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Saturation of iterations of bounded linear operators

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1. Introduction

Let E be a Banach space with norm $\|\cdot\|_{E}$ and let $B[E]$ denote the Banach algebra of all bounded linear operators of E into itself with the usual operator norm $\|\cdot\|_{B[E]}$. Let $\{L_{\alpha}; \alpha\in D\}$ be an approximation process on E, i. e., a net of operators in $B[E]$ converging strongly to the identity operator I on E and $\{x_{\alpha}; \alpha\in D\}$ a net of positive real numbers converging to zero. The trivial class of $\{L_{\alpha}\}$, denoted by $T[E; \{L_{\alpha}\}]$, is defined by

$$T[E; \{L_{\alpha}\}]=\{f\in E; L_{\alpha}(f)=f \text{ for all } \alpha\in D\},$$

which is a closed linear subspace of E and we set

$$S[E; \{L_{\alpha}\}, \{x_{\alpha}\}]=\{f\in E; \|L_{\alpha}(f)-f\|_{E}=O(x_{\alpha})\},$$

which is a linear subspace of E. The net $\{L_{\alpha}\}$ is said to be saturated in E with order $\{x_{\alpha}\}$ if $\|L_{\alpha}(f)-f\|_{E}=o(x_{\alpha})$ implies $f\in T[E; \{L_{\alpha}\}]$ and if there exists an element $f_{0}\in S[E; \{L_{\alpha}\}, \{x_{\alpha}\}]\backslash T[E; \{L_{\alpha}\}]$, and in this case $S[E; \{L_{\alpha}\}, \{x_{\alpha}\}]$ is called the saturation class of $\{L_{\alpha}\}$ (cf. [1], [2], [4], [7] and [10]).

Note that if $\{L_{\alpha}\}$ is saturated in E with order $\{x_{\alpha}\}$ and if $\{y_{\alpha}; \alpha\in D\}$ is a net of positive real numbers for which there exist constants $C>C'>0$ and an element $\alpha_{0}\in D$ such that $C'y_{\alpha}\leqq x_{\alpha}\leqq Cy_{\alpha}$ for all $\alpha\geqq\alpha_{0}$, then $\{L_{\alpha}\}$ is also saturated in E with order $\{y_{\alpha}\}$ and the saturation classes for these two orders are the same.

The problem of saturation may actually consist of two different questions: firstly, the question of whether saturation phenomenon occurs, that is, the establishment of the existence of the saturation order $\{x_{\alpha}\}$ of a given approximation process $\{L_{\alpha}\}$ on E; secondly, the characterization of the saturation class $S[E; \{L_{\alpha}\}, \{x_{\alpha}\}]$ of it.

The purpose of this paper lies in considering the saturation problem for a net $\{T_{\alpha}^{k}; \alpha\in D\}$ of the k-th iterations of operators T_{α} in $B[E]$, where k is an arbitrary fixed positive integer, under certain requirements.

The results of this paper can be applied to approximation processes generated by semigroups of operators in $B[E]$. For the basic theory of semigroups of operators on Banach spaces, we refer to [2] and [6].

2. Relative completions

Here we introduce the concept of relative completion which can be useful for a characterization of the saturation classes.

Let Y be a linear subspace of E with norm $\|\cdot\|_Y$ for which it is continuously embedded in E, i. e., there exists a constant $C>0$ such that $\|f\|_E \leqq C\|f\|_Y$ for all $f \in Y$. In particular, if the constant C equals one, then Y is called a normalized linear subspace of E. Also, if Y is a Banach space under the norm $\|\cdot\|_Y$, then it is called a Banach subspace of E. The relative completion of Y in E, denoted by $Y^{\sim E}$, is defined by

$$Y^{\sim E} = \cup\{\overline{S_Y(\varepsilon)}^E ;\ \varepsilon>0\},$$

where

$$S_Y(\varepsilon)=\{f \in Y ;\ \|f\|_Y \leqq \varepsilon\}$$

and $\overline{S_Y(\varepsilon)}^E$ denotes the closure of $S_Y(\varepsilon)$ in the norm $\|\cdot\|_E$- topology on E. Obviously, f belongs to $Y^{\sim E}$ if and only if there exists a sequence $\{f_n\}$ in Y such that $\sup_n\|f_n\|_Y<\infty$ and $\lim_{n\to\infty}\|f_n-f\|_E=0$.

