Hokkaido Mathematical Journal Vol. 18 (1989), p. 231-243

Radon, Baire, and Borel measures on compact spaces. I

H. H. SCHAEFER

(Received August 16, 1988)

Introduction

There are essentially two approaches to the classical theory of integration on locally compact spaces; one is represented by Halmos [5] (see also [3]), the other by Bourbaki [1]. One way to illuminate their mutual relationship is to employ the theorem of Riesz-Markov [3], to the effect that every Radon measure on a compact space X possesses a unique regular Borel extension, and to imbed the space $B(X)$ of bounded Borel functions into the bidual of $C(X)$. While the Riesz-Markov Theorem itself is a deep, not readily proved result, it does not immediately reveal the finer aspects of the connections between continuous, Baire, and Borel functions or between the corresponding types of measures.

The purpose of this treatise is to make these connections and relations as transparent as possible. The approach is Bourbaki's approach turned upside down, as it were, at least in the beginning: it is (or so the author believes) utterly functional analytic, the key notion being that of a Riesz space. The present first part is concerned essentially with imbedding the Riesz spaces of bounded Baire and Borel functions into $C(X)''$, this latter space provided, in addition to its norm topology, with the topology $o(C(X)'', M(X))$ (see below) under which it is a complete Lebesgue space in the sense of Fremlin [4]. The simultaneous discussion of all Radon, Baire, and Borel measures on X is facilitated by restriction to compact spaces X and to Riesz spaces of bounded functions on X. Extension, where appropriate, to unbounded functions and measures is usually easy. Notation and terminology is standard or follows [6], [7].

A. The General Setting

A. 1 *Notation.* Let X denote a compact space. By $C(X)$, $M(X)$, $\bar{C}(X)$ we will understand, respectively, the Banach space of real continuous functions on X, its dual Banach space, and its bidual. These three spaces are Banach lattices under their natural orderings. Moreover, on

$\bar{C}(X)$ we will consider the topology $\mathfrak{T}=o(\bar{C}(X), M(X))$, i.e., the topology of uniform convergence on the order intervals of $M(X)$.

This topology has many desirable properties: It is order continuous, that is, each order convergent filter $\mathfrak{T}$-converges (to its order limit) ($\mathfrak{T}$ is a *Lebesgue* topology [4]), consistent with the duality $\langle \bar{C}(X), M(X)\rangle$, and its bounded subsets are identical with the norm bounded subsets of $\bar{C}(X)$. Moreover, $\bar{C}(X)$ is Dedekind complete, $\mathfrak{T}$-complete, and contains $C(X)$ as a $\mathfrak{T}$-dense Riesz subspace. (The $\mathfrak{T}$-completeness follows from Grothendieck's theorem [6, IV. 6. 2] in conjunction with the fact that order intervals in $M(X)$ are weakly compact, which is easy to see independently of measure theoretic arguments [7, II. 5. 10].) $\mathfrak{T}$ is occasionally called the *weak Riesz topology* of $\bar{C}(X)$ with respect to $M(X)$.

For reasons explained in the introduction, the sets and Riesz spaces of real functions on X considered in this paper will consist of *bounded* functions only. In particular, L will denote the set (convex cone) of bounded, lower semi-continuous functions $f\geqq 0$ on X, and M will denote the set (convex cone) of all bounded functions $h\geqq 0$ on X. Generally, R will denote a Riesz space (with respect to the pointwise ordering) of bounded, real valued functions on X containing the constant-one function 1; R is a normed Riesz space under the supremum norm.

A. 2 *The Mapping* φ^*. As is well known, the evaluation map $\varphi: C(X)\rightarrow \bar{C}(X)$ is an isometric Riesz isomorphism (for the norm topologies).

2. 1 DEFINITION. *We define the mappings* $\bar{\varphi}: L\rightarrow \bar{C}(X)_+$ *and* $\varphi^*: M\rightarrow \bar{C}(X)_+$, *respectively, as follows*:

(1) $\bar{\varphi}(g)=\sup\{\varphi(f): 0\leqq f\leqq g,\ f\in C(X)\}$

(2) $\varphi^*(h)=\inf\{\bar{\varphi}(g): 0\leqq h\leqq g,\ g\in L\}$.

Note that for each $g\in L$ (*notation* A. 1), $\bar{\varphi}(g)$ *is the* $\mathfrak{T}$-*limit of the directed* ($\leqq$) *set* $\varphi(A)$ *in* $\bar{C}(X)$ *where* $A=\{f\in C(X): 0\leqq f\leqq g\}$; *similarly for* $\varphi^*(h)$.

2. 2 PROPOSITION. *The mapping* $\bar{\varphi}$ *is positive homogeneous and additive on* L; *moreover, it preserves finite suprema and infima.*

PROOF. Clearly, $\bar{\varphi}$ is positive homogeneous. To see that $\bar{\varphi}$ is additive, let $g_1, g_2\in L$ and let a, b, c be arbitrary continuous functions $\geqq 0$ satisfying $a\leqq g_1$, $b\leqq g_2$, $c\leqq g_1+g_2$. If $a\uparrow g_1$, $b\uparrow g_2$ pointwise on X and if c is fixed, then $(a+b)\wedge c\uparrow c$; hence, by Dini's theorem, the latter convergence is uniform. It follows that $\varphi(c) \leqq \sup \varphi(a+b) = \sup \varphi(a) + \sup \varphi(b) = \bar{\varphi}(g_1) + \bar{\varphi}(g_2)$; hence, we obtain

$$\bar{\varphi}(g_1+g_2) \leqq \bar{\varphi}(g_1) + \bar{\varphi}(g_2).$$

On the other hand, since $\varphi(a) + \varphi(b) \leqq \bar{\varphi}(g_1+g_2)$ for all $a \leqq g_1$, $b \leqq g_2$, it follows that $\bar{\varphi}(g_1) + \bar{\varphi}(g_2) \leqq \bar{\varphi}(g_1+g_2)$.

