\circ f^i\circ\psi(v))\leq\varepsilon$ for $i\geq 0$. Thus $\psi^{-1}(y)\in W^s_\varepsilon(v, \tilde{f}, d_1)$ and so $y\in\psi(W^s_\varepsilon(v, \tilde{f}, d_1))$.

LEMMA 4.6. *Let $\alpha_2>0$ be as in Lemma 4.5. Then $\tilde{f}: \boldsymbol{R}^s\times V\to\boldsymbol{R}^s\times V$ has POTP if and only if for $0<\varepsilon<\alpha_2$ there is $\delta>0$ such that for each $x\in K$ $y\in K(0, \delta)+x$ implies $W^s_\varepsilon(y, f, d_0)\cap W^u_\varepsilon(x, f, d_0)\cap\{K(0, \alpha_0)+x\}\neq\phi$.*

If $\tilde{f}: \boldsymbol{R}^s\times V\to\boldsymbol{R}^s\times V$ has POTP, then for $0<\varepsilon<\alpha_2$ there is $\delta>0$ such that $d_1(v, w)<\delta(v, w\in\boldsymbol{R}^s\times V)$ implies $W^s_\varepsilon(v, \tilde{f}, d_1)\cap W^u_\varepsilon(w, \tilde{f}, d_1)\neq\phi$. Since $y\in K(0, \delta)+x(x\in K)$ implies that $d_1(\psi^{-1}(y), \psi^{-1}(x))=d_0(y, x)<\delta$ (by (3.3)), we have $\phi\neq\psi(W^s_\varepsilon(\psi^{-1}(y), \tilde{f}, d_1))\cap\psi(W^u_\varepsilon(\psi^{-1}(x), \tilde{f}, d_1))=W^s_\varepsilon(y, f, d_0)\cap\{K(0, \alpha_0/2))+y\}\cap W^u_\varepsilon(x, f, d_0)\cap\{K(0, \alpha_0/2)+x\}=W^s_\varepsilon(y, f, d_0)\cap W^u_\varepsilon(x, f, d_0)\cap\{K(0, \alpha_0)+x\}$.

The converse follows from the facts that $\tilde{f}$ has canonical coordinates (by Lemma 4.5.) and a hyperbolic metric $\tilde{D}$ and $\boldsymbol{R}^s\times V$ is the complete metric space under $\tilde{D}$.

LEMMA 4.7. *$\tilde{f}: \boldsymbol{R}^s\times V\to\boldsymbol{R}^s\times V$ has POTP.*

As before let $c>0$ be an expansive constant of $\tilde{f}$. Then for $0<\varepsilon_0<c/2$ there is $\delta_0>0$, $0<\delta_1<\delta_0$ and $0<\rho<\delta_1$ such that all the statements of Lemma 2.2 hold. We define the sets A_x, B_x and C_x as

> A_x is the connected component of x in $W^u_{\varepsilon_0,\delta_1}(x, f, d_0)$,
> B_x is that of x in $W^s_{\varepsilon_0,\delta_1}(x, f, d_0)$,
> C_x is that of x in $N(x)$.

By Lemma 2.2.(c) we have $[\,,\,]|_{A_x\times B_x}: A_x\times B_x\to C_x$ is a homeomorphism and $A_x\subset W^u_{\varepsilon_0,\delta_1}(x, f, d_0)\subset W^u_{\varepsilon_0}(x, f, d_0)=W^u_{\varepsilon_0}(x, f, d_0)\cap\{K(0, \alpha_0)+F(0, \alpha_0)+x\}$. Thus we have $A_x\subset W^u_{\varepsilon_0}(x, f, d_0)\cap\{K(0, \alpha_0)+x\}$ since A_x is connected and $x\in A_x$. Similarly $B_x\subset W^s_{\varepsilon_0}(x, f, d_0)\cap\{K(0, \alpha_0)+x\}$. Since $B_\rho(x)\subset N(x)$ by Lemma 2.2(d), we have $K(0, \rho)+x\subset C_x$ and then for $y\in K(0, \rho)+x$ there are $z_1\in A_x$ and $z_2\in B_x$ with $y=[z_1, z_2]$; i.e. $\{y\}=W^s_{\varepsilon_0}(z_1, f, d_0)\cap W^u_{\varepsilon_0}(z_2, f, d_0)\cap C_x$. Thus $z_1\in W^s_{\varepsilon_0}(y, f, d_0)$ *and since* $z_1\in K(0, \alpha_0)+x$, we have $z_1\in W^s_{\varepsilon_0}(y, f, d_0)\cap\{K(0, \alpha_0)+x\}$. Lemma 4.6 implies that $\tilde{f}$ has *POTP*.

