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Comparison of Martin boundaries for Schrödinger operators

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We consider the Martin compactification R_P^* for an admissible Schrödinger operator $-\Delta+P$ on a Riemann surface R with singular but nonnegative potentials P on R and study how R_P^* varies according to a small perturbation of the potential P. One reason of the importance of the study of this kind lies in the following instance. For a detailed study of R_P^* it often occurs the need to construct a potential P such that R_P^* possesses a property given in advance (cf. e.g. **[10]**, **[9]**, **[11]** among many others). In this construction it is easier to seek an appropriate P among potentials which are allowed to be discontinuous than to do among only those that are restricted to be smooth. However our primary concern is about R_P^* with smooth P. Thus one natural procedure may be as follows. First find a P among discontinuous potentials such that R_P^* has a desired property. Then approximate P by smooth potentials, e.g. by $\rho_\varepsilon * P(\varepsilon\downarrow 0)$ with $\rho_\varepsilon *$ the Friedrichs mollifier in a suitable sense (cf. no. 15 below), and then we expect $R_{\rho_\varepsilon * P}^*$ to be identical with R_P^* from the view point of the Martin theory if the approximation is made close enough. This is the motivation of our present study. In this paper we will give a theorem asserting $R_P^*=R_Q^*$ under a certain closeness condition on potentials P and Q on R. Any theorem of this kind (cf. e.g. **[16]**, **[9]**, etc.) would not be considered as natural if it did not imply the following two facts:

a. $R_P^*=R_Q^*$ if $P=Q$ on R outside a compact subset;
b. $R_P^*=R_{\rho_\varepsilon * P}^*$ if ε is small enough in a suitable sense.

Our theorem certainly contains these two facts and especially the validity of the latter of the above must be useful in actual constructions as mentioned above.

After preliminary discussions in nos. 1-4, the main comparison theorem is stated in no. 5 and proven in nos. 6-11. The fact **a** above is deduced in no. 12, and the fact **b** above is stated in no. 13 and proven in nos. 14-15.

Although we state and prove our results for Schrödinger operators

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$-\Delta+P$ on Riemann surfaces R, an obvious modification gives the corresponding results for Schrödinger operators on subregions R of Euclidean space $\boldsymbol{R}^d(d\geqq 2)$.

1. As our basic space we fix an arbitrary open (i. e. noncompact) Riemann surface R. By a *disk* V on R we mean that $\bar{V}$ is compact in R and there exists an associated local parameter z that maps $\bar{V}$ conformally onto the closed disk $|z|\leqq 1$ on the plane. We denote by $V(r)$ $(0<r\leqq 1)$ for agiven disk V the concentric disk corresponding to $|z|<r$ so that $V(1)=V$. Of course $z^{-1}(0)$ is referred to as the center of $V(r)$ and r the radius of $V(r)$. Any point of R can be a center of a disk on R. Consider a *2-form* P on R so that P has an expression $P(z)dxdy$ on any disk V with $z=x+iy$ its associated local parameter and with a function $P(z)$ of z on V. We say that P is nonnegative (positive, resp.), $P\geqq 0$ $(P>0$, resp.) in notation, if $P(z)\geqq 0$ $(P(z)>0$, resp.) on V for every disk V on R. Measurability of P is similarly defined using *local Lebesgue measure* $dm(z)=dxdy$. As usual we denote by $L^p_{loc}(R)$ $(1\leqq p\leqq\infty)$ the class of all measurable real 2-forms P on R such that the integral of $|P(z)|^p$ over V with respect to the measure $dm(z)$ $=dxdy$ is finite for $1\leqq p<\infty$ or the essential supremum of $|P(z)|$ on V with respect to dm is finite for $p=\infty$ for every disk V on R. Note that $L^p_{loc}(R)$ is a class of 2-forms, not of functions. We denote by $K_{loc}(R)$ the class of measurable real 2-forms P on R such that

$$\lim_{r\to 0}\left(\sup_{z\in V(r)}\int_{V(r)}\log\frac{1}{|z-\zeta|}|P(\zeta)|dm(\zeta)\right)=0 \tag{1.1}$$

for every disk V on R. By an application of the Hölder inequality and the integrability of $\|\log|\zeta\|^q$ $(1\leqq q<\infty)$ over any finite disk of the plane we easily see that

$$L^p_{loc}(R)<K_{loc}(R)<L^1_{loc}(R)\ (1<p\leqq\infty) \tag{1.2}$$

where $<$ means the strict inclusion. The letter K suggests Kato who first explicitly considered the class ([**6**], also [**15**], [**13**], [**1**], [**14**]). For a class $\mathscr{F}$ of functions or 2-forms we denote by $\mathscr{F}^+$ the subclass $\{f\in\mathscr{F}: f\geqq 0\}$.

2. We denote by Δ the laplacian on R so that Δ has the local expression $4\partial^2/\partial z\partial\bar{z}=\partial^2/\partial x^2+\partial^2/\partial y^2$ on any disk V on R with $z=x+iy$ its local parameter. A *Schrödinger operator* on R is the expression $-\Delta+P(z)$ where $P=P(z)dxdy$ is referred to as its *potential* which is here a 2-form on R. No confusion is expected for the above term potential with the potential theoretic "potential" (cf. [**4**]). We mean in this paper by a *solution* u of a stationary Schrödinger equation

(2.1) $\quad (-\Delta+P(z))u(z)=0$

on R a *continuous* function u on R satisfying (2.1) above in the sense of distribution, i. e. $uP\in L^1_{\mathrm{loc}}(R)$ and

$$-\int_R u(z)\Delta\varphi(z)dm(z)+\int_R u(z)\varphi(z)P(z)dm(z)=0$$

for every test function φ in $C_0^\infty(R)$. Using the same letter P as the potential, we denote by $P(\Omega)$ the class of solutions of (2.1) on an open subset Ω of R. Then P also defines a *sheaf* of solutions of (2.1) on R. The significance of the class $K_{\mathrm{loc}}(R)$ reveals itself in the following

FACT. *Let P be a nonnegative measurable 2-form on R. The pair (R, P) with P being considered as a sheaf on R is a Brelot harmonic space if and only if $P\in K_{\mathrm{loc}}(R)$.*