We now show that $\bar{\varphi}(g_1 \vee g_2) = \bar{\varphi}(g_1) \vee \bar{\varphi}(g_2)$. Since $\bar{\varphi}$ is obviously isotone, we at once obtain $\bar{\varphi}(g_1) \vee \bar{\varphi}(g_2) \leqq \bar{\varphi}(g_1 \vee g_2)$. Now let $a \leqq g_1$, $b \leqq g_2$, $c \leqq g_1 \vee g_2$ be in $C(X)^+$. Then for fixed c we have, as before, $(a \vee b) \wedge c \uparrow c$ uniformly on X whence $\bar{\varphi}(g_1) \vee \bar{\varphi}(g_2) \geqq \varphi(c)$; this implies $\bar{\varphi}(g_1) \vee \bar{\varphi}(g_2) \geqq \bar{\varphi}(g_1 \vee g_2)$. Because $\bar{\varphi}(g_1) \vee \bar{\varphi}(g_2) \leqq \bar{\varphi}(g_1 \vee g_2)$ trivially, it follows that $\bar{\varphi}$ preserves finite suprema. We omit the similar proof that $\bar{\varphi}$ preserves finite infima as well.

2.3 PROPOSITION. *The linear hull $R = L - L$ of L is a Riesz space ; the linear extension $\tilde{\varphi}$ of $\bar{\varphi}$ to R is a Riesz isomorphism and an isometry (for the norm topologies on R and $\bar{C}(X)$, respectively).*

PROOF. First, because of the identity $|f-g| = f \vee g - f \wedge g$ and because L is a sublattice of $\boldsymbol{R}^X$, it follows that R is a Riesz space. Second, by the properties of $\bar{\varphi}$ established in 2.2, the definition $\tilde{\varphi}(f-g) := \bar{\varphi}(f) - \bar{\varphi}(g)$ yields a linear map. Now the above identity and the lattice preserving properties of $\bar{\varphi}$ imply that

$$\begin{aligned}\tilde{\varphi}(|f-g|) &= \bar{\varphi}(f \vee g) - \bar{\varphi}(f \wedge g) = \bar{\varphi}(f) \vee \bar{\varphi}(g) - \bar{\varphi}(f) \wedge \bar{\varphi}(g) \\ &= |\bar{\varphi}(f) - \bar{\varphi}(g)| = |\tilde{\varphi}(f-g)|,\end{aligned}$$

so $\tilde{\varphi}$ is a Riesz homomorphism. Thus to see that $\tilde{\varphi}$ is injective, it suffices to show that $0 \leqq g \leqq f$, g, $f \in L$, and $\tilde{\varphi}(f-g) = 0$ force $f = g$. If not, there exists $t_0 \in X$ such that $g(t_0) < f(t_0)$, while $\bar{\varphi}(f) = \bar{\varphi}(g)$. Considering the evaluation functional $\delta_0 \in M(X)$, defined by $\delta_0(f) = f(t_0)$ for $f \in C(X)$, we observe that δ_0 has a unique $\mathfrak{T}$-continuous extension to $\bar{C}(X)$ (A.1). Since $\bar{\varphi}(f) = \lim_A \varphi(a)$ and $\bar{\varphi}(g) = \lim_B \varphi(b)$ for $\mathfrak{T}$, where $A = \{0 \leqq a \leqq f,\ a \in C(X)\}$ and $B = \{0 \leqq b \leqq g,\ b \in C(X)\}$, it follows that $\lim_A a(t_0) = \lim_B b(t_0)$, contrary to our assumption.

It remains to show that $\tilde{\varphi}$ is an isometry for the norm topologies. If we denote by 1 the constant-one function on X and by e the unit of $\bar{C}(X)$, we have $\varphi(1) = e$. Now let $0 \leqq h \in R$. If $0 \leqq h \leqq \lambda 1$ then $0 \leqq \tilde{\varphi}(h) \leqq \lambda e$, because $\tilde{\varphi}$ is positive ; thus $\|h\| \leqq \lambda$ implies $\|\tilde{\varphi}(h)\| \leqq \lambda$. This means that $\|\tilde{\varphi}(h)\| \leqq \|h\|$. Conversely, if $0 \leqq \tilde{\varphi}(h) \leqq \lambda e$, then $\tilde{\varphi}(h \wedge \lambda 1) = \tilde{\varphi}(h) \wedge \lambda e = \tilde{\varphi}(h)$, $\tilde{\varphi}$ being a Riesz homomorphism. But $\tilde{\varphi}$ is injective so $h \wedge \lambda 1 = h$ whence $0 \leqq h \leqq \lambda 1$; thus $\|h\| \leqq \|\tilde{\varphi}(h)\|$. Since the norms on R and $\bar{C}(X)$ are lattice norms, it follows that $\tilde{\varphi}$ is an isometry.

We now turn to the mapping φ^* (Def. 2.1) which extends φ to all bounded real functions $h\geqq 0$ on X. It turns out that φ^* is a convex map $M \to \bar{C}(X)$ which preserves finite suprema.