We are in a position to prove Proposition 4.1. Applying Lemma 4.3, we see that $\bar{f}: \boldsymbol{R}^n\to\boldsymbol{R}^n$ is expansive under the metric $\bar{d}$. To see *POTP*, let $\bar{d}$ and $\alpha_1>0$ as in (3.8). Then we can find $\varepsilon>0$ such that

$$\bar{d}(p, q)\leq\varepsilon\ (p, q\in R^n)\ \text{implies}\ \max\{\bar{d}(\bar{f}(p), \bar{f}(q)), \bar{d}(\bar{f}^{-1}(p), \bar{f}^{-1}(q))\}\leq\alpha_1/3.$$

Choose $\delta>0$ small enough and let $\{p_i\}$ be a δ-pseudo orbit of $\bar{f}$. If $y_i=\pi(p_i)$ (where $\pi: \boldsymbol{R}^n\to\boldsymbol{R}^s\times V$ is the projection), then $\{y_i\}$ is a δ-pseudo orbit of $\tilde{f}$. Since $\tilde{f}$ has *POTP*, there is $z\in\boldsymbol{R}^s\times V$ such that $d_1(\tilde{f}^i(z), y_i)\leq\varepsilon$ for all i.

The fact that $d_1(z, y_0)\leq\varepsilon$ ensures the existence of $q_0\in B_{\alpha_1}(p_0)$ such that $\pi(q_0)=z$ where $B_{\alpha_1}(p_0)=\{p\in\boldsymbol{R}^n: \bar{d}(p_0, p)\leq\alpha_1\}$. Then we have $\bar{d}(\bar{f}^i(q_0), p_i))\leq\varepsilon$ for all i (this is shown by induction on i).

If $\bar{d}(\bar{f}^{i-1}(q_0), p_{i-1})\leq\varepsilon$ for $i\geq 1$, then $\bar{d}(\bar{f}^i(q_0), \bar{f}(p_{i-1}))\leq\alpha_1/3$. Since $\bar{d}(\bar{f}(p_{i-1}), p_i)\leq\delta\leq\alpha_1/3$, we have $\bar{d}(\bar{f}^i(q_0), p_i)\leq\alpha_1$ and so

$$\begin{aligned}\bar{d}(\bar{f}^i(q_0), p_i)&=d_1(\pi\circ\bar{f}^i(q_0), \pi(p_i))=d_1(\tilde{f}^i\circ\pi(q_0), y_i)\\&=d_1(\tilde{f}^i(z), y_i)\leq\varepsilon \qquad (i\geq 0).\end{aligned}$$

Similarly the same conclusion is proved for $i<0$. Furthermore expansiveness and *POTP* are independent of a uniformly equivalent metric. Thus the proof of Proposition 4.1 is completed.

§ 5. The existence of semi-conjugacy maps.

Assume that $f(0)=0$. As we saw in Fact 3.9 and 3.12 for $\lambda_0>0$ there exist $\delta_f>0$, continuous maps $\bar{\varphi}_f: (C(\delta_f)\times\boldsymbol{Z}^t)\times\boldsymbol{R}^n\to B(0, \lambda_0)$ and $\bar{\varphi}_{f^{-1}}: (C(\delta_f)\times\boldsymbol{Z}^t)\times\boldsymbol{R}^n\to B(0, \lambda_0)$, and an automorphism $\bar{\sigma}_f: \boldsymbol{R}^n\to\boldsymbol{R}^n$ such that

$$\begin{aligned}&\bar{f}(v+l)=\bar{f}(v)+\bar{\varphi}_f(l, v)+\bar{\sigma}_f(l) \text{ and}\\&\bar{f}^{-1}(v+l)=\bar{f}^{-1}(v)+\bar{\varphi}_{f^{-1}}(l, v)+\bar{\sigma}_f^{-1}(l)\\&\qquad\qquad (l\in C(\delta_f)\times\boldsymbol{Z}^t,\ v\in\boldsymbol{R}^n).\end{aligned}\tag{5.1}$$

Thus we have

$$\begin{aligned}\|\bar{\sigma}_f-\bar{f}\|_{\boldsymbol{R}^n}&=\sup\{d_n(\bar{\sigma}_f(v), \bar{f}(v)): v\in\boldsymbol{R}^n\}\\&=M<\infty.\end{aligned}\tag{5.2}$$