First we remark the following. Let V be any disk on R. We embed V into the plane $\boldsymbol{C}$ in the natural sense and difine a function $F_V(z)$ to be $P(z)$ on V and 0 on $\boldsymbol{C}\setminus V$. Then the logarithmic potential

$$\int_C \log\frac{1}{|z-\zeta|}F_V(\zeta)dm(\zeta)$$

is continuous on $\boldsymbol{C}$ for every V if and only if (1.1) holds for every V or equivalently $P\in K_{\mathrm{loc}}(R)$. Now the sufficiency of the condition of the above fact is found in [**2**] (see also [**1**]). The necessity of the condition is not difficult to prove though nontrivial but we omit it here since we only use the sufficiency part of the above fact in this paper. Therefore we henceforth only consider Schrödinger operators of the following kind:

(2.2) $\quad -\Delta+P(z),\ P=P(z)dxdy\in K_{\mathrm{loc}}(R)^+.$

3. In view of the above fact we can now freely use the local potential theory of Brelot (cf. e. g. [**4**], [**7**], etc.) for our (R, P) with (2.2). A subregion Ω is said to be *nice* if it is relatively compact and every point of $\partial\Omega$ is regular with respect to the usual classical harmonic Dirichlet problem. Then nice regions are regular with respect to any (R, P) (cf. [**2**]). We denote by P_f^Ω the solution of the P-Dirichlet problem on nice Ω with boundary values $f\in C(\partial\Omega)$, i. e. $P_f^\Omega\in C(\bar{\Omega})\cap P(\Omega)$ with $P_f^\Omega|\partial\Omega=f$. On any nice region Ω of R there exists the *P-Geen function* (i. e. the Green function of Ω for the harmonic structure (R, P)) $G_P^\Omega(\cdot, w)$ with its pole $w\in\Omega$ (cf. [**2**]). Here $G_P^\Omega(\cdot, w)$ is so normalized as to satisfy

$$(-\Delta+P(\cdot))G_P^\Omega(\cdot, w)=\delta_w \text{ (the Dirac measure at } w).$$

The property of $G_P^\Omega(z, w)$ is reduced to that of $G_0^\Omega(z, w)$, the classical harmonic Green function, by

$$(3.1)\qquad G_P^\Omega(z, w)+\int_\Omega G_0^\Omega(z, \zeta)P(\zeta)G_P^\Omega(\zeta, w)dm(\zeta)=G_0^\Omega(z, w)$$

so that e. g. we see that $G_P^\Omega(z, w)=O(-\log|z-w|)$ as $z, w\to w_0\in\Omega$. Here and hereafter we follow the usual loose convention to denote the generic point of R and its local parameter by the same letter. Another important consequence of (3.1) is the symmetry of G_P^Ω : $G_P^\Omega(z, w)=G_P^\Omega(w, z)$ for z and w in Ω with $z\neq w$ (cf. [2]).

Since $\{G_P^\Omega(\cdot, w)\}_{\Omega\uparrow R}$ is increasing, either it converges to the P-Green function $G_P(\cdot, w)=G_P^R(\cdot, w)$ on R almost uniformly of $R\backslash\{w\}$ or it diverges to $+\infty$ almost uniformly on R. In the former case we say that P is *hyperbolic* (of. e. g. [**12**]). In view of

$$(3.2)\qquad \int_\Omega G_P^\Omega(\cdot, \zeta)P(\zeta)dm(\zeta)=1-P_1^\Omega$$

we see that P is hyperbolic if $P\neq 0$, or more precisely if the measure of the set $\{z\in R : P(z)\neq 0\}$ is not zero. We may express this as

$$m(\{z : P(z)\neq 0\})>0.$$

If 0 is hyperbolic (in this case R is said to be hyperbolic in the Riemann surface theory), then by (3.1), any $P\in K_{\mathrm{loc}}(R)^+$ is hyperbolic. If 0 is not hyperbolic (in this case R is said to be parabolic in the above field), then, by the above remark, any $P\in K_{\mathrm{loc}}(R)^+$ is hyperbolic when and only when $P\neq 0$ (i. e. $m(\{z : P(z)\neq 0\})>0$). In any case, therefore, any potential in $K_{\mathrm{loc}}(R)^+$ greater than or equal to a hyperbolic potential in $K_{\mathrm{loc}}(R)^+$ is also hyperbolic.

Suppose that Q is also a potential in $K_{\mathrm{loc}}(R)^+$. We have

$$G_Q^\Omega(z, w)=G_P^\Omega(z, w)+\int_\Omega G_Q^\Omega(z, \zeta)(P(\zeta)-Q(\zeta))G_P^\Omega(\zeta, w)dm(\zeta).$$

In view of (1.1) and (3.2) we have the following *resolvent equation* as the limiting case of the above displayed formula if P and Q are hyperbolic :

$$(3.3)\qquad \begin{aligned} G_Q(z, w)&=G_P(z, w)+\int_R G_Q(z, \zeta)(P(\zeta)-Q(\zeta))G_P(\zeta, w)dm(\zeta)\\ &=G_P(z, w)+\int_R G_P(z, \zeta)(P(\zeta)-Q(\zeta))G_Q(\zeta, w)dm(\zeta).\end{aligned}$$

Here the last identity of the above follows from the symmetry of G_P and G_Q.

4. Take a hyperbolic potential P in $K_{\mathrm{loc}}(R)^+$. We fix throughout the paper a reference point a in R. The *P-Martin kernel* $K_P(z, w)$ on R is defined as follows. First for $(z, w) \in R \times R$ we set

$$(4.1) \qquad K_P(z, w) = \begin{cases} G_P(z, w)/G_P(a, w) & (a \neq w) \\ 0 & (a = w,\ z \neq w) \\ 1 & (a = w = z). \end{cases}$$