2.4 THEOREM. *Let $h, k\in M$ and let $0\leqq \alpha, \beta\in \boldsymbol{R}$. Then $\varphi^*(\alpha h)=\alpha\varphi^*(h)$, $\varphi^*(\alpha h+\beta k)\leqq \alpha\varphi^*(h)+\beta\varphi^*(k)$, and $\varphi^*(h\vee k)=\varphi^*(h)\vee\varphi^*(k)$. Moreover, if (f_n) is an increasing, uniformly bounded sequence in M and if $\lim_n f_n(t)=: f(t)$ for all $t\in X$, then $\varphi^*(f)=\lim_n \varphi^*(f_n)$ in $(\bar{C}(X), \mathfrak{T})$. Finally, if $\varphi^*(h)=0$ then $h=0$.*

PROOF. We recall Def. 2.1. It is clear that $\varphi^*(\alpha h)=\alpha\varphi^*(h)$ for all $h\in M$ and $\alpha\geqq 0$. Further, if $f, g\in L$ satisfy $f\geqq h$, $g\geqq k$ then $f+g\in L$ and $f+g\geqq h+k$. Hence $\bar{\varphi}(f)+\bar{\varphi}(g)=\bar{\varphi}(f+g)\geqq\varphi^*(h+k)$; since this holds for all pairs $f, g\in L$ satisfying $f\geqq h$, $g\geqq k$, we obtain $\varphi^*(h)+\varphi^*(k)\geqq\varphi^*(h+k)$.

We now show that for all $h, k\in M$, we have $\varphi^*(h\vee k)=\varphi^*(h)\vee\varphi^*(k)$. Clearly, $\varphi^*(h)\vee\varphi^*(k)\leqq\varphi^*(h\vee k)$, since φ^* is isotone. On the other hand, if f, g are arbitrary functions in L satisfying $f\geqq h$, $g\geqq k$ then $f\vee g\geqq h\vee k$ and, by 2.2, $\bar{\varphi}(f\vee g)=\bar{\varphi}(f)\vee\bar{\varphi}(g)\geqq\varphi^*(h\vee k)$. Thus we obtain $\varphi^*(h)\vee\varphi^*(k)\geqq\varphi^*(h\vee k)$ by continuity of the lattice operations in $(\bar{C}(X), \mathfrak{T})$.

The next assertion is easily obtained from [1, Chap. IV, § 1, Thm. 3] but we include its proof for completeness. Suppose then that (f_n) is an increasing uniformly bounded sequence in M with pointwise limit f. Let U denote any closed, convex, circled, solid $\mathfrak{T}$-neighborhood of 0 in $\bar{C}(X)$. By Def. 2.1, for each $n\in\boldsymbol{N}$ there exists $g_n\in L$, $g_n\geqq f_n$, such that $\bar{\varphi}(g_n)\in\varphi^*(f_n)+2^{-(n+1)}U$. Then if $\bar{g}_n:=\sup\{g_\nu : 1\leqq\nu\leqq n\}$, $(\bar{g}_n)$ is an increasing sequence with pointwise limit $g\in L$. Since $\bar{\varphi}(g_n)=\sup\{\bar{\varphi}(g_\nu) : 1\leqq\nu\leqq n\}$ by 2.2, from a standard formula ([7], II. 1.4(6)) we obtain

$$0\leqq\bar{\varphi}(\bar{g}_n)-\varphi^*(f_n)\leqq\sum_{\nu=1}^{n}[\bar{\varphi}(g_\nu)-\varphi^*(f_\nu)]\in\frac{1}{2}U.$$

Through an application of Dini's theorem, from Def. 2.1 it follows that $\bar{\varphi}(g)=\sup_n\bar{\varphi}(g_n)=\mathfrak{T}\text{-}\lim_n\bar{\varphi}(\bar{g}_n)$; thus there exists $n_0\in\boldsymbol{N}$ such that $\bar{\varphi}(g)\in\bar{\varphi}(\bar{g}_n)+\frac{1}{2}U$ for all $n\geqq n_0$. This implies

$$0\leqq\bar{\varphi}(g)-\varphi^*(f_m)\in U \quad (m\geqq n_0)$$

and hence $\bar{\varphi}(g)-\varphi^*(f)\in U$, since U is solid and $\bar{\varphi}(g)\geqq\varphi^*(f)\geqq\varphi^*(f_m)$. But since U is closed, we also have $0\leqq\bar{\varphi}(g)-\sup_n\varphi^*(f_n)\in U$ and this shows that $\varphi^*(f)=\sup_n\varphi^*(f_n)$.

Finally, if $\varphi^*(h)=0$, by 2.1 (2) we have $\lim_B \bar{\varphi}(g)=0$ for the directed $(\geqq)$ set $B=\{g\in L: g\geqq h\}$. Denoting by $\delta_t(t\in X)$ the $\mathfrak{T}$-continuous extension of the evaluation functional $f\to f(t)$ $(f\in C(X))$, we obtain $\langle\bar{\varphi}(g), \delta_t\rangle=g(t)$ for each $g\in L$ (Def. 2.1). Thus $\lim_B g(t)=0$ for each $t\in X$ which shows that $h=0$.

2.5 REMARK. In general, it is not true that $\varphi^*(h\wedge k)=\varphi^*(k)\wedge\varphi^*(k)$ for arbitrary functions $h, k\in M$; but it is easy to see that the latter equality holds whenever $\varphi^*(h+k)=\varphi^*(h)+\varphi^*(k)$. Thus if R is a Riesz space of bounded real functions on X such that φ^* is additive on R_+ then φ^* defines, by linear extension, a Riesz homomorphism of R into $\bar{C}(X)$. It will be the purpose of the following section to study this phenomenon.

A. 3 *An Extension Theorem.* The space R of Prop. 2.3 and $C(X)$ itself furnish instances of Riesz subspaces of $\boldsymbol{R}^X$ on which the map φ^*, by linear extension, defines a Riesz isomorphism (even an isometry) into $\bar{C}(X)$. (Note that on the positive cone of R as well as on $C(X)_+$, φ^* is additive; for R, this is shown in B. 2.1 below.) It will be shown that such a space R, if it contains 1, can be imbedded in a (generally) larger space R^* of the same type.