If $\bar{\sigma}_f: \boldsymbol{R}^n\to\boldsymbol{R}^n$ is hyperbolic, then $\bar{\sigma}_f$ is expansive under the metric d_n and its expansive constant is arbitrary. This fact derives easily that for every $M>0$ there is $\delta_M>0$ such that each M-pseudo orbit $\{x_i\}$ of $\bar{\sigma}_f$ is δ_M-traced by some point x (i.e. $d_n(\bar{\sigma}_f^i(x), x_i)\leq\delta_M$ for i). Using this fact, we have

LEMMA 5.1. [4] *If $\bar{\sigma}_f: \boldsymbol{R}^n\to\boldsymbol{R}^n$ is hyperbolic, then there exists a continuous surjection $\bar{h}: \boldsymbol{R}^n\to\boldsymbol{R}^n$ such that*

(i) $\bar{\sigma}_f\circ\bar{h}=\bar{h}\circ\bar{f}$

(ii) $d_n(\bar{h}(v), v)\leq\delta_M$ *for* $v\in\boldsymbol{R}^n$.

LEMMA 5.2. *The following holds:*

(1) $\bar{h}: \boldsymbol{R}^n\to\boldsymbol{R}^n$ is *d_n-uniformly continuous,*

(2) $\bar{h}(v+l)=l+\bar{h}(v)$ *for* $v\in\boldsymbol{R}^n$ *and* $l\in\{0\}\times\boldsymbol{Z}^t$,

(3) *for* $\lambda_3>0$ *small there is* $\delta_3>0$, *and a continuous map* $\bar{\eta}: C(\delta_3)\times \boldsymbol{R}^n\to B(0, \lambda_3)$ *such that*

$$\bar{h}(v+l)=l+\bar{h}(v)+\bar{\eta}(l, v) \qquad (l\in C(\delta_3),\ v\in \boldsymbol{R}^n).$$

(1) and (2) follow from [4]. To check (3), let M and δ_M be as above. Then we can find $N>0$ such that for $v,\ w\in \boldsymbol{R}^n$, $d_n(\bar{\sigma}_f^j(v),\ \bar{\sigma}_f^j(w))\leq 2\delta_M+\lambda_3$ $(|j|\leq N)$ implies $d_n(v, w)\leq \varepsilon$. Since $f: X\to X$ is d_0-biuniformly continuous, for small $0<\lambda_3'<\lambda_3/2$ such that $d_0(x, y)\leq \lambda_3'$ $(x, y\in X)$ implies $\max_{|j|\leq N} d_0(f^i(x), f^i(y))\leq \lambda_3$. By Fact 3.9, there exists $\delta_N>0$ such that for $|j|\leq N$ there exist a continuous map $\varphi_j: F(0, \delta_N)\times X\to K(0, \lambda_3)$ and a homomorphism $\sigma_j: F(0, \delta_N)\to F(0, \lambda_3')$ so that

$$f^j(x+y)=f^j(x)+\varphi_j(y, x)+\sigma_j(y) \tag{5.3}$$

for $x\in X$ and $y\in F(0, \delta_N)$. Thus there exist a continuous map $\bar{\varphi}_j: (C(\delta_N)\times \boldsymbol{Z}^t)\times(\boldsymbol{R}^s\times \boldsymbol{R}^t)\to B(0, \lambda_3)$ and an automorphism $\bar{\sigma}_j: \boldsymbol{R}^n\to \boldsymbol{R}^n$ which satisfy $\bar{f}^j(v+l)=\bar{f}^j(v)+\bar{\varphi}_j(l, v)+\bar{\sigma}_j(l)$ for $l\in C(\delta_N)\times \boldsymbol{Z}^t,\ v\in \boldsymbol{R}^s\times \boldsymbol{R}^t$. Then we have $\delta_3>0$ such that $\bar{\sigma}_{j|C(\delta_3)\times\{0\}}=\bar{\sigma}^j_{f|C(\delta_3)\times\{0\}}$ for $|j|\leq N$. Indeed, choose $\delta_3>0$ such that $d_0(0, x)<\delta_3$ $(x\in F(0, \delta_f))$ implies $d_0(0, \sigma_f^j(x))<\delta_N$ for $|j|\leq N$. Let $l\in C(\delta_3)$, then we have

$$f(\psi_1(l))=f(0+\psi_1(l))=\varphi_f(\psi_1(l), 0)+\sigma_f(\psi_1(l))$$

and since $d_0(\sigma_f(\psi_1(l), 0)\leq \delta_N$,

$$\begin{aligned} f^2(\psi_1(l))&=f(\varphi_f(\psi_1(l), 0)+\sigma_f\psi_1(l)))\\ &=f(\varphi_f(\psi_1(l), 0))+\varphi_f(\sigma_f(\psi_1(l)),\ \varphi_f(\psi_1(l)), 0)+\sigma_f^2(\psi_1(l)) \end{aligned}$$