For each fixed $z \in R$ the function $K_P(z, \cdot)$ is continuous on R and there exists a unique (up to a homeomorphism, of course) compactification R_P^* of R such that every $K_P(z, \cdot)$ $(z \in R)$ is continuously extended to R_P^* (cf. [3], [5]). We call R_P^* the *Martin compactification* of R for the Schrödinger operator $-\Delta + P$ or simply *P-Martin compactification.* The set $\partial_P R = R_P^* \backslash R$ is the *P-Martin boundary* of R. Then finally $K_P(z, w^*)$ is defined on $R \times R_P^*$ by

$$K_P(z, w^*) = \lim_{w \in R, w \to w^*} K_P(z, w).$$

Clearly $K_P(\cdot, w^*) \in P(R \backslash \{w^*\})$ and $K_P(\cdot, \cdot) \in C(R \times R_P^* \backslash \Delta)$ where here Δ is the diagonal set of $R \times R$. Fix a disk V_a with a its center. Then the topology of R_P^* defined above is metrizable by the following metric

$$d(w_1^*, w_2^*) = \int_{V_a} \frac{|K_P(z, w_1^*) - K_P(z, w_2^*)|}{1 + |K_P(z, w_1^*) - K_P(z, w_2^*)|} dm(z) \ (w_1^*, w_2^* \in R_P^*).$$

We call $u \in P(R)^+$ *minimal* if for any $v \in P(R)^+$ with $v \leqq u$ on R there exists a constant c such that $v = cu$ on R. We denote by $\delta_P R$ the set of points $w^* \in \partial_P R$ such that $K_P(\cdot, w^*)$ is minimal. We call $\delta_P R$ the *P-Martin minimal boundary* of R. Since every necessary tool is available (cf. [2]) to derive the Martin theory [8], we can establish the following result along the line given by Martin (cf, e. g. [5], [4]):

MARTIN THEORY. *The minimal boundary $\delta_P R$ is a G_δ subset of the whole boundary $\partial_P R$ and there exists a bijective correspondence $u \to \mu$ between $P(R)^+$ and the family $\{\mu\}$ of nonnegative Borel measures μ on $\delta_P R$ such that*

$$(4.2) \qquad u = \int_{\delta_P R} K_P(\cdot, w^*) d\mu(w^*) \ (\textit{canonical representation}).$$

Consider two hyperbolic potentials P and Q in $K_{\mathrm{loc}}(R)^+$. If the identity map ι of R onto itself can be extended to a homeomorphism ι_{PQ}^* of R_P^* onto R_Q^*, then we say that R_P^* and R_Q^* are *naturally homeomorphic,* and if moreover $\iota_{PQ}^{*}(\delta_P R)=\delta_Q R$, then we say that R_P^* and R_Q^* are *canonically homeomorphic,* and in this case we may say that $R_P^*=R_Q^*$ not only from the topological but also potential theoretic view point.

5. Let $\{V_j\}_{j\in N}$ be a locally finite covering of R by disks V_j on R. For definiteness we always assume that $a\in V_a\subset \bar{V}_a\subset V_1$, which is entirely inessential. For a potential $P\in K_{\mathrm{loc}}(R)^+$ we consider a family $\{P_j\}_{j\in N}$ of potentials $P_j\in K_{\mathrm{loc}}(R)^+$ such that

$$\begin{cases}\mathrm{supp}\, P_j\subset V_j \quad (j\in \boldsymbol{N}),\\ P=\sum_{j\in N} P_j.\end{cases}$$

In this case we say that $\{P_j\}$ is a *decomposition of P associated with $\{V_j\}$.* Now consider two hyperbolic potentials P and Q in $K_{\mathrm{loc}}(R)^+$. We say that *P rules Q,* or equivalently, *Q is ruled by P,* if there exists a locally finite covering $\{V_j\}$ of R by disks and decompositions $\{P_j\}$ and $\{Q_j\}$ of P and Q respectively associated with $\{V_j\}$ such that

$$\sum_{j\in N}\frac{1}{\inf_{z\in V_j} G_P(a,z)}\sup_{z\in V_j}\int_{V_j} G_P(z,\zeta)|P_j(\zeta)-Q_j(\zeta)|dm(\zeta)<\infty. \tag{5.1}$$

The purpose of this paper is to prove the following comparison criterion for Martin compactifications for two Schrödinger operators:

THE MAIN THEOREM. *Suppose P and Q are hyperbolic potentials in the class $K_{\mathrm{loc}}(R)^+$. If P rules Q, then R_P^* and R_Q^* are canonically homeomorphic.*

The proof will be given below in nos. 6-11.

6. Throughout the proof we fix an exhaustion $\{R_\nu\}_{\nu\in N}$ of R by nice subregions R_ν such that $\bar{V}_1\subset R_1$, $\bar{R}_\nu\subset R_{\nu+1}$ $(\nu\in \boldsymbol{N})$, and $R=\cup_{\nu\in N}R_\nu$. For each $\nu\in \boldsymbol{N}$ we set

$$\alpha(\nu)=\max\{\alpha\in \boldsymbol{N} : \cup_{1\leqq j\leqq \alpha}\bar{V}_j\subset R_\nu\}.$$

Then $\{\alpha(\nu)\}_{\nu\in N}$ is a nondecreasing divergent sequence. We also set

$$\beta(\mu)=\min\{\nu\in \boldsymbol{N} : (U_{j>\alpha(\nu)}\bar{V}_j)\cap \bar{R}_{\mu+1}=\emptyset\}$$

for each $\mu\in \boldsymbol{N}$. Clearly $\beta(\mu)>\mu+1$ and $\bar{R}_{\mu+1}\subset R_\nu$ for each $\nu\geqq\beta(\mu)$.

For simplicity we set $D_j=P_j-Q_j$ and $D=\sum_{j\in N}D_j$ which are all in $K_{\mathrm{loc}}(R)$ but not necessarily nonnegative. Needless to say $D=P-Q$. We also set

$$\begin{cases} \gamma_j = \gamma_j(P) = \inf_{z \in V_j} G_P(a, z) \\ d_j = d_j(P, Q) = \sup_{z \in V_j} \int_{V_j} G_p(z, \zeta)|D_j(\zeta)|dm(\zeta) \end{cases}$$

for each $j \in \boldsymbol{N}$. Since $\sum_{j \in \boldsymbol{N}} d_j/\gamma_j < \infty$ by (5.1), we have

(6.1) $$\lim_{\nu \to \infty} \sum_{j > \alpha(\nu)} d_j/\gamma_j = 0$$

by virtue of the definition of $\alpha(\nu)$.