3.1 DEFINITION. *Let R be a Riesz space, containing* 1, *of bounded real functions on X. We define $\bar{R}_+$ to be the set of all bounded functions $f\geqq 0$ on X which are pointwise limits of an increasing sequence in R_+. Further, we define P^* to be the set of all functions $f\geqq 0$ on X which are pointwise limits of a decreasing sequence in $\bar{R}_+$.*

3.2 NOTE. It is obvious that the sets $\bar{R}_+$ and P^* are convex cones in $\boldsymbol{R}^X$ which are stable under the formation of finite suprema and infima. Thus the sets $R'=\bar{R}_+-\bar{R}_+$ and $R^*=P^*-P^*$ are Riesz subspaces of $\boldsymbol{R}^X$; however, the positive cones of R' and of R^* are, in general, strictly larger than $\bar{R}_+$ and P^*, respectively.

3.3 LEMMA. *Suppose R to be as in* 3.1, *and suppose φ^* to be additive on R_+. Let (f_n) be an increasing sequence in R_+ with $f=\sup_n f_n\in\bar{R}_+$, and let (g_n) be a decreasing sequence in $\bar{R}_+$ with $g=\inf_n h_n\in P^*$. Then $\varphi^*(f)=\sup_n \varphi^*(f_n)$ and $\varphi^*(h)=\inf_n \varphi^*(g_n)$ in $\bar{C}(X)$. In particular, φ^* is additive on P^*.*

PROOF. The validity of the relation $\varphi^*(f)=\sup_n \varphi^*(f_n)(=\lim_n \varphi^*(f_n)$ for $\mathfrak{T}$) is clear from 2.4. This implies, in particular, the additivity of φ^* on $\bar{R}_+$. To show that $\varphi^*(g)=\inf_n \varphi^*(g_n)$ we can suppose that $0\leqq g_n\leqq 1$ for all n, since φ^* is positive homogeneous. We have, for each fixed n,

(1) $\varphi^*(1)=\varphi^*(g_n)+\varphi^*(1-g_n).$

In fact, let (h_k) be an increasing sequence in R_+ such that $g_n=\sup_k h_k$. By hypothesis, $\varphi^*(1)=\varphi^*(h_k)+\varphi^*(1-h_k)$. For $k\to\infty$, the first member $\mathfrak{T}$-converges to $\varphi^*(g_n)$ by 2.4, whence $\varphi^*(1)=\varphi^*(g_n)+\inf_k\varphi^*(1-h_k)$. By subadditivity of φ^*, $\varphi^*(1)\leqq\varphi^*(g_n)+\varphi^*(1-g_n)$. Thus

$$\inf_k \varphi^*(1-h_k)\leqq\varphi^*(1-g_n).$$

But here equality must hold, since $1-h_k\geqq 1-g_n$ for all k, and since φ^* is isotone. This proves (1).

Now we apply the same argument to (1) for $n\to\infty$. This time $\varphi^*(1-g)=\sup_n \varphi^*(1-g_n)$ by 2.4, and using the subadditivity of φ^*again we obtain $\inf_n \varphi^*(g_n)\leqq\varphi^*(g)$. Since $g_n\geqq g$ for all n, equality must hold.

It is now immediately clear that $\varphi^*$is additive on $\overline{R}_+$ and hence on P^*.

3.4 PROPOSITION. *Let R be a Riesz space, containing* 1, *of bounded real functions on X. If φ^* is additive on R_+, then φ^* is additive on R_+^* (where R^* is the Riesz subspace P^*-P^* of $\boldsymbol{R}^X$, Def.* 3.1).

PROOF. We shall prove first that φ^* is additive on the positive cone of the Riesz space $R'=\overline{R}_+-\overline{R}_+$ (cf. Note 3.2). Since every $h\in R'_+$ is of the form $h=f-g$ where f, $g\in\overline{R}_+$ and $g\leqq f$, it suffices (as a simple calculation shows) to show that $\varphi^*(f-g)=\varphi^*(f)-\varphi^*(g)$, since φ^* is additive on $\overline{R}_+$ by 3.3. Thus let f, $g\in\overline{R}_+$ and $g\leqq f$; there exist sequences $f_n\uparrow f$, $g_n\uparrow g$ pointwise where f_n, $g_n\in R_+$. By considering the sequences $f'_n=f_n\vee g_n$, $g'_n=f_n\wedge g_n$ if necessary, we can suppose that $g_n\leqq f_n$ for all n. Now $\varphi^*(f_n-g_n)=\varphi^*(f_n)-\varphi^*(g_n)$ by hypothesis; Thm. 2.4 implies that $\varphi^*(f-g_n)=\varphi^*(f)-\varphi^*(g_n)$ for all n. Now $(f-g_n)$ is a decreasing sequence in $\overline{R}_+$, so by 3.3 we have $\varphi^*(f-g)=\lim_n \varphi^*(f-g_n)$; $\varphi^*(g)=\lim_n \varphi^*(g_n)$ by 2.4. This proves that $\varphi^*(f-g)=\varphi^*(f)-\varphi^*(g)$.

Second, we have to prove the same relation for f, $g\in P^*$ and $g\leqq f$. Again there exist sequences (f_n), (g_n) in $\overline{R}_+$ such that $f_n\downarrow f$, $g_n\downarrow g$ and $g_n\leqq f_n$ for all n. Thus by the first part of the proof, $\varphi^*(f_m-g_n)=\varphi^*(f_m)-\varphi^*(g_n)$ whenever $m\leqq n$. By 2.4 and 3.3, this leads to $\varphi^*(f_m)=\varphi^*(f_m-g)+\varphi^*(g)$. Now $\lim_m \varphi^*(f_m)=\varphi^*(f)$ by 3.3, and so we have

$$\varphi^*(f)=\lim_m \varphi^*(f_m-g)+\varphi^*(g).$$

But again, $\varphi^*(f)\leqq\varphi^*(f-g)+\varphi^*(g)$ since φ^* is subadditive, which implies $\lim_m \varphi^*(f_m-g)=\varphi^*(f-g)$ because φ^* is isotone. Thus, finally, $\varphi^*(f-g)=\varphi^*(f)-\varphi^*(g)$.

