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A Bochner type theorem for compact groups*)

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Introduction

Let G be a compact abelian group and Γ_0 be a fixed subsemigroup of the dual group $\Gamma=\hat{G}$ of G. It is well known that in the case when G is the unit circle S^1 and $\Gamma_0=\boldsymbol{Z}_+$ any complex Borel measure $d\mu$ on G with zero nonpositive Fourier-Stieltjes coefficients $c_{-n}=\int_0^{2\pi} e^{int}d\mu(t)$, $n\in\boldsymbol{Z}_+$, is absolutely continuous with respect to the Haar (i. e. Lebesgue) measure $d\sigma$ on $G=S^1$. This is exactly the famous F. and M. Riesz theorem for analytic measures on the unit circle (e. g. [1]). In the sequel we shall use the following

DEFINITION 1. A pair (G, K) of a compact abelian group G and a subset K of its dual group $\Gamma=\hat{G}$ is said to be a Riesz pair if every finite Borel measure $d\mu$ orthogonal to K (i. e. $\int_G \chi(x)d\mu(x)=0$ for any $\chi\in K$) is absolutely continuous with respect to the Haar measure $d\sigma$ on G.

The F. and M. Riesz theorem says that $(S^1, \boldsymbol{Z}_+)$ is a Riesz pair. As shown by S. Koshi and H. Yamaguchi [3] in the case when $\Gamma_0\cup\Gamma_0^{-1}=\Gamma$ and $\Gamma_0\cap\Gamma_0^{-1}=\{1\}$ an analogue of F. and M. Riesz theorem for analytic measures on a compact connected group G does not hold unless $G=S^1$ and $\Gamma_0=\boldsymbol{Z}_+$ (or $\boldsymbol{Z}_-$). A theorem by I. Glicksberg [2] says that (S^1, Γ_0) is a Riesz pair for any subsemigroup Γ_0 of $\boldsymbol{Z}$, such that $\Gamma_0-\Gamma_0=\boldsymbol{Z}$. Consequently any finite complex Borel measure on S^1 that is orthogonal to such $\Gamma_0\subset\boldsymbol{Z}$ and is singular with respect to the Haar measure on S^1 coincides with the zero measure on S^1. On the other hand according to Bochner's theorem (e. g. [1]) (T^2, K) is a Riesz pair, where T^2 is the two dimensional torus and K is the complement in $\boldsymbol{Z}^2=\hat{\boldsymbol{T}}^2$ of a plane angle less then 2π edged at the origin. Here we extend Glicksberg's theorem and give a general construction of Riesz pairs that generalizes the Bochner's one.

1. Low-complete subsets of partially ordered sets

Let G be a compact abelian group. If Γ_0 is a subsemigroup of its dual

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group $\Gamma=\hat{G}$, such that $\Gamma_0 \cup \Gamma_0^{-1}=\Gamma$ then Γ can be provided in a natural way with a partial ordering (the so called Γ_0-ordering), namely, by defining that a follows b ($a>b$) iff $ab^{-1}\in\Gamma_0$, a, $b\in\Gamma$. This ordering possesses the following properties: $ac>bc$ whenever $a>b$ for any a, b, c from Γ; for every $a\in\Gamma$ either $a>1$ or $1>a$, where both conditions can be fulfilled simultaneously. If in addition $\Gamma_0\cap\Gamma_0^{-1}=\{1\}$ then the Γ_0-ordering is complete, i.e. $a>b>a$ implies always that $a=b$. As mentioned before if a Γ_0-ordering of $\Gamma=\hat{G}$ is complete, then $(S^1, \boldsymbol{Z}_+)$ and $(S^1, \boldsymbol{Z}_-)$ are the only Riesz pairs of type (G, Γ_0).

DEFINITION 2[6]. Let Z be a partially ordered set and let Ω be a subset of Z. Ω is said to be low-complete with respect to the given ordering in Z iff for any subset $Y\subset Z$ that is bounded from below by some element of Ω there exists in $\Omega\backslash Y$ a greatest among all lower boundaries of Y.

EXAMPLE 1. Let $Z=\boldsymbol{Z}^2$ is the standard $\boldsymbol{Z}$-lattice in $\boldsymbol{R}^2$ provided with the partial ordering generated by the semigroup $\Gamma_0=\boldsymbol{Z}^2=\{(n, m)\in\boldsymbol{Z}^2 : n\geqq 0\}$. Here $\Gamma_0\cap\Gamma_0^{-1}=\{(0, n) : n\in Z\}\neq\emptyset$. The set $\Omega=\{(n, m) : n\leqq 0, m=0\}$ is low-complete with respect to the Γ_0-ordering in $\boldsymbol{Z}^2$. Indeed, let Y be a subset of $\boldsymbol{Z}^2$ that is bounded from below by some element $(n, 0)$ of Ω. This simply means that $Y\subset\{(n, m)\in\boldsymbol{Z}^2 : n\geqq n_0\leqq 0\}$ and it is clear that in $\Omega\backslash Y$ there exists a greatest low boundary for Y, namely the point $(n_1, 0)$, where $n_1=\max\ \{n : (n, 0)\notin Y\}$.