We consider one more quantity $k(\mu)$ for each $\mu \in \boldsymbol{N}$ as follows:

$$k(\mu) = k(\mu : X) = \sup\{G_X(z, \zeta) : z \in R_\mu,\ \zeta \in R \backslash R_{\mu+1}\},$$

where $X = P$ or Q. By the definition of $\beta(\mu)$ we have $\bar{R}_{\mu+1} \subset R_\nu$ for $\nu \geqq \beta(\mu)$ and in particular $\bar{R}_\mu \subset R_{\beta(\mu)}$. Thus $k(\mu) < \infty$ and $G_X(z, \zeta) \leqq k(\mu)$ for any $z \in R_\mu$ and $\zeta \in R \backslash R_\nu (\nu \geqq \beta(\mu))$. Therefore $\bar{V}_j \subset R \backslash R_{\mu+1}$ for each $j > \alpha(\nu) \geqq \alpha(\beta(\mu))$ $(\nu \geqq \beta(\mu))$ and we have

$$\int_R G_X(z, \zeta)|D_j(\zeta)|G_P(\zeta, w)dm(\zeta) \leqq k(\mu)d_j \quad ((z, w) \in R_\mu \times \bar{V}_j).$$

Consider the measure $d\mu_z(\zeta) = G_X(z, \zeta)|D_j(\zeta)|dm(\zeta)$ and use the standard notation $G_P\mu_z = \int G_P(\cdot, \zeta)d\mu_z(\zeta)$ for G_P-potentials. Then the above inequality can be rewritten as

$$G_P\mu_z(w) \leqq k(\mu)d_j \quad ((z, w) \in R_\mu \times \bar{V}_j).$$

Since $G_P(a, w) \geqq \gamma_j$ for any $w \in V_j$, we deduce from the above

$$G_P\mu_z(w) \leqq k(\mu)(d_j/\gamma_j)G_P(a, w)$$

for any $w \in V_j$ and in particular for any $w \in \operatorname{supp}\mu_z$. Since $G_P(a, \cdot)$ is positive and P-superharmonic (i.e. superharmonic with respect to the Brelot harmonic space (R, P)), the domination principle which is known to hold for (R, P) (cf. [2]) yields the last displayed inequality for every $w \in R$. We thus have

(6.2) $$\int_R G_X(z, \zeta)|D_j(\zeta)|G_P(\zeta, w)dm(\zeta) \leqq k(\mu)(d_j/\gamma_j)G_P(a, w)$$

for each $(z, w) \in R_\mu \times R$ and each $j > \alpha(\nu)$ $(\nu \geqq \beta(\mu))$. By the definition of $\alpha(\nu)$ we see that $\zeta \in R \backslash R_\nu$ implies $\zeta \in R \backslash V_j (1 \leqq j \leqq \alpha(\nu))$ and then $D_j(\zeta) = 0$ $(1 \leqq j \leqq \alpha(\nu))$ and therefore

$$D(\zeta) = \sum_{j > \alpha(\nu)} D_j(\zeta), \quad |D(\zeta)| \leqq \sum_{j > \alpha(\nu)} |D_j(\zeta)| \quad (\zeta \in R \backslash R_\nu).$$

Hence by (6.2) we have

$$\begin{aligned}&\int_{R\backslash R_\nu} G_X(z, \zeta)|D(\zeta)|G_P(\zeta, w)dm(\zeta)\\ &\qquad \leqq \sum_{j>\alpha(\nu)}\int_R G_X(z, \zeta)|D_j(\zeta)|G_P(\zeta, w)dm(\zeta)\\ &\qquad \leqq k(\mu)[\sum_{j>\alpha(\nu)} d_j/\gamma_j]G_P(a, w)\end{aligned}$$

for each $\nu\geqq\beta(\mu)$. In other words we have for each fixed $\mu\in\boldsymbol{N}$

$$(6.3)\qquad \int_{R\backslash R_\nu} G_X(z, \zeta)|D(\zeta)|K_P(\zeta, w)dm(\zeta)\leqq k(\mu)\sum_{j>\alpha(\nu)} d_j/\gamma_j$$

for each $(z, w)\in R_\mu\times R$ and each $\nu\geqq\beta(\mu)$. Take any $w^*\in\partial_P R$ and any sequence $\{w_n\}$ in R converging to w^*. Consider (6.3) for $w=w_n$ and take the lower limit of both sides as $n\uparrow\infty$. By the Fatou lemma and $K_P(\cdot, w_n)\rightarrow K_P(\cdot, w^*)$ on R $(n\uparrow\infty)$, we have

$$(6.4)\qquad \int_{R\backslash R_\nu} G_X(z, \zeta)|D(\zeta)|K_P(\zeta, w^*)dm(\zeta)\leqq k(\mu)\sum_{j>\alpha(\nu)} d_j/\gamma_j$$

for any $z\in R_\mu$ and any $\nu\geqq\beta(\mu)$. From (6.3) and (6.4) it follows that (6.4) is also valid for any $(z, w^*)\in R_\mu\times R^*$ and any $\nu\geqq\beta(\mu)$.

Now observe that for any $\nu\geqq\beta(\mu)$

$$\begin{aligned}a_\nu(n:z):&=\left|\int_{R_\nu} G_X(z, \zeta)D(\zeta)(K_P(\zeta, w_n)-K_P(\zeta, w^*))dm(\zeta)\right|\\ &\leqq[\sup_{\zeta\in R_\nu}|K_P(\zeta, w_n)-K_P(\zeta, w^*)|]\cdot\int_{R_\nu} G_X(z, \zeta)|D(\zeta)|dm(\zeta).\end{aligned}$$

By (1.1) we see that the term on the most right hand side of the above converges to zero uniformly for z on R_μ and the same is true for $a_\nu(n:z)$. We denote by $b_\nu(n:z)$ and $c_\nu(z)$ the terms on the left hand side of (6.3) for $w=w_n$ and (6.4), respectively. Then by (6.3) and (6.4) we have

$$\begin{aligned}&\left|\int_R G_X(z, \zeta)D(\zeta)K_P(\zeta, w_n)dm(\zeta)-\int_R G_X(z, \zeta)D(\zeta)K_P(\zeta, w^*)dm(\zeta)\right|\\ &\qquad \leqq a_\nu(n:z)+b_\nu(n:z)+c_\nu(z)\\ &\qquad \leqq a_\nu(n:z)+2k(\mu)\sum_{j>\alpha(\nu)} d_j/\gamma_j.\end{aligned}$$