For each $f \in Y^{\sim E}$ we put

$$\|f\|_{Y^{\sim E}}=\inf\{\varepsilon>0 ;\ f \in \overline{S_Y(\varepsilon)}^E\},$$

which defines a norm on $Y^{\sim E}$. Note that if Y is a normalized linear subspace of E, then $Y^{\sim E}$ is a normalized linear subspace of E containing Y and satisfying $\|f\|_{Y^{\sim E}} \leqq \|f\|_Y$ for all $f \in Y$. Moreover, if Y is a Banach subspace of E, then $Y^{\sim E}$ is also a Banach subspace of E. For further basic properties of the relative completions of normalized Banach subspaces, we refer the reader to [1 ; Sec. 2. 2] (cf. [4 ; Sec. 10. 4 and the references mentioned in Sec. 10. 7]).

Let L be a linear operator with domain $D(L)$ and range in E. Then $D(L)$ becomes a normalized linear subspace of E under the graph norm $\|\cdot\|_{D(L)}$ given by

$$\|f\|_{D(L)}=\|f\|_E+\|L(f)\|_E \qquad (f \in D(L)).$$

In particular, if L is a closed operator, then $D(L)$ is a normalized Banach subspace of E.

3. Main results

Let $\{L_\alpha ;\ \alpha \in D\}$ be a net of operators in $B[E]$ and let $\{x_\alpha ;\ \alpha \in D\}$ be a net of positive real numbers with $\lim_\alpha x_\alpha=0$. Let L be a linear operator with domain $D(L)$ and range in E. We say that $\{L_\alpha\}$ satisfies a Voronovskaja condition of type $(x_\alpha; L)$ on $D(L)$ if

$$\lim_\alpha x_\alpha^{-1}(L_\alpha - I) = L, \text{ strongly on } D(L). \tag{1}$$

A sequence $\{R_n; n \geqq 1\}$ of operators in $B[E]$ is called a regularization process on E for $\{L_\alpha\}$ and L if $R_n(E) \subset D(L)$ for each $n \geqq 1$, $R_n L_\alpha = L_\alpha R_n$ for all $\alpha \in D$, $n \geqq 1$, and if $\lim_{n\to\infty} \|R_n(f) - f\|_E = 0$ for every $f \in E$.

From now on let $\{T_\alpha; \alpha \in D\}$ be a net of operators in $B[E]$ satisfying a Voronovskaja condition of type $(x_\alpha; L)$ on $D(L)$ with

$$\limsup_\alpha \|T_\alpha\|_{B[E]} < \infty, \tag{2}$$

and let k be any fixed positive integer.

LEMMA. *If $D(L)$ is dense in E, then $\{T_\alpha^k; \alpha \in D\}$ satisfies a Voronovskaja condition of type $(kx_\alpha; L)$ on $D(L)$ and it is an approximation process on E.*

PROOF. We first notice that $\{T_\alpha^k\}$ satisfies a Voronovskaja condition of type $(kx_\alpha; L)$ on $D(L)$ if and only if

$$\lim_\alpha \|x_\alpha^{-1}(T_\alpha^k(f) - f) - kL(f)\|_E = 0 \text{ for all } f \in D(L). \tag{3}$$

Thus we may now show (3) by induction on k. Suppose that $D(L)$ is dense in E. In view of (1) and (2), the theorem of Banach-Steinhaus guarantees that

$$\lim_\alpha \|T_\alpha(g) - g\|_E = 0 \text{ for every } g \in E. \tag{4}$$

Let $f \in D(L)$ be given. For $k=1$, (3) is identical with (1). Now we assume that

$$\lim_\alpha \|x_\alpha^{-1}(T_\alpha^{k-1}(f) - f) - (k-1)L(f)\|_E = 0. \tag{5}$$

Then we have

$$\begin{aligned} &\|x_\alpha^{-1}(T_\alpha^k(f) - f) - kL(f)\|_E \\ &\leqq \|T_\alpha\|_{B[E]} \|x_\alpha^{-1}(T_\alpha^{k-1}(f) - f) - (k-1)L(f)\|_E \\ &\quad + (k-1)\|T_\alpha(L(f)) - L(f)\|_E + \|x_\alpha^{-1}(T_\alpha(f) - f) - L(f)\|_E, \end{aligned}$$

which converges to zero on account of (1), (2), (4) and (5), and so (3) is true. Since

$$\begin{aligned} \|T_\alpha^k(g) - g\|_E &= \Big\|\sum_{i=0}^{k-1} T_\alpha^i (T_\alpha - I)(g)\Big\|_E \\ &\leqq \sum_{i=0}^{k-1} \|T_\alpha\|_{B[E]}^i \|T_\alpha(g) - g\|_E, \end{aligned}$$

for all $\alpha \in D$ and all $g \in E$, it follows from (2) and (4) that

$$\lim_\alpha \| T^k_\alpha(g) - g \|_E = 0 \qquad (g \in E),$$

i.e., $\{T^k_\alpha\}$ is an approximation process on E. The proof is thus completed.