3.5 THEOREM. *Let R be a Riesz space, containing* 1, *of bounded real functions on X such that the map φ^* is additive on R_+. Then φ^* is additive on the positive functions of the Riesz space $R^*=P^*-P^*$ (Def. 3.1) and, by linear extension, defines an isometric Riesz isomorphism $\tilde{\varphi}: R^* \to \bar{C}(X)$. Moreover, for every monotone sequence (f_n) in R^* with pointwise limit $f \in R^*$, one has $\tilde{\varphi}(f)=\lim_n \tilde{\varphi}(f_n)$ in $(\bar{C}(X), \mathfrak{T})$.*

PROOF. Since $\varphi^*$is additive on R^*_+ by 3.4, its linear extension $\tilde{\varphi}$ is well defined by $\tilde{\varphi}(f-g):=\varphi^*(f)-\varphi^*(g)$ $(f, g \in R^*_+)$. Moreover, by 2.4 we have $\varphi^*(f \vee g)=\varphi^*(f) \vee \varphi^*(g)$ and hence, $\tilde{\varphi}$ is a Riesz homomorphism. To show that $\tilde{\varphi}$ is injective it suffices, therefore, to see that $\varphi^*(f)=0$ $(f \in R^*_+)$ implies $f=0$; this is the last assertion of 2.4. The proof that $\tilde{\varphi}$ is an isometry (for the sup-norm on R^* and the standard norm on $\bar{C}(X)=C(X)''$, respectively) is exactly the same as in 2.3. Finally, for increasing sequences the final assertion follows from 2.4. For decreasing sequences (g_n) (where, without restriction of generality, we can assume that $0 \leqq g_n \leqq 1$), Thm. 2.4 applies to the sequence $(1-g_n)$; but $\varphi^*(1-g_n)=\varphi^*(1)-\varphi^*(g_n)$ by additivity of φ^*.

3.6 NOTE. It should be observed that the point functionals δ_t (or rather, their $\mathfrak{T}$-continuous extensions to $\bar{C}(X)$) separate the range space $\tilde{\varphi}(R^*)$.

B. Baire and Borel Functions

B.1 *Baire Classes.* Let ω_1 denote the smallest uncountable ordinal. The *Baire classes* $\tilde{B}_\alpha$ $(\alpha<\omega_1)$ are traditionally defined as follows: Let $\tilde{B}_0:=C(X)$ and, for each ordinal $\beta<\omega_1$, let $\tilde{B}_\beta$ denote the set of all functions $f \in \boldsymbol{R}^X$ which are pointwise limits of uniformly bounded sequences in $\bigcup_{\alpha<\beta} \tilde{B}_\alpha$. $\tilde{B}(X):=\bigcup_{\alpha<\omega_1} \tilde{B}_\alpha$, which is obviously a Riesz subspace of $\boldsymbol{R}^X$, is then called the space of *bounded Baire functions* on X.

1.1 NOTE. The boundedness condition is not essential for the previous construction; if the Baire class C_α $(\alpha<\omega_1)$ is defined by $C_0=C(X)$ and C_β the set of all pointwise limits of sequences in $\bigcup_{\alpha<\beta} C_\alpha$, then $f \in C_\alpha$ iff $|f| \wedge n1 \in \tilde{B}_\alpha$ for each $n \in \boldsymbol{N}$. However, for our purposes it will be essential to restrict attention to bounded functions.

1.2 PROPOSITION. *Each $\tilde{B}_\alpha(\alpha<\omega_1)$ is a Banach lattice under the sup-norm.*

PROOF. It is clear from the definition that each $\tilde{B}_\alpha$ is a Riesz subspace of $\boldsymbol{R}^X$, containing 1 and consisting of bounded functions only. To show that $\widetilde{B}_\alpha$ $(1\leqq\alpha<\omega_1)$ is closed under uniform convergence, it suffices to prove that $f\in\widetilde{B}_\alpha$ whenever $f=\sum_{n=1}^{\infty}g_n$, where $g_n\in\widetilde{B}_\alpha$ for all n and where $\|g_n\|<2^{-(n+1)}$. By hypothesis, for each n there exists a sequence $(f_{nk})_{k\in N}$ such that $f_{nk}\in\bigcup_{\beta<\alpha}\widetilde{B}_\beta$ and $\lim_k f_{nk}(t)=\sum_{\nu=1}^{n}g_\nu(t)$, all $t\in X$. Clearly, we can assume that $\|f_{\nu,k}-f_{\nu-1,k}\|\leqq\|g_\nu\|$ $(\nu\geqq 2)$. Now let $\varepsilon>0$ be given, $t\in X$ fixed. There exists n_0 such that $|f(t)-\sum_{\nu=1}^{n}g_\nu(t)|\leqq\varepsilon$ for $n\geqq n_0$. For each n, there exists $k_n\geqq n$ such that for $k\geqq k_n$,

$$|f_{n,k}(t)-\sum_{\nu=1}^{n}g_\nu(t)|\leqq\varepsilon$$

Finally, for all $k\geqq k_n$ we have $\|f_{kk}-f_{nk}\|<2^{-n}$. It follows that $|f_{kk}(t)-f(t)|<2\varepsilon+2^{-n}$, whence the sequence (f_{kk}) converges pointwise to f.

We now show that the mapping φ^* (Def. A. 2. 1) is additive on $\widetilde{B}(X)_+$; for the proof, we recall the definition of the Riesz space R^* (Def. A. 3. 1 and Note A. 3. 2).