EXAMPLE 2. Let now $Z=\boldsymbol{Z}^2$ is provided with the partial ordering generated by the semigroup $\Gamma_0=\{(n, m)\in\boldsymbol{Z}^2 : m\leq\sqrt{2}n\}$. Here $\Gamma_0\cap\text{-}\Gamma_0=\{0\}$ The set $\Omega=\{(n, m)\in\boldsymbol{Z}^2 : n\leqq 0, |m|\leqq -n\}$ is low-complete with respect to the Γ_0-ordering in $\boldsymbol{Z}^2$. Indeed let Y be a subset of $\boldsymbol{Z}^2$ that is bounded from below by some element $(n_0, m_0)\in\Omega$. This means that $Y\subset\{(n, m)\in\boldsymbol{Z}^2 : m\leqq\sqrt{2}(n-n_0)+m_0\}$, i.e. Y lies on the right hand side of the line $\lambda : y=\sqrt{2}(x-n_0)+m_0$. If λ_1 is the rightest possible line parallel to λ, so that Y lies on the right hand side of λ_1, then $\lambda_1\cap\{(x, y)\in\boldsymbol{R}^2 : x\leqq 0, |y|=-x\}$ is a finite segment from λ_1 and it is easy to see that there are points from $\Omega\backslash Y$ that are closest to λ_1. That it will be only one closest to λ_1 point in $\Omega\backslash Y$ follows from the fact that the line $y=\sqrt{2}x$ contains only one point (namely 0) from $\boldsymbol{Z}^2$.

EXAMPLE 3. In the previous example one can take Q to be any subset of $\boldsymbol{R}^2$, which intersections with every line parallel to $y=\sqrt{2}\,x$ are bounded segments and to define Ω to be $Q\cap\boldsymbol{Z}^2$, or, equivalently, all the sets Ω-(n, m), where $(n, m)\in\Omega$, to be finite.

2. Main results

The next theorem is an extension of the mentioned at the beginning Glicksberg's theorem.

THEOREM 1. *Let G be a compact abelian group, let Γ_0 be a fixed subsemigroup of the dual group $\Gamma=\hat{G}$ of G, for which $\Gamma_0\cup\Gamma_0^{-1}=\Gamma$, $\Gamma_0\cap\Gamma_0^{-1}=\{1\}$ and let Σ be a nonempty subset of $\Gamma\backslash\Gamma_0$ that is low-complete with respect to the Γ_0-ordering in Γ. Then every finite complex Borel measure $d\mu$ on G that is orthogonal to the set $K=\Gamma\backslash\Sigma$ and is singular with respect to the Haar measure $d\sigma$ on G coincides with the zero measure on G.*

PROOF. Assume that $d\mu\neq 0$. Then $d\mu$ is not orthogonal to Γ by the uniqueness theorem for Fourier-Stieltjes transforms. Let $Y=\{\chi\in\Gamma:$ $\int_G\chi_1(g)\,d\mu(g)=0$ for every $\chi_1>\chi\}$. Note that Y contains every $\chi\in\Gamma$ that follows some element of Y. Also Y contains the whole semigroup Γ_0. On the other hand Y is bounded from below by some element of Σ because in the opposite case every element of Σ will follow some element of Y and consequently will belong to Y in contradiction with $d\mu\not\perp\Gamma$. Since Σ is a low-complete subset of Γ there will exist in $\Sigma\backslash Y$ an element that is biggest among all low boundaries of Y, say δ. Then we have $\delta(\Gamma_0\backslash\{1\})\subset Y$. To see this assume $\delta\cdot\chi\notin Y$ for some $\chi\in\Gamma_0\backslash\{1\}$. Therefore there exists a $\chi_1\in\Gamma_0$ such that $\int_G\chi_1(g)\chi(g)\delta(g)\,d\mu(g)\neq 0$. Thus $\chi_1\chi\delta\in\Sigma\backslash Y$ because $d\mu$ is orthogonal to $\Gamma\backslash\Sigma$ and because of the definition of Y. Since $\chi_1\chi\delta>\chi\delta$, $\chi_1\chi\delta$ is not a low boundary of Y. Consequently $\chi_1\chi\delta$ follows some element of Y and henceforth $\chi_1\chi\delta\in Y$ by the definition of Y. But this is a contradiction. Hence $\chi\delta\in Y$ for every $\chi\in\Gamma_0\backslash\{1\}$, i. e. $\delta\Gamma_0\backslash\{1\}\subset Y$, wherefrom $\int_G\chi(g)\delta(g)\,d\mu(g)=0$ for every $\chi\in\Gamma_0\backslash\{1\}$. Denote by $d\nu$ the complex measure $d\nu=\delta d\mu$ on G. We have:

$$(1)\qquad \int_G\chi(g)\,d\nu(g)=\int_G\chi(g)\delta(g)\,d\mu(g)=0$$

for every $\chi\in\Gamma_0\backslash\{1\}$. Put $d\tilde{\nu}=\delta d\mu-d\sigma$. Then $\int_G\delta(g)\,d\mu(g)=0$ by the Helson-Lowdenslager theorem [6] because $\int_G\chi(g)\,d\tilde{\nu}(g)=0$ for each $\chi\in\Gamma_0\backslash\{1\}$ and $d\tilde{\nu}_s=\delta d\mu$. This implies $\delta\in Y$. But this is a contradiction. The theorem is proved.

The next theorem generalizes Bochner's theorem.

THEOREM 2. *Let G be a fixed compact abelian group, let Ξ be a family of subsemigroups $\{\Gamma_\alpha\}_{\alpha\in\mathfrak{A}}$ of its dual group $\Gamma=\hat{G}$ such that $\Gamma_\alpha\cup\Gamma_\alpha^{-1}=\Gamma$ for every $\alpha\in\mathfrak{A}$ and let $\delta_\alpha\in\Gamma_\alpha^{-1}$ for every $\alpha\in\mathfrak{A}$. If the complement $\Sigma=\Gamma\backslash K$ of the set $K=\bigcup_{\alpha\in\mathfrak{A}}\delta_\alpha\Gamma_\alpha$ is low-complete with respect to the Γ_0-ordering, generated by some semigroup Γ_0 from Ξ with $\Gamma_0\cap\Gamma_0^{-1}=\{1\}$, then every finite Borel measure on G that is orthogonal to K is absolutely continuous with respect to the Haar measure $d\sigma$ on G.*

Theorem 2 means simply that under above conditions (G, K) is a Riesz pair.

PROOF. Let $d\mu$ be a finite Borel measure on G that is orthogonal to the set K. Then $d\mu\perp\delta_\alpha\Gamma_\alpha$ for each $\alpha\in\mathfrak{A}$ and thatswhy the measure $d\nu_\alpha=\delta_\alpha d\mu$ is orthogonal to the semigroup Γ_α for each $\alpha\in\mathfrak{A}$. As shown by Yamaguchi [4] both absolutely continuous $((d\nu_\alpha)_a)$ and singular $((d\nu_\alpha)_s)$ components of the measure $d\nu_\alpha$ with respect to $d\sigma$ are orthogonal to Γ_α, i.e. $(d\nu_\alpha)_a\perp\Gamma_\alpha$, $(d\nu_\alpha)_s\perp\Gamma_\alpha$. If $d\mu=d\mu_a+d\mu_s$ is the Lebesgue decomposition of $d\mu$, then $\delta_\alpha d\mu_s\perp\Gamma_\alpha$ since $\delta_\alpha d\mu_s=(\delta_\alpha d\mu)_s=(d\nu_\alpha)_s\perp\Gamma_\alpha$. Hence $d\mu_s\perp\delta_\alpha\Gamma_\alpha$ for any $\alpha\in\mathfrak{A}$ and consequently $d\mu_s\perp K$ for $K=\bigcup_{\alpha\in\mathfrak{A}}\delta_\alpha\Gamma_\alpha$. Now G, $d\mu_s$, $\Sigma=\Gamma\backslash K$ and Γ_0 satisfy the conditions of Theorem 1 and thatswhy $d\mu_s=0$. Hence $d\mu=d\mu_a$.
Q. E. D.

In the case when $\Gamma_\alpha\cap\Gamma_\alpha^{-1}=\{1\}$ Theorem 2 is proved in[6]. Bochner's theorem and its n-dimensional version for Borel measures on the n-dimensional torus T^n is a simple corollary from Theorem 2. Actually we can obtain the following:

COROLLARY 1. *Let L be a closed convex set in $\boldsymbol{R}^n$ that is contained entirely in some half-space E_0 of $\boldsymbol{R}^n$ with $\lambda\cap\boldsymbol{Z}^n=\{0\}$, where λ is the $(n$-$1)$-dimensional boundary of E_0 and such that the intersections of L with all $(n$-$1)$-dimensional spaces parallel to λ are bounded. Then every finite complex Borel measure on the n-dimensional torus T^n with vanishing outside L Fourier-Stieltjes coefficients is absolutely continuous with respect to the Haar measure $d\sigma$ on T^n.*