We set

$$S = S[E\,;\,\{T^k_\alpha\}, \{kx_\alpha\}],$$

which is a normalized Banach subspace of E under the norm $\|\cdot\|_S$ given by

$$\|f\|_S = \|f\|_E + \sup_\alpha (kx_\alpha)^{-1} \| T^k_\alpha(f) - f \|_E \qquad (f \in S).$$

Let $Ker(L)$ denote the kernel of L, i.e.,

$$Ker(L) = \{f \in D(L)\,;\, L(f) = 0\}.$$

THEOREM 1. *The following statements hold*:

(a) *Suppose that L is a closed operator and that there exists a regularization process $\{R_n\}$ on E for $\{T_\alpha\}$ and L. Let f and g be elements in E. Then f belongs to $D(L)$ and $L(f) = g$ if and only if*

$$(6) \qquad \liminf_\alpha \| (kx_\alpha)^{-1} (T^k_\alpha(f) - f) - g \|_E = 0.$$

In particular, f belongs to $Ker(L)$ if and only if

$$\liminf_\alpha (kx_\alpha)^{-1} \| T^k_\alpha(f) - f \|_E = 0.$$

(b) *If there is a regularization process $\{R_n\}$ on E for $\{T_\alpha\}$ and L, then*

$$(7) \qquad S \subset \{f \in E\,;\, \|(LR_n)(f)\|_E = O(1)\} \subset D(L)^{\sim E}.$$

(c) *If L is a closed operator, then*

$$(8) \qquad D(L)^{\sim E} \subset S.$$

PROOF. (a) Since

$$(9) \qquad R_n(h) \in D(L) \qquad (h \in E,\ n \geqq 1)$$

and

$$(10) \qquad \lim_{n \to \infty} \| R_n(h) - h \|_E = 0 \qquad (h \in E),$$

$D(L)$ is dense in E. Therefore, by Lemma, $\{T^k_\alpha\}$ satisfies a Voronovskaja condition of type $(kx_\alpha\,;\, L)$ on $D(L)$. If $f \in D(L)$ and $L(f) = g$, then by (3) we immediately obtain (6). Conversely, assume that (6) is fulfilled. By (10) and the uniform boundedness principle, there exists a constant $C > 0$ such that

(11) $$\|R_n\|_{B[E]} \leqq C \quad \text{for all } n \geqq 1.$$

Since

$$R_n T_\alpha^k = T_\alpha^k R_n \quad (n \geqq 1, \ \alpha \in D)$$

by induction on k, we have

(12) $$\begin{aligned} &\|(kx_\alpha)^{-1}(T_\alpha^k(R_n(f)) - R_n(f)) - R_n(g)\|_E \\ &\leqq \|R_n\|_{B[E]} \|(kx_\alpha)^{-1}(T_\alpha^k(f) - f) - g\|_E, \end{aligned}$$

which together with (6) implies

$$\liminf_\alpha \|(kx_\alpha)^{-1}(T_\alpha^k(R_n(f)) - R_n(f)) - R_n(g)\|_E = 0.$$

Thus in view of (3), (9) and (10), setting $f_n = R_n(f)$, we conclude

$$f_n \in D(L) \ (n \geqq 1), \ \lim_{n\to\infty} \|f_n - f\|_E = 0 \text{ and } \lim_{n\to\infty} \|L(f_n) - g\|_E = 0.$$

Hence f belongs to $D(L)$ and $L(f) = g$, since L is a closed operator.