By the choice of λ_3' we have

$$f(\varphi_f(\psi_1(l), 0))+\varphi_f(\sigma_f)\psi_1(l)),\ \varphi_f(\psi_1(l), 0))\in K(0, \lambda_3)$$

and

$$f^2(\psi_1(l))=\varphi_2(\psi_1(l), 0)+\sigma_2(\psi_1(l))\in K(0, \lambda_3)\oplus F(0, \lambda_3)$$

from which

$$\sigma_f^2\circ\psi_1(l)=\sigma_2\circ\psi_1(l).$$

Since $\bar{\sigma}_{f|\boldsymbol{R}^s\times\{0\}}=\tilde{\sigma}_{f|\boldsymbol{R}^s\times\{0\}}$ and $\tilde{\sigma}_f$ is an extension of $\psi_1^{-1}\circ\sigma_f\circ\psi_1$, the conclusion is obtained. Repeating in this fashion, we obtain the conclusion for $|j|\leq N$.

By (5.3) we have

$\bar{f}^j(v+l)=\bar{\sigma}_f^j(l)+\bar{f}^j(v)+\bar{\varphi}_j(l, v)$ for $v\in \boldsymbol{R}^n,\ l\in C(\delta_3)$, and since $\{\bar{f}^j(v)\}$, $\{\bar{f}^j(v+l)\}$ are M-pseudo orbits of $\bar{\sigma}_f$, we can find points $w,\ w'\in$

$\boldsymbol{R}^n$ satisfying $d_n(\bar{f}^j(v), \bar{\sigma}_f^j(w')) \leq \delta_M$, $d_n(\bar{f}^j(v+l), \bar{\sigma}_f^j(w)) \leq \delta_M$, $\bar{h}(v)=w'$ and $\bar{h}(v+l)=w$. Put $\tilde{w}=w-l$, then we have

$$\begin{aligned} & d_n(\bar{\sigma}_f^j(w'), \bar{\sigma}_f^j(\tilde{w})) \\ & \leq d_n(\bar{\sigma}_f^j(w'), \bar{f}^j(v)) + d_n(\bar{f}^j(v), \bar{f}^j(v)+\bar{\varphi}_j(l, v)) \\ & \qquad\qquad + d_n(\bar{f}^j(v)+\bar{\varphi}_j(l, v), \bar{\sigma}_f^j(w-l)) \\ & \leq 2\delta_M + \lambda_3 \end{aligned}$$

and hence $d_n(w', \tilde{w}) \leq \varepsilon$. Since $\tilde{w}-w'$ depends on v and l, letting $\bar{\eta}(l, v) = \tilde{w}-w'$, a map $\bar{\eta}: C(\delta_3) \times \boldsymbol{R}^n \to B(0, \lambda_3')$ is defined and so

$$\begin{aligned} \bar{h}(v+l) = w = \tilde{w}+l = w' + \bar{\eta}(l, v) + l \\ = \bar{h}(v) + l + \bar{\eta}(l, v) \end{aligned}$$

from which the continuity of $\bar{\eta}$ is obtained.

§ 6. Proof of Theorem.

Since $f: X \to X$ is expansive and has *POTP* and $f(0)=0$, $\bar{f}: \boldsymbol{R}^n \to \boldsymbol{R}^n$ satisfies one of the following cases:

(1) $\bar{f}$ and $\bar{f}^{-1}$ are not positively expansive,

(2) one of $\bar{f}$ and $\bar{f}^{-1}$ is positively expansive.