PROOF. As a closed convex set, L is an intersection of certain family of closed half-spaces E_α, $\alpha\in\mathfrak{A}$, i.e. $L=\bigcap_{\alpha\in\mathfrak{A}}E_\alpha$. Without loss of generality we can assume that the boundary of E_α contains some point (say Z_α) from $\boldsymbol{Z}^n$ for every α and that E_0 belongs to this family. For semigroups $\Gamma_\alpha=(Z_\alpha-E_\alpha)\cap\boldsymbol{Z}^n$ we have: $0\in\Gamma_\alpha$, $\Gamma_\alpha\cap-\Gamma_\alpha=\{0\}$ for each $\alpha\in\mathfrak{A}$. For $K=\boldsymbol{Z}^n\backslash(-L)$ we get: $K=-(\boldsymbol{Z}^n\backslash L)=-(\boldsymbol{Z}^n\backslash\bigcap_{\alpha\in\mathfrak{A}}E_\alpha)=-\bigcup_{\alpha\in\mathfrak{A}}(\boldsymbol{Z}^n\backslash E_\alpha)=-\bigcup_{\alpha\in\mathfrak{A}}(\boldsymbol{Z}^n\backslash$

$(Z_\alpha-\Gamma_\alpha))=\bigcup_{\alpha\in\mathfrak{A}}(\boldsymbol{Z}^n\backslash\Gamma_\alpha-Z_\alpha))$. The set $\Sigma=\boldsymbol{Z}^n\backslash K=\boldsymbol{Z}^n\cap L$ is low complete with respect to the Γ_0-ordering on Γ. Indeed, let Y be a bounded from below subset of $\boldsymbol{Z}^n$. This means that $Y\subset-E_0+Z_1$ for some point $Z_1\in\boldsymbol{Z}^n$. Let $Z_2\in\boldsymbol{Z}^n$ be such that $Y\subset-E_0+Z_2$, but $Y\not\subset-E_0+Z$, $Z\in\boldsymbol{Z}^n$, $Z>Z_2$. From the hypotheses it follows that $-(E_0+Z_2)\cap Z^n$ is a finite set and consequently, since $\lambda\cap\boldsymbol{Z}^n=\{0\}$, there exists a unique elment $Z_3\in(\boldsymbol{Z}^n\cap L)\backslash Y$ that is closest to $(\lambda+Z_2)\cap L$ amongst all elements of $\boldsymbol{Z}^n\cap L$, λ being the boundary of E. It is clear that Z_3 is the biggest amongst all low boundaries of Y belonging to $(\boldsymbol{Z}^n\cap L)\backslash Y$. The proof now terminates by applying Theorem 2.

COROLLARY 2. *Let F be a real linear functional of $\bigoplus_{n=1}^{\infty}\boldsymbol{R}$ and let L be a closed convex set in $\bigoplus_{n=1}^{\infty}\boldsymbol{R}$ such that: (i) $F(Z)\geqq 0$ on L; (ii)* Ker $F\cap\bigoplus_{n=1}^{\infty}\boldsymbol{Z}=\{0\}$; *(iii) the set $L\cap\{Z\in\bigoplus_{n=1}^{\infty}\boldsymbol{Z}:\alpha=F(Z)\}$ is finite for every positive number α. Then $(T^\infty,(\bigoplus_{n=1}^{\infty}\boldsymbol{Z})\backslash L)$ is a Riesz pair.*

EXAMPLE. Let $\{y_k\}_{k=1}^{\infty}$ be a fixed sequence of linearly independent over $\boldsymbol{Z}$ positive numbers and let F be the linear functional on $\bigoplus_{n=1}^{\infty}\boldsymbol{R}$, defined as: $F(x_1,\dots,x_k,\dots)=\sum_{k=1}^{\infty}y_kx_k$ (note that at most finite many of x_k are different from 0). Clearly Ker $F\cap\bigoplus_{n=1}^{\infty}\boldsymbol{Z}=(0,\dots,0,\dots)$ and thatswhy each of the sets $\{Z\in\bigoplus_{n=1}\boldsymbol{Z}:F(Z)=\alpha\}$ contains at most one point from $\bigoplus_{n=1}^{\infty}\boldsymbol{Z}$, α being a positive number. Hence for any closed convex set L in $\{Z\in\bigoplus_{n=1}\boldsymbol{R}:F(Z)\geqq 0\}$ the set $L\cap\{Z\in\bigoplus_{n=1}\boldsymbol{Z}:F(Z)=\alpha\}$ is finite for each $\alpha>0$. Therefore $(T^\infty,(\bigoplus_{n=1}^{\infty}\boldsymbol{Z})L)$ is a Riesz pair, according to Corollary 2.

Note, that in the considered in [7] general case when $\Sigma\subset\Gamma\backslash\Gamma_0$ and the sets $(\Sigma-\chi)\cap\Gamma_0$ are finite for all $\chi\in\Sigma$, the set Σ is low-complete with respect to the given complete Γ_0-ordering of Γ.

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