(b) Let f be any element in S. Then there exist a constant $M > 0$ and an element $\alpha_0 \in D$ such that $\|T_\alpha^k(f) - f\|_E \leqq Mkx_\alpha$ for all $\alpha \geqq \alpha_0$. Therefore, taking $g = 0$ in (12), (11) gives

$$(kx_\alpha)^{-1} \|T_\alpha^k(R_n(f)) - R_n(f))\|_E \leqq CM \quad (\alpha \geqq \alpha_0, \ n \geqq 1),$$

which yields

$$\|L(R_n(f))\|_E \leqq CM \quad (n \geqq 1)$$

on account of (3). Thus we get the first inclusion of (7). Next, let f be any element in E with $\|(LR_n)(f)\|_E = O(1)$. Then, taking $h = f$ in (9) and (10), it follows that f belongs to $D(L)^{\sim E}$, since by (11)

$$\begin{aligned} \|R_n(f)\|_{D(L)} &= \|R_n(f)\|_E + \|L(R_n(f))\|_E \\ &= O(1) + O(1) = O(1). \end{aligned}$$

Hence the second inclusion of (7) holds.

(c) Suppose that L is a closed operator. Then $D(L)$ becomes a Banach space under the graph norm $\|\cdot\|_{D(L)}$. Therefore, in view of (3) and the uniform boundedness principle, there exist an element $\beta_0 \in D$ and a constant $C_1 > 0$ such that

(13) $$(kx_\alpha)^{-1} \|T_\alpha^k(f) - f\|_E \leqq C_1 \|f\|_{D(L)}$$

for all $f \in D(L)$ and all $\alpha \geqq \beta_0$. Now, let g be any element in $D(L)^{\sim E}$, and thus there is a sequence $\{g_n ; n \geqq 1\}$ of elements in $D(L)$ and a constant $C_2 > 0$ such that $\|g_n\|_{D(L)} \leqq C_2$ for every $n \geqq 1$ and $\lim_{n\to\infty} \|g_n - g\|_E = 0$. Replacing

f by g_n in (13), and letting n tend to infinity, we have

$$(k\chi_\alpha)^{-1}\|T_\alpha^k(g)-g\|_E \leqq C_1C_2 \qquad (\alpha \geqq \beta_0),$$

which implies that g belongs to S. Thus the inclusion (8) holds, and the proof of the theorem is complete.

COROLLARY 1. *Assume that L is a closed operator and that there exists a regularization process $\{R_n\}$ on E for $\{T_\alpha\}$ and L. Then*

$$S=\{f\in E\,;\ \|(LR_n)(f)\|_E=O(1)\}=D(L)^{\sim E}.$$

In particular, if E is reflexive, then

$$S=\{f\in E\,;\ \|(LR_n)(f)\|_E=O(1)\}=D(L).$$

This result follows from Theorem 1 (b), (c) and [4; Proposition 10. 4. 3] (cf. [1; p. 15 (h)]).

REMARK 1. Let $\{T_{\alpha,\lambda}\,;\ \alpha\in D,\ \lambda\in\Lambda\}$ be a family of operators in $B[E]$, where Λ is an arbitrary index set. Then in the same way as the author [9; Definitions 1 and 3], the saturation property and the Voronovskaja condition are defined for the family $\{T_{\alpha,\lambda}\}$, and the statements analogous to Theorem 1 and Corollary 1 can be obtained for the family $\{T_{\alpha,\lambda}^k\}$ of the k-iterations of $T_{\alpha,\lambda}$. Actually, the results of the author [9] can be extended to approximation processes of iterations of multiplier operators in Banach spaces. We omit the details.

4. Applications

Let $\{W(t)\,;\ t\geqq 0\}$ be a strongly continuous semigroup of operators in $B[E]$, i. e., a family of operators in $B[E]$ satisfying the following conditions:

(i) $W(t+u)=W(t)W(u) \qquad (t,\ u\geqq 0)$,

(ii) $W(0)=I$,

(iii) The map $t\rightarrow W(t)$ is continuous on $[0,\infty)$ in the strong operator topology on $B[E]$, i. e.,

$$\lim_{t\to u}\|W(t)(f)-W(u)(f)\|_E=0$$

for each $f\in E$ and for every $u\geqq 0$.

We define

$$G(f)=\lim_{t\to 0+0}(1/t)(W(t)(f)-f), \tag{14}$$

whenever the limit exists in the sense of strong convergence and let $D(G)$

denote the set of all elements $f \in E$ for which the strong limit in (14) exists. Evidently, $D(G)$ is a linear subspace of E and G is a linear operator of $D(G)$ into E. This operator G is called the infinitesimal generator of the semigroup $\{W(t)\}$.