1. 3 THEOREM. *The mapping φ^* is additive on $\widetilde{B}(X)_+$ and, by linear extension, defines an isometric Riesz isomorphism of the Banach lattice $\widetilde{B}(X)$ onto a Dedekind σ-complete Riesz subspace of $\overline{C}(X)$ which transforms bounded, pointwise convergent sequences into $\mathfrak{T}$-convergent sequences.*

PROOF. Clearly, $\widetilde{B}(X)$ is complete under the sup-norm. To see that φ^* is additive on $\widetilde{B}_+$, we let $R_0:=C(X)$ and define the transfinite sequence R_α of Riesz subspaces of $\boldsymbol{R}^X$ by $R_\alpha:=(\bigcup_{\beta<\alpha}R_\beta)^*$ (Note A. 3. 2) for every ordinal α, $1\leqq\alpha<\omega_1$.

It is easily verified from the definitions of $\widetilde{B}_\alpha$ and of R^* that $\widetilde{B}_\alpha\subset R_\alpha$ for each ordinal $\alpha<\omega_1$. On the other hand, $R_\alpha\subset\widetilde{B}_{\alpha+1}$ for each limit ordinal $\alpha<\omega_1$. Thus we have $\widetilde{B}(X)=\bigcup_{\alpha<\omega_1}\widetilde{B}_\alpha=\bigcup_{\alpha<\omega_1}R_\alpha$. It now follows from Thm. A. 3. 5 by transfinite induction that φ^* is additive on $\widetilde{B}(X)_+$ and that the linear extension $\tilde{\varphi}$ of φ^* maps bounded, pointwise convergent sequences onto $\mathfrak{T}$-convergent sequences in $\overline{C}(X)$. In fact, if (f_n) is bounded and $f_n\rightarrow f$ pointwise on X then (f_n) order converges in $\widetilde{B}(X)$; thus $(\tilde{\varphi}(f_n))$ order converges and hence $\mathfrak{T}$-converges in $\overline{C}(X)$. In particular, $\tilde{\varphi}(\widetilde{B}(X))$ is Dedekind σ-complete as a Riesz subspace of $\overline{C}(X)$. Of course, we also have $\widetilde{B}(X)=\widetilde{B}(X)^*$ (Note A. 3. 2.).

1.4 COROLLARY. *Under the linear extension of φ^*, $\tilde{B}(X)$ can be identified with a Banach sublattice of $\bar{C}(X)$. Under this identification, we obtain $\langle f, \delta_t\rangle = f(t)$ $(f \in \tilde{B}(X),\ t \in X)$ where δ_t denotes the $\mathfrak{T}$-continuous extension of $f \to f(t)$ $(f \in C(X))$.*

PROOF. The first assertion is immediate from 1.3. Now since $R_0 = C(X)$, the second assertion holds for all $f \in R_1$, in view of A.3.5. The rest follows by transfinite induction over all ordinals $\alpha < \omega_1$.

1.5 REMARK. An alternative to using transfinite induction for establishing the properties of $\tilde{B}(X)$ is given by an application of Zorn's Lemma. In fact, consider the class $\mathscr{R}$ of all Riesz spaces of bounded functions on X that contain $C(X)$ as a Riesz subspace and on whose positive cone φ^* is additive. Then $\mathscr{R}$ is inductively ordered under inclusion ($R \subset S$ meaning that R is a Riesz subspace of S). Thus $\mathscr{R}$ contains a maximal element $\hat{R}$; from Thm. A.3.5 it follows that $\hat{R} = \hat{R}^*$. Clearly $\tilde{B}(X) \subset \hat{R}$ and $\hat{R}$ enjoys the same properties as those stated for $\tilde{B}(X)$ in 1.3, 1.4 above. It is likely but not obvious that necessarily $\hat{R} = B(X)$ (Def. 2.4 below).

B.2 *Borel Classes.* Borel functions on a topological space are usually defined through measurability properties (see [3], [5] and B.3 below) but we prefer a definition (Def. 2.4) that closely parallels the definition of Baire functions given in the previous section B.1.

We recall that L denotes the set (convex cone) of all bounded, lower semi-continuous functions $\geqq 0$ on X (A.1). It was shown in A.2.3 that $R = L - L$ is a Riesz subspace of $\boldsymbol{R}^X$ and that by linear extension, the map $\bar{\varphi}$ (Def. A.2.1) defines an isometric Riesz isomorphism $R \to \bar{C}(X)$. We supplement this by the following lemma.

2.1 LEMMA. *The mapping φ^* is additive on $R_+ = \{f \in R : f \geqq 0\}$.*

PROOF. Since $\varphi^*(f) = \bar{\varphi}(f)$ for all $f \in L$ and since $\bar{\varphi}$ is additive on L by A.2.2, it suffices to show that $\varphi^*(g_1 - g_2) = \varphi^*(g_1) - \varphi^*(g_2)$ whenever $g_1, g_2 \in L$ and $g_2 \leqq g_1$. If $h := g_1 - g_2$ then $h = \inf_B (g_1 - f)$ where $B := (f \in C(X) : 0 \leqq f \leqq g_2\}$. Now $g_1 - f \in L$ for each $f \in B$, and $\varphi^*(g_1 - f) = \bar{\varphi}(g_1 - f) = \bar{\varphi}(g_1) - \bar{\varphi}(f)$; thus by Def. A.2.1 $\inf_B \varphi^*(g_1 - f) = \bar{\varphi}(g_1) - \sup_B \bar{\varphi}(f) = \bar{\varphi}(g_1) - \bar{\varphi}(g_2) = \varphi^*(g_1) - \varphi^*(g_2)$.

By definition of φ^*, $\varphi^*(h) \leqq \inf_B \varphi^*(g_1 - f) = \varphi^*(g_1) - \varphi^*(g_2)$. But $g_1 = g_2 + h$ implies $\varphi^*(g_1) \leqq \varphi^*(g_2) + \varphi^*(h)$ by subadditivity of φ^* A.2.4, hence $\varphi^*(h) \geqq \varphi^*(g_1) - \varphi^*(g_2)$. It follows that $\varphi^*(g_1 - g_2) = \varphi^*(g_1) - \varphi^*(g_2)$.