From now on we give the proof of the case (1). Since $\bar{\sigma}_f: \boldsymbol{R}^n \to \boldsymbol{R}^n$ is hyperbolic, $\boldsymbol{R}^n$ splits into the direct sum $\boldsymbol{R}^n = \bar{L}^s(0) \oplus \bar{L}^u(0)$ of $\bar{\sigma}_f$-invariant subspaces $\bar{L}^s(0)$ and $\bar{L}^u(0)$ where $\bar{L}^s(0)$ is the sum of eigenspaces of $\bar{\sigma}_f$ which correspond to eigenvalues whose absolute value is smaller than one and $\bar{L}^u(0)$ is that of eigenvalues whose absolute value is greater than one. Let $\bar{L}^\sigma(x)$ ($\sigma = s, u$) denote the translation of $\bar{L}^\sigma(0)$ to x. For $\sigma = s, u$ the family $\mathscr{G}^\sigma = \{\bar{L}^\sigma(x) : x \in \boldsymbol{R}^n\}$ is a generalized foliation of $\boldsymbol{R}^n$. Let $\mathscr{F}^s$ and $\mathscr{F}^u$ be as in Lemma 2.6. Since $\bar{h}: \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a d_n-uniformly continuous surjection and $\bar{\sigma}_f \circ \bar{h} = \bar{h} \circ \bar{f}$ holds (by § 5), we have $\bar{h}(\mathscr{F}^\sigma) = \mathscr{G}^\sigma$ ($\sigma = s, u$), and since an expansive constant of $\bar{f}$ is arbitrary, $\bar{h}$ is injective and $\bar{h}^{-1}$ is continuous. Since $\bar{f}$ is expansive and has POTP (by Fact 3.8 and Proposition 4.1), we can choose positive numbers δ_0, δ_1 and ρ as in Lemma 2.2. Then the following holds.

LEMMA 6.1. *$\bar{h}(W^\sigma(x, \bar{f})) = \bar{L}^\sigma(\bar{h}(x))$ for $x \in \boldsymbol{R}^n$ and $\sigma = s, u$.*

Assume that there exists $x \in \mathrm{R}^n$ such that $\bar{h}(W^s(x, \bar{f})) \subsetneqq \bar{L}^s(\bar{h}(x))$, then there exists a subset $A \subset \boldsymbol{R}^n$ such that $\bigcup_{y \in A} \bar{h}(W^s(y, \bar{f})) = \bar{L}^s(\bar{h}(x))$ since $\bar{h}$ is surjective. By the continuity of $\bar{h}^{-1}$, $\bigcup_{y \in A} W^s(y, \bar{f})$ is connected in $\boldsymbol{R}^n$, and so there exist $y_1, y_2 \in A$ with $W^s(y_1, \bar{f}) \neq W^s(y_2, \bar{f})$ and $z_1 \in W^s(y_1, \bar{f})$ and $z_2 \in W^s(y_2, \bar{f})$ with $d_n(z_1, z_2) \leq \delta_0$. Since $W^u(z_1, \bar{f}) \cap W^s(z_1, \bar{f})$ and $W^u(z_1, \bar{f}) \cap W^s(z_2, \bar{f})$ are one point sets, $\bar{h}(W^u(z_1, \bar{f}) \cap$

$W^s(z_1, \bar{f})) = \bar{L}^u(\bar{h}(z_1)) \cap \bar{L}^s(\bar{h}(z_1)) = \bar{L}^u(\bar{h}(z_1)) \cap \bar{L}^s(\bar{h}(x)) = \bar{h}(W^u(z_1, \bar{f}) \cap W^s(z_2, \bar{f}))$ and so $W^s(z_1, \bar{f}) \cap W^s(z_2, \bar{f}) \neq \phi$ which is a contradiction.

By Lemma 6. 1, the following is easily checked.

LEMMA 6. 2. *Let $x, y \in \boldsymbol{R}^n$. Then $W^u(x, \bar{f}) \cap W^s(y, \bar{f})$ is the set of one point.*

By the definition of $\bar{d}$, $\bar{f}$ satisfies the condition $(*)$ of Lemma 2. 4. Thus $\bar{f}$ has a hyperbolic metric $\bar{D}$ which is uniformly equivalent to d_n, and $\mathscr{F}^s$ and $\mathscr{F}^u$ are generalized foliations. By Proposition 4. 2 there is a continuous function

$$\bar{D}^\sigma : \cup\{W^\sigma(x, \bar{f}) \times W^\sigma(x, \bar{f}) : x \in \boldsymbol{R}^n\} \to \boldsymbol{R} \quad (\sigma = s, u).$$

LEMMA 6. 3. *For any $\varepsilon_4 > 0$ there exists $\delta_4 > 0$ such that if $d_n(x, y) \leq \delta_4$ $(y \in W^\sigma(x, \bar{f}))$ then $\bar{D}^\sigma(x, y) \leq \varepsilon_4$ $(\sigma = s, u)$.*