The Cesàro mean operators $C(t)$ $(t>0)$ are defined by

$$C(t)(f)=(1/t)\int_0^t W(u)(f)\,du \qquad (f \in E). \tag{15}$$

Then we have

$$(1/t)(W(t)(f)-f)=C(t)(G(f)) \qquad (f \in D(G),\ t>0), \tag{16}$$

and so

$$Ker(G)=T[E\,;\ \{W(t)\}].$$

Also, it can be seen that

$$C(t)(f) \in D(G),\ \lim_{t \to 0+0}\|C(t)(f)-f\|_E=0 \qquad (f \in E) \tag{17}$$

and

$$W(t)C(u)=C(u)W(t),\ C(t)C(u)=C(u)C(t) \qquad (t,\ u>0). \tag{18}$$

In particular, (17) implies that $D(G)$ is dense in E. Moreover, G is a closed operator. Indeed, suppose that $f_n \in D(G)$ $(n \geqq 1)$, $\lim_{n\to\infty}\|f_n-f\|_E=0$ and $\lim_{n\to\infty}\|G(f_n)-g\|_E=0$. Then, using (16) and the continuity of $W(t)$, $C(t)$ we obtain

$$\begin{aligned}(1/t)(W(t)(f)-f)&=\lim_{n\to\infty}(1/t)(W(t)(f_n)-f_n)\\&=\lim_{n\to\infty}C(t)(G(f_n))=C(t)(g).\end{aligned}$$

Therefore, the latter fact of (17) yields $f \in D(G)$ and $G(f)=g$, which shows that G is closed.

For further extensive properties of semigroups of operators on Banach spaces, the reader may consult [2] and [6].

Now let $\{y_\alpha\,;\ \alpha \in D\}$ be a net of positive real numbers converging to zero and $\{n_\alpha\,;\ \alpha \in D\}$ a net of positive integers satisfying $\lim_\alpha n_\alpha y_\alpha=0$. Setting

$$W_\alpha=(W(y_\alpha))^{n_\alpha} \qquad (\alpha \in D),$$

we have, from the condition (i), $W_\alpha=W(n_\alpha y_\alpha)$ for all $\alpha \in D$.

THEOREM 2. *The following assertions hold*:

(a) *Let f and g be elements in E. Then f belongs to $D(G)$ and*

$G(f)=g$ if and only if

$$\lim\inf_{\alpha}\|(n_{\alpha}y_{\alpha})^{-1}(W_{\alpha}(f)-f)-g\|_{E}=0.$$

In particular, $f\in Ker(G)$, or equivalently, $f\in T[E\,;\,\{W_{\alpha}\}]$ if and only if

$$\lim\inf_{\alpha}(n_{\alpha}y_{\alpha})^{-1}\|W_{\alpha}(f)-f\|_{E}=0.$$

(b) *$\{W_{\alpha}\}$ is saturated in E with order $\{n_{\alpha}y_{\alpha}\}$ and its saturation class coincides with $D(G)^{\sim E}$. In particular, if E is reflexive, then $S[E\,;\,\{W_{\alpha}\},\{n_{\alpha}y_{\alpha}\}]=D(G)$.*

PROOF. The net $\{W_{\alpha}\}$ satisfies a Voronovskaja condition of type $(n_{\alpha}y_{\alpha}\,;\,G)$ on $D(G)$ with G being a closed operator. Also, there exists an element $f_0\in D(G)\setminus T[E\,;\,\{W_{\alpha}\}]$, and thus $\|W_{\alpha}(f_0)-f_0\|_{E}=O(n_{\alpha}y_{\alpha})$. Furthermore, defining the operator R_n in $B[E]$ by

(19) $$R_n(f)=C(1/n)\qquad(f\in E,\ n\geqq 1),$$

(17) and (18) imply that the sequence $\{R_n\}$ is a regularization process on E for $\{W_{\alpha}\}$ and G. Therefore, the desired results follow from Theorem 1 (a) and Corollary 1.