2.2 PROPOSITION. *Let L_0 denote the convex conical hull of the set of* *characteristic functions* $\{\chi_G : G$ *open in* $X\}$. *Then* $R_0 := L_0 - L_0$ *is a Riesz subspace of* $\boldsymbol{R}^X$, *identical to the linear span of* $\{\chi_A : A \in \mathfrak{A}\}$ *where* $\mathfrak{A}$ *denotes the field of subsets of* X *generated by all open sets.*

PROOF. It is tedious but not difficult to see that $\mathfrak{A}$ is the set of all finite unions $A = \bigcup_{i=1}^{n} A_i$ (for some $n \in \boldsymbol{N}$ dependent on A) where $A_i = F_i \cap G_i$ for suitable closed subsets F_i and open subsets $G_i (i=1, \dots, n)$ of X. If $n=2$ then $\chi_A = \chi_{A_1} \vee \chi_{A_2}$ and $\chi_{A_1} + \chi_{A_2} = \chi_A + \chi_B$ where $B = A_1 \cap A_2 = F \cap G$ for suitable closed F and open G. But $\chi_{A_i} = \chi_{G_i} - \chi_{G_i \cap F_i'}$ ($F_i' = X \backslash F_i$, $i=1, 2$) and it follows that $\chi_A = \sum \chi_{G_i} + \chi_G - (\sum \chi_{G_i \cap F_i'} + \chi_{G \cap F'})$; for $n>2$ we obtain a similar representation of χ_A by induction on the number of summands. It is now clear that the Riesz space of $\mathfrak{A}$-simple functions equals $L_0 - L_0$.

2.3 PROPOSITION. *Let* B_0 *denote the uniform closure of the Riesz subspace* $L_0 - L_0$ *of* $\boldsymbol{R}^X$ *(Prop. 2.2). Then* B_0 *is a Banach lattice which contains* $C(X)$ *and on whose positive cone the map* φ^* *is additive.*

PROOF. It is easy to see that each $f \in C(X)$ can be approximated, uniformly on X, by $\mathfrak{A}$-simple functions; hence B_0 is a Banach lattice (sup-norm) containing $C(X)$. Since φ^* is additive on the positive cone of $R = L - L$ (Prop. 2.1), it is clearly additive on $(L_0 - L_0)_+$. By A.3.5, φ^* is additive on the positive cone of $R^* := (L_0 - L_0)^*$. But clearly, B_0 is a Riesz subspace of R^*, and hence the proposition is proved.

If we want to construct the space of bounded Borel functions on X in a manner analogous to the construction of $\widetilde{B}(X)$ (Section B.1), the Banach lattice B_0 is a good candidate to start with (see 2.8 below). Again, we denote by ω_1 the smallest uncountable ordinal.

2.4 DEFINITION. *Let* α *denote an ordinal,* $1 \leqq \alpha < \omega_1$. *Denoting by* $B_0(X)$ *the Banach lattice* B_0 *defined above (Prop. 2.3) we define, by transfinite induction,* $B_\alpha(X)$ *to be the set of all functions* $f \in \boldsymbol{R}^X$ *that are pointwise limits of uniformly bounded sequences in* $\bigcup_{\beta<\alpha} B_\beta(X)$. $B_\alpha(X)$ *will be called the Borel class (over* X*) of order* α, *and* $B(X) := \bigcup_{\alpha<\omega_1} B_\alpha(X)$ *the space of bounded Borel functions on* X.

2.5 PROPOSITION. *Each* $B_\alpha(X)$ $(\alpha < \omega_1)$, *as well as* $B(X)$, *is a Banach lattice (under the sup-norm).*

PROOF. It is at once clear from Def. 2.4 that each $B_\alpha(X)$, as well as $B(X)$, is a Riesz subspace of $\boldsymbol{R}^X$, containing 1. The fact that each $B_\alpha(X)$ is norm complete follows verbatim as in the proof of 1.2 above; for $B(X)$

it is nearly trivial that it is closed (in $\boldsymbol{R}^X$) under the topology of uniform convergence on X.

We now obtain the exact analogue of Thm. 1.3 for the Banach lattice $B(X)$ of all bounded Borel functions on X.

2.6 THEOREM. *The mapping φ^* is additive on $B(X)_+$ and, by linear extension, defines an isometric Riesz isomorphism of $B(X)$ onto a Dedekind σ-complete Riesz subspace of $\bar{C}(X)$ which transforms bounded, pointwise convergent sequences into $\mathfrak{T}$-convergent sequences.*

PROOF. The proof exactly parallels the proof of Thm. 1.3, except that we have to construct the transfinite sequence $(R_\alpha)_{\alpha<\omega_1}$ of Riesz spaces R_α not starting with $C(X)$ but with $B_0(X)$ (Prop. 2.3). Recalling Def. A.3.1 and Note A.3.2, we let $R_0 := B_0(X)$ and, for each ordinal α, $1 \leqq \alpha < \omega_1$, define R_α to be $(\bigcup_{\beta<\alpha} R_\beta)^*$. It then follows, again by transfinite induction, that $B_\alpha(X) \subset R_\alpha$ for each $\alpha < \omega_1$. Thm. A.3.5 implies that φ^* is additive on $B(X)_+$, and the remaining steps are identical to those in the proof of Thm. 1.3.