Since $f : X \to X$ is expansive, by Lemma 2. 4 there is a metric D for X such that the following holds:

(i) D is uniformly equivalent to d_0 and

(ii) there are $\kappa > 0$, $a' \geq 1$ and $0 < \lambda' < 1$ such that for $x \in X$

$$D(f^j(x), f^j(y)) \leq a'\lambda'^j D(x, y) \quad (y \in W^s_\kappa(x, f, D))$$
$$D(f^{-j}(x), f^{-j}(y)) \leq a'\lambda'^j D(x, y) \quad (y \in W^u_\kappa(x, f, D))$$

for all $j \geq 0$.

We can assume that there exists $r > 0$ such that $\kappa \leq r$ and if $\bar{D}(x, y) \leq r (x, y \in \boldsymbol{R}^n)$ then $D(\psi(x), \psi(y)) = \bar{D}(x, y)$. As Proposition 4. 2, we can construct a continuous function

$$D^\sigma : \cup\{(D^\sigma_\kappa(x) \cap K(x, \alpha_0)) \times (D^\sigma_\kappa(x) \cap K(x, \alpha_0))) : x \in X\} \to \boldsymbol{R}$$

such that if $v \in \bar{D}^\sigma_\kappa(u)$ $(u, v \in \boldsymbol{R}^n)$ then $D^\sigma(\psi(u), \psi(v)) = \bar{D}^\sigma(u, v)$ and if $\psi(v) \in D^\sigma_\kappa(\psi(u)) \cap K(\psi(u), \alpha_0)$ then $D^\sigma(\psi(u), \psi(v)) = \bar{D}^\sigma(u, v)$ $(\sigma = s, u)$. Here $\bar{D}^\sigma_\kappa(u)$ and $D^\sigma_\kappa(x)$ denote the connected component of u in $W^\sigma_\kappa(u, \bar{f}, \bar{D})$ and that of x in $W^\sigma_\kappa(x, f, D)$. It is easily checked that for any $\varepsilon_4 > 0$ there exists $\delta'_4 > 0$ such that $D(x, y) \leq \delta'_4 (y \in D^\sigma_\kappa(x) \cap K(x, \alpha_0))$ implies $D^\sigma(x, y) \leq \varepsilon_4$. Choose $\rho > \delta_4 > 0$ such that $d_n(u, v) \leq \delta_4$ implies $\bar{D}(u, v) \leq \delta'_4$. Since an expansive constant of $\bar{f}$ is arbitrary, for every $u \in \boldsymbol{R}^n$, $B_\rho(u) \cap W^\sigma(u, \bar{f})$ is connected and so $d_n(u, v) \leq \delta_4 (v \in W^\sigma(u, \bar{f}))$ implies $v \in W^\sigma_\kappa(u, \bar{f}, \bar{D})$. Thus $D(\psi(u), \psi(v)) \leq \delta'_4$ and $\psi(v) \in D^\sigma_\kappa(\psi(u)) \cap K(\psi(u), \alpha_0)$, and then $\bar{D}^\sigma(u, v) = D^\sigma(\psi(u), \psi(v)) \leq \varepsilon_4$.

LEMMA 6. 4. *For any $M_1 > 0$ there exists $M'_1 > 0$ such that if $d_n(u, v) \leq M_1$ $(v \in W^\sigma(u, \bar{f}))$ then $\bar{D}^\sigma(u, v) \leq M'_1$ $(\sigma = s, u)$.*

Let ε_4 and δ_4 be as in Lemma 6.3 and fix $u\in\boldsymbol{R}^n$. Since $d_n(\bar{h}^{-1}(v), v)\leq\delta_M$ for $v\in\boldsymbol{R}^n$, we have

$$B_{M_1}(u)\cap W^s(u, \bar{f})\subset\bar{h}^{-1}(\bar{L}^s(u))\cap B_{M_1+\delta_M}(\bar{h}(u))\cap B_{M_1+2\delta_M}(u).$$

Since $B_{M_1+2\delta_M}(u)$ is compact, there exists a sequence $\{u_i\}_{i=1}^k\subset\boldsymbol{R}^n$ such that $\bigcup_{i=1}^k B_{\delta_4}(u+u_i)\supset B_{M_1+2\delta_M}(u)$, and hence $\bar{D}^\sigma(u, v)\leq k\varepsilon_4$ for $v\in B_{M_1}(u)\cap W^s(u, \bar{f})$.