As an immediate consequence of Theorem 2 (b) we have the following:

COROLLARY 2. *The net $\{W(y_{\alpha})\,;\,\alpha\in D\}$ is saturated in E with order $\{y_{\alpha}\}$ and its saturation class is equal to $D(G)^{\sim E}$. In particular, if E is reflexive, then $S[E\,;\,\{W(y_{\alpha})\},\{y_{\alpha}\}]=D(G)$.*

REMARK 2. For any $f\in E$, $t\geqq 0$ we define

$$K(t,f)=\inf\{\|f-g\|_{E}+t\|G(g)\|_{E}\,;\,g\in D(G)\},$$

which is called the K-functional of f and plays an important role in the study of intermediate spaces (cf. [2 ; Chap. III]). Using this quantity we have

(20) $$\begin{aligned}S[E\,;\,\{W(t)\},\{t\}](=\{f\in E\,;\,\|W(t)(f)-f\|_{E}\\ =O(t)\ (t\to 0+0)\})=D(G)^{\sim E}=X[E\,;\,G],\end{aligned}$$

where

$$X[E\,;\,G]=\{f\in E\,;\,K(t,f)=O(t)\ (t\to 0+0)\}$$

(cf. [2 ; Proposition 3.4.1 and Theorem 3.4.3]).

From now on let k be any fixed positive integer. Putting

$$C_{\alpha}=C(y_{\alpha})\qquad(\alpha\in D)$$

with the Cesàro mean operators given by (15), we have

THEOREM 3. *The following statements hold*:

(a) *Let f and g be elements in E. Then f belongs to $D(G)$ and $(1/2)G(f)=g$ if and only if*

$$\liminf_\alpha \|(ky_\alpha)^{-1}(C_\alpha^k(f)-f)-g\|_E=0.$$

In particular, $f\in Ker(G)$, or equivalently, $f\in T[E;\{C_\alpha^k\}]$ if and only if

$$\liminf_\alpha (ky_\alpha)^{-1}\|C_\alpha^k(f)-f\|_E=0.$$

(b) *$\{C_\alpha^k\}$ is saturated in E with order $\{ky_\alpha\}$ and its saturation class is identical with $D(G)^{\sim E}$, or equivalently, $X[E;G]$. In particular, if E is reflexive, then $S[E;\{C_\alpha^k\},\{ky_\alpha\}]=D(G)=X[E;G]$.*

PROOF. By [3; Lemma 1], the net $\{C_\alpha\}$ satisfies a Voronovskaja condition of type $(y_\alpha;(1/2)G)$ on $D(G)$ with G being a closed operator. Note that there exist an element $f_0\in D(G)\backslash T[E;\{C_\alpha^k\}]$, and so $\|C_\alpha^k(f_0)-f_0\|_E=O(ky_\alpha)$. Moreover, in view of (17), (18) and (19), the sequence $\{R_n\}$ becomes a regularization process on E for $\{C_\alpha\}$ and $(1/2)G$. Therefore, the desired claims follow from Theorem 1 (a), Corollary 1 and (20).

As an immediate consequence of Theorem 3 we obtain the following:

COROLLARY 3. *$\{C_\alpha\}$ is saturated in E with order $\{y_\alpha\}$ and its saturation class coincides with $D(G)^{\sim E}$, or equivalently, $X[E;G]$. In particular, if E is reflexive, then $S[E;\{C_\alpha\},\{y_\alpha\}]=D(G)=X[E;G]$.*

Next let us consider the Abel mean operators defined by

$$A(\omega)(f)=\omega\int_0^\infty \exp(-\omega t)W(t)(f)\,dt \qquad (f\in E,\ \omega>\omega_0),$$

where

$$\omega_0=\inf\{(1/t)\log\|W(t)\|_{B[E]};\ t>0\}$$
$$=\lim_{t\to\infty}(1/t)\log\|W(t)\|_{B[E]}.$$

Let $\{z_\alpha;\ \alpha\in D\}$ be a net of positive real numbers satisfying $z_\alpha>\omega_0$ for all $\alpha\in D$ and $\lim_\alpha z_\alpha=+\infty$. Then setting

$$A_\alpha=A(z_\alpha) \qquad (\alpha\in D),$$

we get the following.