We denote by $A(n:z)$ the term on the most left hand side of the above. Then for each fixed $\mu\in\boldsymbol{N}$

$$\limsup_{n\to\infty}(\sup_{z\in R_\mu} A(n:z))\leqq 2k(\mu)\sum_{j>\alpha(\nu)} d_j/\gamma_j$$

since $\lim_{n\to\infty}(\sup_{z\in R_\mu} a_\nu(n:z))=0$. By (6.1) we conclude that

$$\lim_{n\to\infty}(\sup_{z\in R_\mu} A(n:z))=0.$$

By the arbitrariness of $\mu\in\boldsymbol{N}$ we finally conclude that

$$\text{(6.5)}\qquad \lim_{n\to\infty}\int_R G_X(z,\zeta)D(\zeta)K_P(\zeta,w_n)dm(\zeta)$$
$$=\int_R G_X(z,\zeta)D(\zeta)K_P(\zeta,w^*)dm(\zeta)$$

almost uniformly for z in R, where we again recall that $X=P$ or Q.

7. We introduce a function $C(w^*)=C(w^*:P,Q)$ defined on R_P^* by

$$\text{(7.1)}\qquad C(w^*)=1+\int_R G_Q(a,\zeta)D(\zeta)K_P(\zeta,w^*)dm(\zeta).$$

First for $w\in R\backslash\{a\}$ we see by (3.3) that

$$C(w)=G_Q(a,w)/G_P(a,w)$$

and therefore $C(w)$ is positive and continuous on $R\backslash\{a\}$. In view of (6.4) and (6.5) for $X=Q$ we can easily see that $C(w^*)$ is continuous at any point of $w^*\in\partial_P R$. Therefore $C(w^*)$ is continuous on $R_P^*\backslash\{a\}$ and $C(w^*)\geqq 0$ there. In this connection we need to consider the following exceptional set

$$E=E_{PQ}=\{w^*\in\partial_P R: C(w^*)=C(w^*:P,Q)=0\}.$$

Let w^* be any point in $\partial_P R$ and $\{w_n\}$ and sequence in R converging to w^*. Then $C(w_n)\to C(w^*)$ $(n\uparrow\infty)$. By (3.3) we have

$$C(w_n)K_Q(z,w_n)=K_P(z,w_n)+\int_R G_Q(z,\zeta)D(\zeta)K_P(\zeta,w_n)dm(\zeta).$$

This with (6.5) shows that if $w^*\in\partial_P R\backslash E$, then $\{w_n\}$ is also a Cauchy sequence (i.e. a fundamental sequence) in R embedded in R_Q^* converging to a point in R_Q^*, say $\sigma(w^*)$, determined only by w^* independent of the choice of $\{w_n\}$. By defining $\sigma(w)=w$ for every $w\in R$ we obtain

$$\text{(7.2)}\qquad C(w^*)K_Q(z,\sigma(w^*))=K_P(z,w^*)+\int_R G_Q(z,\zeta)D(\zeta)K_P(\zeta,w^*)dm(\zeta)$$

for any $z\in R$ and any $w^*\in R_P^*$ except for the case $z=w^*=a$, where we understand that the left hand side of the above is zero if $w^*\in E$ for which $\sigma(w^*)$ is of course undefined; we have

$$0=K_P(z,w^*)+\int_R G_Q(z,\zeta)D(\zeta)K_P(\zeta,w^*)dm(\zeta)$$

for any $z \in R$ and any $w^* \in E$. It is easy to see that the mapping

$$w^* \to \sigma(w^*) = \sigma(w^* : P, Q)$$

from $R_P^* \backslash E$ to R_Q^* is the identity on R and continuous on $R_P^* \backslash E$, and moreover if $E = \emptyset$, then the mapping is surjective.

8. Before proceeding further based on (7.2) we need to consider an auxiliary potential W on R defined by the following:

$$W = \sum_{j \in \boldsymbol{N}} (P_j + |P_j - Q_j|).$$

Clearly $W \in K_{\mathrm{loc}}(R)^+$. Since $W \geqq P$ (and of course $W \geqq Q$), we see that W is also hyperbolic (see the remark after (3.2)). Let $W_j = P_j + |P_j - Q_j|$ ($j \in \boldsymbol{N}$). Then $\{W_j\}$ is a decomposition of W associated with $\{V_j\}$ fixed at the beginning. Here observe that $|P_j - W_j| = |P_j - Q_j|$ and therefore (5.1) implies that W is also ruled by P. Hence $\sigma(\cdot) = \sigma(\cdot : P, W)$, $C(\cdot) = C(\cdot : P, W)$, and $E = E_{PW}$ can also be defined. By (7.2) we have

$$\begin{aligned}(8.1) \qquad & C(w^*) K_W(z, \sigma(w^*)) \\ & \qquad = K_P(z, w^*) + \int_R G_W(z, \zeta)(P(\zeta) W(\zeta)) K_P(\zeta, w^*) dm(\zeta)\end{aligned}$$

for any $z \in R$ and any $w^* \in R_P^*$ except for $z = w^* = a$, where as in (7.2), we understand that the left hand side of the above is zero if $w^* \in E_{PW}$ which will be seen right below to be empty.

We now maintain that $C(w^*) > 0$ ($w^* \in \partial_P R$) so that $E_{PW} = \emptyset$. Recall that $C(w^*) \geqq 0$ ($w^* \in \partial_P R$). Contrary to the assertion assume that $C(w^*) = 0$ for some w^* in $\partial_P R$. For simplicity set $u = K_P(\cdot, w^*)$ and

$$d\mu(\zeta) = (W(\zeta) - P(\zeta)) K_P(\zeta, w^*) dm(\zeta).$$

By (3.3) we see that $G_W(z, \zeta) \leqq G_P(z, \zeta)$ since $W \geqq P$. Thus we have by (8.1)

$$u(z) \leqq G_P \mu(z) < \infty \quad (z \in R).$$

The last inequality follows from (6.4) established for P and W with $X = P$. Since $G_P \mu$ is a P-potential on R and u is nonnegative and P-harmonic on R, we must have $u \equiv 0$ on R contradicting $u(a) = 1$.