PROPOSITION 6.5. *$\bar{h}^{-1}$ is d_n-uniformly continuous.*

Since $\bar{f}:\boldsymbol{R}^n\to\boldsymbol{R}^n$ is expansive, if we established the following:

for $\varepsilon>0$ there is $N>0$ such that $d_n(\bar{f}^j(x), \bar{f}^j(y))\leq 3\delta_M$ for $|j|\leq N$ implies $d_n(x, y)\leq\varepsilon$, (6.1)

then by the continuity of $\bar{\sigma}_f$, we can find $\delta>0$ such that $d_n(x, y)\leq\delta$ implies $d_n(\bar{\sigma}_f^j(x), \bar{\sigma}_f^j(y))\leq\delta_M$ for $|j|\leq N$. By the facts that $\bar{h}^{-1}\circ\bar{\sigma}_f=\bar{f}\circ\bar{h}^{-1}$ and $d_n(\bar{h}^{-1}(x), x)\leq\delta_M(x\in\boldsymbol{R}^n)$, we have

$$d_n(\bar{f}^j\circ\bar{h}^{-1}(x), \bar{\sigma}_f^j(x))+d_n(\bar{f}^j\circ\bar{h}^{-1}(y), \bar{\sigma}_f^j(y))\leq 2\delta_M,$$

and so

$$d_n(\bar{f}^j\circ\bar{h}^{-1}(x), \bar{f}^j\circ\bar{h}^{-1}(y))\leq 3\delta_M \text{ for } |j|\leq N,$$

from which $d_n(\bar{h}^{-1}(x), \bar{h}^{-1}(y))\leq\varepsilon$. Therefore $\bar{h}^{-1}$ is d_n-uniformly continuous.

From now on we give the proof of (6.1). Since $d_n(\bar{h}(x), x)\leq\delta_M$ for $x\in\boldsymbol{R}^n$ and $\bar{\sigma}_f\circ\bar{h}=\bar{h}\circ\bar{f}$, $W^\sigma(x, \bar{f})\subset B_{2\delta_M}(\bar{L}^\sigma(\bar{h}(x))$ for $x\in\boldsymbol{R}^n$, and hence there exists $M_1>0$ such that

if $d_n(x, y)\leq 3\delta_M$ $(x, y\in\boldsymbol{R}^n)$ then $\max\{d_n(x, W^s(x, \bar{f})\cap W^u(y, \bar{f})), d_n(y, W^s(x, \bar{f})\cap W^u(y, \bar{f}))\}\leq M_1$. (6.2)

By Proposition 4.2, for $\varepsilon>0$ there exists $\varepsilon'>0$ such that $\bar{D}^\sigma(x, y)\leq\varepsilon'$ implies $d_n(x, y)\leq\varepsilon/2$, by Lemma 6.4 there exists $M_1'>0$ such that $d_n(x, y)\leq M_1$ $(y\in W^\sigma(x, \bar{f}))$ implies $\bar{D}^\sigma(x, y)\leq M_1'$ $(\sigma=s, u)$. Let a_2 and λ_2 be as in Proposition 4.2 and choose $N>0$ such that $a_2\lambda_2^{-N}M_1'\leq\varepsilon'$. Suppose $d_n(\bar{f}^i(x), \bar{f}^i(y))\leq 3\delta_M$ for $|i|\leq N$, then for $|i|\leq N$ there exists $z_i\in\boldsymbol{R}^n$ such that $W^s(\bar{f}^i(x), \bar{f})\cap W^u(\bar{f}^i(y), \bar{f})=\{z_i\}$ (by Lemma 6.2), and it is clear that $\bar{f}^i(z_0)=z_i$ for $|i|\leq N$. By (6.2) we have $d_n(\bar{f}^N(y), z_N)\leq M_1$ and hence $\bar{D}^u(\bar{f}^N(y), z_N)\leq M_1'$ and so $\bar{D}(y, z_0)\leq a_2\lambda_2^N M_1'\leq\varepsilon'$ from which $d_n(y, z_0)\leq\varepsilon/2$. Similary we have $d_n(x, z_0)\leq\varepsilon/2$ and so $d_n(x, y)\leq\varepsilon$.