2.7 COROLLARY. *Under the linear extension of φ^*, $B(X)$ can be identified with a Banach sublattice of $\bar{C}(X)$. Under this identification, we obtain $\langle f, \delta_t\rangle = f(t)$ $(f \in B(X),\ t \in X)$ where δ_t denotes the $\mathfrak{T}$-continuous extension of $f \to f(t)$ $(f \in C(X))$, and $B(X)$ is sequentially $\mathfrak{T}$-complete.*

2.8 REMARKS. We would have obtained the same Riesz space $B(X)$ if in Def. 2.4, we had replaced B_0 by its dense Riesz subspace $L_0 - L_0$ (Prop. 2.2). Starting the definition by transfinite induction with B_0 rather than $L_0 - L_0$, however, seems to be more satisfactory not so much because B_0 is norm complete but because it contains $C(X)$.

The essential difference in the transfinite build-up of $\tilde{B}(X)$ (Baire functions) and $B(X)$ (Borel functions) lies exactly in the fact that in general, there exist open sets $G \subset X$ whose characteristic function χ_G is not the (pointwise) upper envelope of a countable family of continuous functions (see 3.2 below). Indeed, if each χ_G is (for example, if X is metrizable) then $\tilde{B}(X) = B(X)$. This observation is of some importance for characterizing the regularity of finitely additive Borel measures (see Section C below).

Furthermore, it should be pointed out that in general, the Baire classes $\tilde{B}_\alpha(X)$ (respectively, the Borel classes $B_\alpha(X)$) are not necessarily pairwise distinct. A simple example is furnished by a compact space X which is countable with a finite number of non-isolated points. Finally, Remark 1.5 above applies equally in the present situation, with an obvious minor modification.

B. 3 *Baire and Borel Sets.* Let Y be a set. There is a close relationship between rings and fields of subsets of Y and Riesz subspaces of $\boldsymbol{R}^Y$ which is, however, rarely mentioned in standard measure theoretic texts (see, for example, [1] Chap. II). Thus if R is a Riesz subspace of $\boldsymbol{R}^Y$, then the family $\mathfrak{A}\subset 2^Y$ of all $A\subset Y$ for which $\chi_A\in R$, is a ring; if, in addition, $1\in R$ then $\mathfrak{A}$ is a field (algebra).
Conversely, if $\mathfrak{A}$ is a field then the linear span of $\{\chi_A : A\in\mathfrak{A}\}$ is a Riesz subspace R of R^Y; we obviously have $R=R_+-R_+$ where R_+ is the convex conical hull of $\{\chi_A : A\in\mathfrak{A}\}$. However, often a Riesz space R is given as $R=P-P$, where P is a convex subcone of R_+ which is stable under either finite suprema or finite infima (taken in $\boldsymbol{R}^Y$); compare A. 2.3, A. 3. 2, A. 3. 5, B. 2. 2.

For later reference, let us agree on this definition. As always, X denotes a compact space.

3. 1 Definition. *A subset $A\subset X$ is called a Baire (Borel) set if for some ordinal $\alpha<\omega_1$, $\chi_A\in\tilde{B}_\alpha(X)$ (respectively, $\chi_A\in B_\alpha(X)$). The smallest ordinal with this property is called the Baire (Borel) order of A.*

Since the Riesz spaces $\tilde{B}(X)$ and $B(X)$ of bounded Baire (respectively, Borel) functions are Dedekind σ-complete, by B. 1. 3 and B. 2. 6, it is clear that the fields $\tilde{\mathfrak{B}}$ and $\mathfrak{B}$ of all Baire (respectively, Borel) sets are σ-fields. From the constructions of $\tilde{B}(X)$ and $B(X)$ it is also clear that $\tilde{\mathfrak{B}}$ and $\mathfrak{B}$ are the σ-fields generated by all open F_σ-sets (all open sets), respectively; this is their customary definition.

3. 2 Proposition. *The fields $\tilde{\mathfrak{B}}$ and $\mathfrak{B}$ of all Baire (respectively, Borel) subsets of X are σ-fields satisfying $\tilde{\mathfrak{B}}\subset\mathfrak{B}$; for $\tilde{\mathfrak{B}}=\mathfrak{B}$, it is necessary and sufficient that each open subset of X be the union of countably many compact subsets.*

Proof. The first assertion is clear from the preceding. Second, if $G\subset X$ is open and f is a Baire function such that $S:=\{t: |f(t)|>0\}\subset G$, then by transfinite induction on the Baire class of f it follows that $S\subset\bigcup_1^\infty F_n$ for some sequence (F_n) of compact subsets of G. Hence χ_G is Baire iff G is an F_σ-set. (Cf. [5], p. 221 Thm. D.)

In particular, $\widetilde{\mathfrak{B}}=\mathfrak{B}$ iff every Borel set of order zero is a Baire set of order not larger than one, or equivalently, iff $B_0(X)\subset\widetilde{B}_1(X)$.

Literature

[1] BOURBAKI, N., Eléments de Math., Livre VI: Intégration, Chap. 1-4. Act. Sci. Ind. No. 1175, Hermann & Cie (Paris 1952).

[2] BOURBAKI, N., Eléments de Math., Livre VI: Intégration, Chap. 5. Act. Sci. Ind. No. 1244, Hermann & Cie (Paris 1956).

[3] DUNFORD, N. and SCHWARTZ, J. T., Linear Operators. Part I: General Theory. New York: Wiley 1958.

[4] FREMLIN, D. H., Topological Riesz Spaces and Measure Theory. Cambridge University Press 1974.

[5] HALMOS, P. R., Measure Theory. New York-Toronto-London: D. van Nostrand 1950.

[6] SCHAEFER, H. H., Topological Vector Spaces. New York-Berlin-Heidelberg-Tokyo: Springer 1971.

[7] SCHAEFER, H. H., Banach Lattices and Positive Operators. Berlin-Heidelberg-New York: Springer 1974.

[8] SEMADENI, Z., Banach Spaces of Continuous Functions, Vol. I. Warszawa: Polish Scientific Publishers 1971.

Mathematisches Institut
der Universität Tübingen