The function $C(\cdot)$ is thus positive and continuous on $\partial_P R$ and therefore the infimum of $C(\cdot)$ on $\partial_P R$ is strictly positive. A fortiori there exists a constant $q \in [1, \infty)$ such that

$$q^{-1} \leqq G_W(a, w)/G_P(a, w) \leqq 1 \quad (w \in R \backslash V_1).$$

The condition (5.1), the above inequality, and the identities $|W_j-P_j|=|P_j-Q_j|$ $(j\in\boldsymbol{N})$ together imply that

$$\sum_{j\in N}\frac{1}{\inf_{z\in V_j} G_W(a,z)}\sup_{z\in V_j}\int_{V_j} G_W(z,\zeta)|W_j(\zeta)-P_j(\zeta)|dm(\zeta)<\infty,$$

i.e. P is also ruled by W. Observe that $|W_j-Q_j|\leqq 2|P_j-Q_j|=2|W_j-P_j|$. Hence the above last displayed inequality is also valid if we replace $|W_j(\zeta)-P_j(\zeta)|$ by $|W_j(\zeta)-Q_j(\zeta)|$. Thus Q is also ruled by W. If we can show that R_P^* and R_Q^* are canonically homeomorphic to R_W^*, then R_Q^* is canonically homeomorphic to R_P^*.

9. By the reduction made in no. 8 we can henceforth assume that $P\geqq Q$ on R to prove that R_Q^* is canonically homeomorphic to R_P^* under the assumption (5.1). We now develope our discussion based upon (7.2) where $C(w^*)=C(w^*:P,Q)$ again. Recall that $C(\cdot)$ is continuous on $R_P^*\backslash\{a\}$ and further $C(\cdot)\geqq 1$ which follows from (7.1) in view of our additional assumption $D(\zeta)=P(\zeta)-Q(\zeta)\geqq 0$.

Since $E_{PQ}=\emptyset$ as we have just seen above, we can now conclude based upon what we have seen in no. 7 that

$$w^*\to\sigma(w^*)=\sigma(w^*:P,Q)$$

is a surjective and continuous mapping of R_P^* to R_Q^*. Here we maintain that it is *injective*. For the purpose we take $w_i^*\in\partial_P R$ $(i=1,2)$ such that $\sigma(w_1^*)=\sigma(w_2^*)$ and consider

$$u:=C(w_2^*)K_P(\cdot,w_1^*)-C(w_1^*)K_P(\cdot,w_2^*)$$

which belongs to $P(R)$. By (7.2) we obtain

$$u(z)+\int_R G_Q(z,\zeta)D(\zeta)u(\zeta)dm(\zeta)=0\quad(z\in R).$$

Set $d\mu(\zeta)=D(\zeta)|u(\zeta)|dm(\zeta)$. Again by (7.2) we see that $G_Q\mu$ is finite on R. Therefore we have

(9.1) $\qquad |u(z)|\leqq G_Q\mu(z)\quad(z\in R).$

Since u in P-harmonic on R, $|u|$ is P-subharmonic on R. For any nice region Ω, $P_{|u|}^{\Omega}\leqq Q_{|u|}^{\Omega}$ on Ω because of $P\geqq Q$ and the identity

$$Q_{|u|}^{\Omega}=P_{|u|}^{\Omega}+\int_{\Omega}G_Q^{\Omega}(\cdot,\zeta)(P(\zeta)-Q(\zeta))P_{|u|}^{\Omega}(\zeta)dm(\zeta).$$

Hence we see that $|u|$ is also Q-subharmonic on R because $|u|\leqq P_{|u|}^{\Omega}\leqq Q_{|u|}^{\Omega}$

for every nice region Ω of R (of. e. g. [4]). The relation (9.1) says that the nonnegative Q-subharmonic function $|u|$ is dominated by the Q-potential $G_Q\mu$ on R. Therefore we must have $|u|=0$, i. e.

$$C(w_2^*)K_P(z, w_1^*)=C(w_1^*)K_P(z, w_2^*)$$

for any $z\in R$. Setting $z=a$ we see that $C(w_2^*)=C(w_1^*)$ and a fortiori $K_P(z, w_1^*)=K_P(z, w_2^*)$ for any $z\in R$, i. e. $w_1^*=w_2^*$. Thus we have seen that the mapping $w^*\to\sigma(w^*:P, Q)$ gives a *natural* homeomorphism between R_P^* and R_Q^*. The task left is to show that it is *canonical.*

10. We now maistain that a mapping $S:P(R)^+\to Q(R)^+$ can be defined by

$$(10.1)\qquad Su=u+\int_R G_Q(\cdot, \zeta)D(\zeta)u(\zeta)dm(\zeta);$$

S is additive and homogeneous (i. e. $S(\lambda u)=\lambda Su$ for positive numbers λ) and bijective so that $P(R)^+$ and $Q(R)^+$ are isomorphic as convex cones.

First we show that Su given by (10.1) is well defined and $Su\in Q(R)^+$ for any $u\in P(R)^+$. In general the integral on the right hand side of (10.1) may diverge and hence we have to show its integrability first. For the purpose we take an arbitrary $u\in P(R)^+$. Take any measure μ on $\partial_P R$ such that $u=K_P\mu$ on R. Such a μ is unique if we restrict them to canonical measures μ, i. e. μ with $\mu(\partial_P R\backslash\delta_P R)=0$, as is seen by (4.2). Nevertheless we take *any* μ on $\partial_P R$ with $u=K_P\mu$ on R. Let μ_1 be the measure on $\partial_Q R$ constructed from μ by

$$(10.2)\qquad d\mu_1(w_1^*)=C(\sigma^{-1}(w_1^*))d\mu(\sigma^{-1}(w_1^*))\ (w_1^*\in\partial_Q R).$$

Note that even if we take μ to be canonical, it cannot be concluded at the present stage that μ_1 is canonical. This is the reason why we take *any* μ not necessarily canonical. Now integrate both sides of (7.2) with respect to $d\mu$ on $\partial_P R$. Apply the change of variables using (10.2) to the left hand side and the Fubini theorem to the right hand side of the resulting identity of the above integration we obtain

$$(10.3)\qquad K_Q\mu_1=K_P\mu+\int_R G_Q(\cdot, \zeta)D(\zeta)K_P\mu(\zeta)dm(\zeta).$$

By recalling $K_P\mu=u$, the above identity shows that Su is well defined and $Su=K_Q\mu_1\in Q(R)^+$.