Let $\bar{h}:\boldsymbol{R}^n\to\boldsymbol{R}^n$ be as above and define a map $\tilde{h}:\boldsymbol{R}^s\times V\to\boldsymbol{R}^s\times V$ by $\tilde{h}=$

$\pi_0\circ \bar{h}\circ\pi_0^{-1}$ where $\pi_0=\pi_{|R^s\times[0,1)}$. Since $\bar{h}$ is d_n-biuniformly continuous and $\bar{d}$ is uniformly equivalent to d_n, $\bar{h}$ is $\bar{d}$-biuniformly continuous. Moreover we have the following:

(1) $\tilde{\sigma}_f\circ\tilde{h}=\tilde{h}\circ\tilde{f}$ and

(2) $d_1(\tilde{h}(v), v)\leq\delta'_M$ for $v\in \boldsymbol{R}^s\times V$

where $\tilde{\sigma}_f=\pi_0\circ\bar{\sigma}_f\circ\pi_0^{-1}$ and δ'_M is a positive number such that $d_n(x, y)\leq\delta_M$ implies $\bar{d}(x, y)\leq\delta'_M$. Define a map h: $\psi_1(\boldsymbol{R}^s)\oplus V\rightarrow\psi_1(\boldsymbol{R}^s)\oplus V$ by $h=\psi\circ\tilde{h}\circ\psi$, then h is bijective since so is ψ.

LEMMA 6.6. *h: $\psi_1(\boldsymbol{R}^s)\oplus V\rightarrow\psi_1(\boldsymbol{R}^s)\oplus V$ and its inverse are both d_0-uniformly continuous.*

As the proof of Lemma 5.2(iii), for any $\lambda_3>0$ there exist $\delta_3>0$ and a continuous map $\tilde{\eta}: C(\delta_3)\times(\boldsymbol{R}^s\times V)\rightarrow\tilde{B}(0, \lambda_3)$ such that $\tilde{h}(v+l)=l+\tilde{h}(v)+\tilde{\eta}(l, v)$ $(l\in C(\delta_3),\ v\in\boldsymbol{R}^s\times V)$. Since $\tilde{h}$ is d_1-uniformly continuous, so is $\tilde{\eta}$. And there exists $\delta_3\geq\delta'_3>0$ such that $d_1(x, y)\leq\delta'_3$ implies $d_1(\tilde{h}(x), \tilde{h}(y))\leq\lambda_3$. If $d_0(x, y)\leq\delta'_3$ then there exist $l, l'\in C(\delta_3)$ and $v, v'\in\boldsymbol{R}^s\times V$ such that $\psi(v+l)=x$, $\psi(v'+l')=y$ and $d_1(v, v')\leq\delta'_3$. Then $\tilde{h}(v+l)-\tilde{h}(v'+l)=(l-l')+(\tilde{h}(v)-\tilde{h}(v'))+(\tilde{\eta}(l, v)-\tilde{\eta}(l', v'))$, $d_1(\tilde{h}(v), \tilde{h}(v'))\leq\lambda_3$ and $d_1(\tilde{\eta}(l, v), \tilde{\eta}(l', v'))\leq 2\lambda_3$. Since $d_0(\psi(l), \psi(l'))\leq 2\delta_3$, we have $d_0(h(x), h(y))\leq 3\lambda_3+2\delta_3\leq 5\lambda_3$ and so h is d_0-uniformly continuous.

The d_0-uniformly continuity of h^{-1} is obtained in this fashion.

Lemma 6.6 ensures the existence of a homeomorphism of X which is denoted by the same symbol, and $h\circ f=\sigma_f\circ h$ holds. Therefore the case (1) was concluded.

The conclusion of the case (2) follows from Fact 1.3 and therefore Theorem 1 was proved.

The proof of Theorem 2 is done as follows. Since $\dim(X)=1$ by the assumption, one of the maps $\bar{f}$: $\boldsymbol{R}\rightarrow\boldsymbol{R}$ and $\bar{f}^{-1}$: $\boldsymbol{R}\rightarrow\boldsymbol{R}$ must be positively expansive. For, assume that both $\bar{f}$ and $\bar{f}^{-1}$ are not positively expansive. Then we have that $D^s(x)\neq\{x\}$ and $D^u(x)\neq\{x\}$ be Lemma 2.5. Since $D^\sigma(x)$ is connected, $D^s(x)$ contains an open interval I. Take $y\in I$ and then $D^u(y)$ contains an open interval J. Thus $y\in I\cap J\subset D^s(x)\cap D^u(y)$. But $D^s(x)\cap D^u(y)$ is the set of one point, thus a contradiction.

Therefore the conclusion follows from Fact 1.3.

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Department of Mathematics
Tokyo Metropolitan University