THEOREM 4. *The following results hold*:

(a) *Let f and g be elements in E. Then f belongs to $D(G)$ and*

$G(f)=g$ if and only if

$$\liminf_\alpha \|(z_\alpha/k)(A_\alpha^k(f)-f)-g\|_E=0.$$

In particular, $f\in Ker(G)$, or equivalently, $f\in T[E\,;\,\{A_\alpha^k\}]$ if and only if

$$\liminf_\alpha (z_\alpha/k)\|A_\alpha^k(f)-f)\|_E=0.$$

(b) *$\{A_\alpha^k\}$ is saturated in E with order $\{k/z_\alpha\}$ and its saturation class is equal to $D(G)^{\sim E}$, or equivalently, $X[E\,;\,G]$. In particular, if E is reflexive, then $S[E\,;\,\{A_\alpha^k\},\{k/z_\alpha\}]=D(G)=X[E\,;\,G]$.*

PROOF. $\{A_\alpha\}$ satisfies a Voronovskaja condition of type $(1/z_\alpha\,;\,G)$ on $D(G)$ with G being a closed operator (cf. [2; Proposition 2.5.2 (b)], [5; Proposition 2.1 (b)]), and there is an element $f_0\in D(G)\backslash T[E\,;\,\{A_\alpha^k\}]$, and hence $\|A_\alpha^k(f_0)-f_0\|_E=O(k/z_\alpha)$. Furthermore, since

$$C(t)A(\omega)=A(\omega)C(t) \qquad (t>0,\ \omega>\omega_0),$$

(17) yields that the sequence $\{R_n\}$ given by (19) is a regularization process on E for $\{A_\alpha\}$ and G. Thus the desired assertions follow from Theorem 1 (a), Corollary 1 and (20).

REMARK 3. In [10] we established the saturation theorem for a net of iterations of positive linear operators on the Banach lattice of all real-valued continuous functions on a compact Hausdorff space. Applying Theorem 1 (b), (c) and Corollary 1, we are able to characterize the saturation class of it. Also, the results of this paper can be applied to iterations of the Bernstein operators and the strongly continuous semigroup of Markov operators treated in [10; Sec. 4] (cf. [8]).

Finally, we mention some concrete examples of strongly continuous semigroups of contractions on the classical function spaces.

EXAMPLE 1 (Translation operators): Let $E=C(-\infty,\infty)$ denote the Banach space of all bounded uniformly continuous functions f on $(-\infty,\infty)$ with the supremum norm

$$\|f\|_\infty=\sup\{|f(x)|\,;\,-\infty<x<\infty\}.$$

Defining $\{T(t)\,;\,t\geqq 0\}$ by

$$(T(t)(f))(x)=f(x+t) \qquad (f\in E,\ -\infty<x<\infty),$$

the infinitesimal generator G is given by $G(f)=f'$ with domain

$$D(G)=\{f\in E\,;\,f'\in E\}.$$

EXAMPLE 2 (Gauss-Weierstrass operators): Let $1 \leqq p < \infty$, and let $E = L^p(-\infty, \infty)$ denote the Banach space of all Lebesgue measurable functions f on $(-\infty, \infty)$ such that $|f(x)|^p$ is integrable on $(-\infty, \infty)$ with the norm

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{1/p}.$$

We define $\{U(t) ; t \geqq 0\}$ by $U(0) = I$ and

$$(U(t)(f))(x) = (\pi t)^{-1/2} \int_{-\infty}^{\infty} f(x-y) \exp(-y^2/t) dy$$
$$(f \in E,\ t > 0,\ -\infty < x < \infty).$$

Then its infinitesimal generator G is given by $G(f) = (1/4) f''$ with domain

$$D(G) = \{f \in E ; f, f' \in AC\ (-\infty, \infty) \text{ and } f', f'' \in E\},$$

where $AC\ (-\infty, \infty)$ denotes the space of all absolutely continuous functions on $(-\infty, \infty)$.

EXAMPLE 3. (Cauchy-Poisson operators): Let $E = L^p(-\infty, \infty)$, $1 \leqq p < \infty$, and define $V(0) = I$,

$$(V(t)(f))(x) = (t/\pi) \int_{-\infty}^{\infty} f(x-y)/(t^2+y^2) dy$$
$$(f \in E,\ t > 0,\ -\infty < x < \infty).$$

Then its infinitesimal generator G is given by $G(f) = -(f^{\sim})'$ with domain

$$D(G) = \{f \in E ; f^{\sim} \in AC\ (-\infty, \infty) \text{ and } (f^{\sim})' \in E\},$$

where $f^{\sim}$ denotes the Hilbert trasform of f, defined by

$$f^{\sim}(x) = -(1/\pi) \lim_{\varepsilon \to 0+0} \int_{\varepsilon}^{\infty} \{f(x+y) - f(x-y)\}/y dy.$$

For further details of these operators, we refer to [2; Chap. IV] where additional examples can be found.

References

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