It is obvious that S is additive and homogeneous. To see that S is *surjective* we take an arbitrary $u_1\in Q(R)^+$. There exists a measure μ_1 on $\partial_Q R$ such that $u_1=K_Q\mu_1$. Construct a measure μ on $\partial_P R$ from μ_1 by

$$(10.4)\qquad d\mu(w^*)=C(w^*)^{-1}d\mu_1(\sigma(w^*))\ (w^*\in\partial_P R).$$

Since μ_1 is, conversely, obtained from μ by (10.2) we can use (10.3) to conclude that $u_1=Su$ with $u=K_P\mu\in P(R)^+$. By a similar fashion as in the proof of the injectivity of σ in no. 9 we can also show that S is *injective.*

11. Finally we show that $\sigma(\delta_P R)=\delta_Q R$ so that $\sigma=\sigma(\cdot:P,Q)$ is canonical. First take any $w_0^*\in\delta_P R$. Let μ_1 be the canonical measure on $\delta_Q R$ associated with $K_Q(\cdot,\sigma(w_0^*))$ in (4.2) : $K_Q(\cdot,\sigma(w_0^*))=K_Q\mu_1$. Let μ be the measure on $\partial_P R$ constructed from μ_1 *by* (10.2). Then, as in no. 10, we have (10.3) : $K_Q\mu_1=S(K_P\mu)$ or $K_Q(\cdot,\sigma(W_0^*))=S(K_P\mu)$. On the other hand, (7.2) with $w^*=w_0^*$ takes the form $C(w_0^*)K_Q(\cdot,\sigma(w_0^*))=S(K_P(\cdot,w_0^*))$ or $K_Q(\cdot,\sigma(w_0^*))=S(C(w_0^*)^{-1}K_P(\cdot,w_0^*))$. Therefore

$$S(K_P\mu)=S(C(w_0^*)^{-1}K_P(\cdot,w_0^*)).$$

By the injectiveness of S we see that

$$K_P\mu=C(w_0^*)^{-1}K_P(\cdot,w_0^*)$$

on R. Therefore $K_P\mu$ is minimal in $P(R)^+$. Then $\operatorname{supp}\mu$ must consist of a single point in $\delta_P R$ (see p. 254 of [**5**]). Then, by (10.4) or equivalently by (10.2), $\operatorname{supp}\mu_1$ also consists of a single point which belongs to $\delta_Q R$ since μ_1 is canonical. From $K_Q(\cdot,\sigma(w_0^*))=K_Q\mu_1$ it now follows that $\sigma(w_0^*)\in\delta_Q R$. We have thus seen that $\sigma(\delta_P R)\subset\delta_Q R$.

The proof for $\sigma(\delta_P R)\supset\delta_Q R$ or $\sigma^{-1}(\delta_Q R)\subset\delta_P R$ goes in a similar fashion as above. Namely, take any $w_1^*\in\delta_Q R$. Let $K_P(\cdot,\sigma^{-1}(w_1^*))K_P\mu$ be the canonical representation with μ a measure on $\delta_P R$. Let μ_1 be the measure on $\partial_Q R$ constructed from μ by (10.2). Using (10.3) and (7.2) with $w^*=\sigma^{-1}(w_1^*)$ we see as above

$$K_Q\mu_1=S(K_P\mu)=S(K_P(\cdot,\sigma^{-1}(w_1^*))=C(\sigma^{-1}(w_1^*))K_Q(\cdot,w_1^*).$$

Hence $K_Q\mu_1$ is minimal in $Q(R)^+$ and thus $\operatorname{supp}\mu_1$ consists of a single point in $\delta_Q R$ as above and therefore $\operatorname{supp}\mu$ consists of a single point which belongs to $\delta_P R$ since μ is canonical. Thus $\sigma^{-1}(w_1^*)\in\delta_P R$ so that $\sigma^{-1}(\delta_Q R)\subset\delta_P R$.

We have shown that R_P^* and R_Q^* are naturally homeomorphic in no. 9 and that $\sigma(\delta_P R)=\delta_Q R$ right now. A fortiori R_P^* and R_Q^* are canonically homeomorphic.

The proof of the main theorem is herewith complete. □

12. Suppose P and Q are hyperbolic potentials in $K_{\mathrm{loc}}(R)^+$ and $P=Q$ outside a compact subset of R. Then the terms in the summation of (5.1) are zero except for a finite number of terms and therefore (5.1) is clearly satisfied. Thus P rules Q and we have the following

TRIVIAL COROLLARY. *If hyperbolic potentials P and Q in $K_{\mathrm{loc}}(R)^+$ are identical on R except for a compact subset of R, then R_P^* and R_Q^* are canonically homeomorphic.*

One must be careful in applying the above result in the following point. Namely, there can exist potentials P and Q in $K_{\mathrm{loc}}(R)^+$ such that P is hyperbolic and P rules Q and yet Q is not hyperbolic. There can even exist two potentials P and Q in $K_{\mathrm{loc}}(R)^+$ identical on R except for a compast subset of R such that P is hyperbolic and Q is not hyperbolic. The latter occurs only when R is parabolic (see the remark after (3.2)) for $Q\equiv 0$ and P such that $\{z\in R : P(z)\neq 0\}$ is of positive m-measure and compact in R.

13. As mentioned in the introduction the following corollary of our main theorem will be useful in the construction of a C^∞ hyperbolic potential P such that R_P^* possesses a property assigned in advance; we only have to construct a P in $K_{\mathrm{loc}}(R)^+$ that may be fairly wildly discontinuous. We now state the following

NONTRIVIAL COROLLARY. *There exists a nonnegative C^∞ hyperbolic potential Q for any given byperbolic potential P in $K_{\mathrm{loc}}(R)^+$ such that R_P^* and R_Q^* are canonically homeomorphic. Moreover Q can be chosen close enough to P in a sense that will become clear in the proof below.*

The proof will be given in two steps below in nos. 14-15.

14. Let $\{\varphi_j\}_{j\in N}$ be an arbitrary partition of unity on R and $\{V_j\}_{j\in N}$ be an arbitrary locally finite covering of R by disks such that $\mathrm{supp}\,\varphi_j\subset V_j$ $(j\in N)$. We then fix these throughout the proof. Set $P_j=\varphi_jP$ $(j\in N)$. Then $\{P_j\}_{j\in N}$ is a decomposition of P associated with $\{V_j\}$. Let Q_{jn} be the potential (i. e. 2-form on R) defined by

$$\begin{cases}\mathrm{supp}\,Q_{jn}\subset V_j\\ Q_{jn}(z)dxdy=[\min(P_j(z),\ n)]dxdy\ \ (z=x+iy\in V_j)\end{cases}$$

for each $j\in N$ and $n\in N$. Clearly $Q_{jn}\in L^\infty_{\mathrm{loc}}(R)^+$. Consider a sequence $\{u_n\}_{n\in N}$ of functions u_n on R given by

$$u_n(z)=\int_R G_P(z,\ \zeta)(P_j(\zeta)-Q_{jn}(\zeta))dm(\zeta)$$

for each fixed $j\in N$. In view of (1.1), $u_n\in C(R)$. By the Fatou lemma, we see that $u_n\downarrow 0$ $(n\uparrow\infty)$ at each $z\in R$. By the Dini theorem $\{u_n\}$ converges to zero uniformly on V_j. Hence we can find an $n(j)\in \boldsymbol{N}$ for each j such that

$$\sup_{z\in V_j}\int_{V_j} G_P(z,\zeta)(P_j(\zeta)-Q_{jn(j)}(\zeta))dm(\zeta)\leqq 2^{-j}\inf_{z\in V_j} G_P(a,z).$$

Clearly

$$(14.1)\qquad Q=\sum_{j\in \boldsymbol{N}} Q_{jn(j)}$$

is in $L_{\mathrm{loc}}^{\infty}(R)^{+}$ and hyperbolic along with P (see the remark after (3.2)) and ruled by P. Hence R_P^* and R_Q^* are canonically homeomorphic by our main theorem. Therefore to prove the above corollary we may assume that $P\in L_{\mathrm{loc}}^{\infty}(R)$. Here we view that Q in (14.1) can be made as close to P as we wish by taking $(n(1), n(2), \cdots, n(j), \cdots)$ "large" enough.

15. We retain the decomposition $\{P_j\}$ of P associated with $\{V_j\}$ as above in no. 14 but this time $P_j(z)$ is m-essentially bounded on V_j for each j. As usual we take $\rho(z)$ to be $A\cdot\exp(-1/(1-|z|^2))$ on $V_j:|z|<1$ and 0 on $R\setminus V_j$ where A is so chosen as to satisfy $\int_{V_j}\rho(z)dm(z)=1$. For any positive ε less than $\mathrm{dis}(\partial V_j, \mathrm{supp}\,\varphi_j)$ we set $\rho_\varepsilon(z)=\varepsilon^{-2}\rho(z/\varepsilon)$ on V_j and 0 on $R\setminus V_j$ and apply the so called Friedrichs mollifier $\rho_\varepsilon *$ to P_j in the sense that $\mathrm{supp}\,\rho_\varepsilon * P_j\subset V_j$ and $\rho_\varepsilon * P_j=(\rho_\varepsilon * P_j(z))dxdy$ on V_j where

$$\rho_\varepsilon * P_j(z)=\int_{V_j}\rho_\varepsilon(z-\zeta)P_j(\zeta)dm(\zeta).$$

Then $\rho_\varepsilon * P_j$ is a nonnegative C^∞ 2-form on R and, for any $p\in[1,\infty)$,

$$\lim_{\varepsilon\downarrow 0}\int_{V_j}|\rho_\varepsilon * P_j(\zeta)-P_j(\zeta)|^p dm(\zeta)=0.$$

Take an arbitrary $p>1$ and its conjugate q(i.e. $1/p+1/q=1$). By the Hölder inequality

$$\int_{V_j} G_P(z,\zeta)|P_j(\zeta)-\rho_\varepsilon * P_j(\zeta)|dm(\zeta)$$
$$\leqq\left(\int_{V_j} G_P(z,\zeta)^q dm(\zeta)\right)^{1/q}\cdot\left(\int_{V_j}|P_j(\zeta)-\rho_\varepsilon * P_j(\zeta)|^p dm(\zeta)\right)^{1/p}.$$

Thus we can find an admissible $\varepsilon_j>0$ such that

$$\left(\int_{V_j}|P_j(\zeta)-\rho_{\varepsilon_j} * P_j(\zeta)|^p dm(\zeta)\right)^{1/p}$$
$$\leqq\frac{1}{2^j}\left(\sup_{z\in V_j}\int_{V_j} G_P(z,\zeta)^q dm(\zeta)\right)^{-1/q}\inf_{z\in V_j} G_P(a,z).$$

We then consider the C^∞ potential

$$(15.1)\qquad Q=\sum_{j\in N}\rho_{\varepsilon_j}*P_j$$

which is seen to be hyperbolic along with P (see the remark after (3.2)). Thus the above Q is the desired potential. Here we view that Q can be made as close to P as we wish by taking $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_j, \cdots)$ "small" enough. □

Combining notations in (14.1) and (15.1) it may be impressive to use the notation $\rho_\varepsilon*P$ for any potential $P\in K_{\mathrm{loc}}(R)^+$ to mean

$$(15.2)\qquad \rho_\varepsilon*P=\sum_{j\in N}\rho_{\varepsilon_j}*Q_{jn(j)}$$

where ε is understood to be the infinite vector

$$\varepsilon=(\varepsilon_1, 1/n(1), \varepsilon_2, 1/n(2), \cdots, \varepsilon_j, 1/n(j), \cdots).$$

By giving the componentwise ordering and understanding $\varepsilon\downarrow 0$ (zero vector) as the convergence in this order we may summarize conclusions in nos. 14 and 15 as

RESTATEMENT. *For an arbitrarily given potential P in $K_{\mathrm{loc}}(R)^+$, $\rho_\varepsilon*P$ is a nonnegative C^∞ potential on R and if ε is sufficiently close to 0, then $R^*_{\rho_\varepsilon*P}=R^*_P$ (canonically homeomorphic).*

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